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Comments on Field Line Resonances and Micropulsations

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Summary

Features and limitations of a simple plasma model exhibiting hydromagnetic transverse mode field line resonances are discussed in relation to geomagnetic micropulsations.

Introduction

Recently a fair degree of success has been attained in understanding features of micropulsation signals using a remarkably simple model of the magnetospheric plasma (Southwood 1974a, b). This paper uses the model to compute a solution to a specific problem: excitation of a dissipative, inhomogeneous plasma by a monochromatic source. Comments are then made on the form of the solution and the limitations of the model.

The model

The model chosen by Southwood (1974a) is simple in the extreme but has the virtue of retaining one important feature of finite inhomogeneous plasma behaviour in the hydromagnetic regime, namely that, at a fixed frequency, only on isolated magnetic shells can standing waves satisfying the transverse mode dispersion relation exist. The magnetic field is taken to be uniform in the $\hat{z}$ direction and the plasma is contained between conducting boundaries at $z = \pm a$. Periodic boundary conditions are applied in the $\hat{y}$ direction and the plasma density, $\rho(x)$ (and hence Alfvén speed) is taken to be a function only of x. Small disturbances may be taken to vary as

$$\exp(i\lambda y + ikz + i\omega t)$$

in y, z, where allowed values of k are discrete. Field line resonance for frequency, ω, is said to occur at the point where

$$\omega^2 = k^2 A^2(x) = \omega_A{}^2(x), \text{ say.}$$

For frequencies $\omega^2 < \lambda^2 A^2{}_{\min}$ where $A_{\min}$ is the minimum value of the Alfvén speed in the plasma the shells where $\omega^2 = \omega_A{}^2(x)$ play a very important role in determining the long term response of the plasma. The equation governing the component of electric field in the $\hat{y}$ direction is approximately

$$\frac{d^2 E_y}{dx^2} + \frac{dK^2/dx}{(K^2-k^2)} \frac{dE_y}{dx} - \lambda^2 E_y = 0 \qquad (1)$$

where $K^2 = \omega^2/A^2(x)$, Southwood (1974a) (note there is a sign error in this paper in equations (4) and (5)) and if ρ varies linearly with x we have

$$\frac{d^2 E_y}{dx^2} + \frac{1}{x-x_0}\frac{dE_y}{dx} - \lambda^2 E_y = 0 \tag{2}$$

where $x = x_0$ is the resonant shell. In the vicinity of this shell

$$E_y \propto \ln(x-x_0)$$

and

$$\frac{dE_y}{dx} \propto \frac{1}{x-x_0} \propto E_x$$

as the general solution of (1) is

$$AK_0(\lambda(x-x_0)) + BI_0(\lambda(x-x_0))$$

where K_0 and I_0 are modified Bessel functions. The result of this singular behaviour is that in the vicinity of x_0 the long time response of the system is

$$E_x \propto \cos \omega_A(x_0)\, t$$

$$E_y \propto \frac{\sin \omega_A(x_0)\, t}{t}$$

(Radoski 1974) so that as $t \to \infty$ each shell oscillates independently at its 'Alfvén frequency' $\omega_A(x)$ with plasma motion entirely in the $\hat{y}$ direction. Eventually this motion would be limited by dissipative effects as Burghes, Kendall & Sweet (1969) point out. This asymptotic behaviour would be expected whatever source had been introduced for a finite length of time though the source would determine how the amplitude of oscillation would vary with shell. Once the source is removed each shell would apparently return to oscillating at its own Alfvén frequency.

If a source is continuously present and some dissipation is present a rather different story emerges. Including, say, a finite conductivity, σ, modifies equation (1) to

$$\frac{d^2 E_y}{dx^2} + \frac{dK^2/dx}{K^2 - k^2\,(1+i\alpha)}\frac{dE_y}{dx} - \lambda^2 E_y = 0 \tag{3}$$

where

$$\alpha = \frac{c^2\,\omega}{4\pi\sigma A^2} \qquad c = \text{velocity of light}$$

(for a slightly more complicated but completely analogous case see McClay (1973)). Now let us introduce a source at $x = b$ such that

$$E_y(b) = E_0\, e^{i\omega_0 t} \tag{4}$$

and assume that at $x = a$

$$E_y(a) = 0 \qquad a < x_0 < b. \tag{5}$$

For a monotonic density distribution, (3) can be approximately written, putting $\varepsilon = \alpha k^2/(dK^2/dx) \simeq (c^2\,\omega_0/4\pi\sigma\, A^2(x_0))/(d(\ln K^2)/dx)$

$$\frac{d^2 E_y}{dx^2} + \frac{1}{x-x_0-i\varepsilon}\frac{dE_y}{dx} - \lambda^2 E_y = 0$$

and the solution satisfying the boundary conditions (4) and (5) is, taking the convention

$\lambda > 0$, ω_0 positive or negative

$$E_y = \frac{E_0 \, e^{i\omega_0 t}}{W(a, b, x_0)} \left(\frac{K_0(\lambda(x_0 - x + i\varepsilon))}{K_0(\lambda(x_0 - a))} - \frac{I_0(\lambda(x_0 - x + i\varepsilon))}{I_0(\lambda(x_0 - a))} \right)$$

where

$$W(a, b, x_0) = \frac{K_0(\lambda(x_0 - b))}{K_0(\lambda(x_0 - a))} - \frac{I_0(\lambda(b - x_0))}{I_0(\lambda(x_0 - a))}$$

if ε is small ($\ll b - x_0, x_0 - a$).
Now, if $|\varepsilon\lambda| \ll 1$

$$K_0(\lambda(x_0 - b)) = K_0(\lambda(b - x_0) - i\pi \,\mathrm{sign}\,(\varepsilon\lambda) \, I_0(\lambda(b - x_0))$$

because of the branch point behaviour at $x = x_0$ in $K_0(\lambda(x_0 - x))$ and so if $|\lambda(b - x_0)|$, $|(\lambda(x_0 - a))| \gg 1$

$$W(a, b, x_0) \simeq \frac{-i\pi \,\mathrm{sign}\,(\varepsilon\lambda) \, I_0(\lambda(b - x_0))}{K_0(\lambda(x_0 - a))}$$

and

$$E_y(x, t) \simeq \frac{E_0 \, e^{i\omega_0 t} K_0(\lambda(x_0 - x + i\varepsilon))}{-i\pi \,\mathrm{sign}\,(\varepsilon\lambda) \, I_0(\lambda(b - x_0))}$$

$$\simeq \frac{E_0 \, e^{i\omega_0 t} K_0(|\lambda(x_0 - x + i\varepsilon)|)}{-i\pi \,\mathrm{sign}\,(\varepsilon\lambda) \, I_0(\lambda(b - x_0))}$$

$$+ \frac{E_0 \, e^{i\omega_0 t} I_0(\lambda(x_0 - x))}{\pi I_0(\lambda(b - x_0))} \cot^{-1} \left(\frac{x_0 - x}{|\varepsilon|} \right).$$

Now there is strong evidence (Southwood 1974a, b; Chen & Hasegawa 1974a), that a significant source of micropulsations is surface waves generated on the magnetopause by Kelvin–Helmholtz instability. Such waves will propagate with a definite sense across the magnetic field (Eastwards after 1200 LT, Westwards before). An appropriate source variation is therefore

$$\cos(\lambda y - \omega_0 t)$$

regarding y as corresponding to East–West direction. For such a source, taking the case of $dK^2/dx > 0$ and noting $\varepsilon < 0$ we have approximately

$$E_y(x, t) = \frac{E_0 \sin(\lambda y - \omega_0 t) \, K_0(|\lambda(x - x_0 + i|\varepsilon|)|)}{\pi I_0(\lambda(b - x_0))}$$

$$+ \frac{E_0 \cos(\lambda y - \omega_0 t) \, I_0(\lambda(x - x_0))}{\pi I_0(\lambda(b - x_0))} \cot^{-1} \left(\frac{x_0 - x}{|\varepsilon|} \right). \quad (6)$$

The other component of electric field present in the disturbance $E_x(x, t)$ is approximately given by

$$i\lambda E_x(x, t) = \frac{dE_y}{dx}(x, t) \quad \text{(Southwood 1974a)}$$

and so in the above case we have

$$E_x(x,t) = \frac{E_0 \cos(\lambda y - \omega_0 t) K_1(|\lambda(x-x_0+i|\varepsilon|)|)}{\pi I_0(\lambda(b-x_0))} \frac{(x-x_0)}{((x-x_0)^2+\varepsilon^2)^{\frac{1}{2}}}$$
$$+ \frac{E_0 \sin(\lambda y - \omega_0 t) I_1(\lambda(x-x_0))}{\pi I_0(\lambda(b-x_0))} \cot^{-1}\left(\frac{x_0-x}{|\varepsilon|}\right)$$
$$+ \frac{E_0 \sin(\lambda y - \omega_0 t) I_0(\lambda(x-x_0))}{\lambda \pi I_0(\lambda(b-x_0))} \frac{|\varepsilon|}{(x_0-x)^2+\varepsilon^2} \quad (7)$$

A salient feature of this solution is that the disturbance on the source side of the resonance can be split into two parts. One part in phase with the source and a part in quadrature with the source. The part in phase cuts off within a scale $|\varepsilon|$ on the other side of the resonance but is clearly the dominant part in the source vicinity and in this vicinity the disturbance is elliptically polarized with major axis in the x or y direction (depending on the size of λ). Now on the source side of the resonance there will occur a region where the parts of the disturbance in quadrature and in phase with the source will have comparable magnitude. Here tilts in the polarization ellipse will occur as Lanzerotti, Fukunishi & Chen (1974) have pointed out and observed in pulsation data. They also note a similar tilting effect was observed by Van–Chi *et al.* (1968). These tilts have an interesting physical significance which we describe later. In the resonance vicinity changes in polarization take place on a scale $|\varepsilon|$. The fact that E_y varies logarithmically while the function describing E_x has a pole at $x = x_0$ suggests that unless λ is very large actually at resonance E_x would be the dominant perturbation and this charges in phase by 180° in the resonance vicinity. The dominant part of E_y in this vicinity varies smoothly.

Limitations

The features of the simple model described here appear to have some counterpart in the actual magnetospheric situation as observations indicate (Craven & Lawrie 1975; Lanzerotti *et al.* 1974; Chen & Hasegawa 1974a; Southwood 1974a, b) and in this respect it is very important to realize its limitations as well as its implications for construction of a more realistic model.

First, let us note that the resonance scale is ludicrously small if estimated on the basis of magnetospheric direct conductivity. In this respect note that

$$\varepsilon \sim \gamma/(d\omega_A(x_0)/dx)$$

where γ^{-1} is the damping time scale for a hydromagnetic wave due to finite conductivity. In practise Alfvén waves in the magnetosphere can interact with hot particles in bounce resonance with the disturbance. This was first noted by Dungey (1965) and discussed further by Southwood, Dungey & Etherington (1969) and Southwood (1973). The latter references were primarily concerned with the possibility of wave generation by this means but in the absence of strong hot particle gradients, damping would be the natural result of resonance. In addition the ionosphere may provide a source of damping though electric amplitudes are liable to be small in the ionosphere.

Further comments on ε are due. For the theory presented here to be valid, $|\varepsilon|$ must exceed the disturbance amplitude as the treatment is a linear one. It should be noted that for a standing wave along a field line the maximum field line displacement is of order $LR_E(b/B)$ where b is magnetic amplitude and LR_E is the equatorial radial distance. b/B of a few per cent is commonly observed in the magnetosphere (see e.g. Cummings, O'Sullivan & Coleman 1969).

Next let us note that the treatment given here gives the response of the system on a time in excess of $1/\gamma$ as the response of the system at other frequencies would damp on this scale. Also it should be noted if the source were removed one might expect each shell of the system to return to oscillating at its own Alfvén frequency (on the basis of Radoski's (1974) study of the asymptotic response of a very similar system). Whether this need be significant could be questioned on the ground that the bulk of energy is deposited in a region $\sim \gamma/(d\omega_A/dx)$ and so the damping time scale is of order the scale significant phase differences would be observable in the resonant vicinity. Observations do suggest that one does observe signals well away from resonance however.

As a last remark on limitations we should finally note that interpretation of ground data is complicated by the existence of the atmosphere and ionospheric layers. Recent calculations of the expected differences in polarization behaviour of large scale disturbances on the ground and in the magnetosphere by Hughes (1974) suggest that magnetospheric patterns may be turned through approximately 90° when seen on the ground. Similar conclusions were reached by Inoue (1973). Fast variations (on horizontal scales of less than ionospheric vertical scales ($\sim$ 50 km)) are liable to be strongly shielded by the ionosphere and so small scale structure in the magnetosphere may not be observable on the ground (Hughes, private communication.)

Discussion

In spite of the many obvious limitations of an oversimplified model it certainly appears to be of importance in its success in explaining features of micropulsations and in clarifying the physics of the resonance phenomenon. In the latter respect the existence of a well-defined signal in quadrature with the source and the absorption of energy in the vicinity of resonance (see Southwood 1974a, Chen & Hasegawa 1974b) are very reminiscent of any system pumped at a resonance frequency. (e.g. a damped harmonic oscillator pumped at its resonant frequency oscillates in quadrature with the pump. Energy is absorbed by whatever dissipative effect provides the damping.) This might lead one tentatively to identify the part of the disturbance in quadrature with the source as an ' eigenmode of the system. In this sense the system has an infinite number of ' eigenmodes ' which in turn explains why the system will not reveal an eigenmode in the normal sense without inclusion of a monochromatic source. To discriminate one eigenmode from neighbours would require infinite time if eigenfrequencies are continuously distributed due to the uncertainty (or indeterminacy) principle. In practise finite amplitude effects (or the requirement that on short enough scales the plasma should oscillate coherently) might be expected to provide an inherent bandwidth or separation of eigenfrequencies.

Energy flow

We have already remarked that tilts in horizontal polarisation ellipse are predicted by the model described here. They are predicted on the source side of the resonance and occur because parts of the disturbance in phase and in quadrature with the source occur here. Not surprisingly in view of the comments in the previous section they are intimately related to the energy flow in the wave. As a result tilts observed in ground micropulsations can indicate the direction of energy flow in the magnetospheric disturbance quite generally.

Since a standing structure is expected along the ambient field we are interested in the Poynting flux across the field. In the hydromagnetic regime the component of

Poynting flux perpendicular to **B** can be written

$$\left(\frac{c\mathbf{E}\times\mathbf{b}}{4\pi}\right)_{\perp} = \mathbf{u}\frac{\mathbf{B}\,.\,\mathbf{b}}{4\pi} \tag{8}$$

where **u** is the field line velocity. Now taking the hydromagnetic momentum equation in the form

$$\rho\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\nabla\frac{\mathbf{B}\,.\,\mathbf{b}}{4\pi} + \frac{\mathbf{B}\,.\,\nabla\mathbf{b}}{4\pi}$$

where $\boldsymbol{\xi}$ is field line displacement and noting that

$$\mathbf{b}_{\perp} = B\frac{\partial \boldsymbol{\xi}}{\partial z}, \qquad \mathbf{u} = \frac{\partial \boldsymbol{\xi}}{\partial t}$$

we see that

$$\left(-\rho\omega^2 + \frac{B^2 k^2}{4\pi}\right)u_y = -\frac{\partial^2}{\partial t\,\partial y}\left(\frac{Bb_z}{4\pi}\right) \tag{9}$$

and

$$\frac{\partial u_x}{\partial z} = \frac{1}{B}\frac{\partial b_x}{\partial t} \tag{10}$$

$$\frac{\partial u_y}{\partial z} = \frac{1}{B}\frac{\partial b_y}{\partial t} \tag{11}$$

Now let us assume that $\omega^2 > B^2k^2/4\pi\rho = \omega_A{}^2(x)$ (this corresponds to the region between source and resonance in the case whose solution is given earlier (equations (6) and (7)) and take the case of the wave moving in the $\hat{y}$ direction. Equation (9) shows that u_y and b_z are in phase. Equations (10) and (11) then indicate (remembering the standing structure along the field) that if we suppose that when b_y is maximum b_x is positive when u_y is maximum u_x is positive. Since b_z and u_y are in phase this means that the average value of $(\mathbf{E}\times\mathbf{b})_x$ is positive. (Also, since b_z and u_y are in phase, (8) shows that there is a mean Poynting flux in the direction of propagation.) For interpretation of ground observations the Hughes rotation needs to be remembered (Hughes 1974) and so generally one may conclude if a wave is propagating eastward (westward) and its horizontal magnetic polarization ellipse seen on the ground in the Northern Hemisphere is tilted into the NE–SW (NW–SE) quadrant and the observer is thought to be above the latitude of the resonant shell for the frequency of interest (so that $\omega^2 > \omega_A{}^2(x)$) energy is flowing southward. In the opposite case, when $\omega_A{}^2(x) > \omega^2$, the tilts reverse for southward flow of energy. In the magnetosphere the tilts of the magnetic polarization ellipse perpendicular to the field would be opposite to those on the ground while the electric polarization ellipse in the magnetosphere would be tilted the same way as the magnetic one on the ground as

$$E_x \propto -u_y$$
$$E_y \propto u_x.$$

It can be checked that the tilts present in the solution given in equations (7) and (8) square with energy flow in the negative $\hat{x}$ direction.

In this section the physical arguments presented are only valid outside the immediate resonance vicinity. Within this region, $|x-x_0| < |\varepsilon|$ the phase relationship between u_y and b_z changes rapidly with x exhibiting a total change of π on the scale $2|\varepsilon|$.

Finally, a practical point, the presence of the plasmapause jump in number density (and so Alfvén speed) means that in reality for a given frequency 2 or 3 resonance regions may exist (outside, at and/or inside the plasmasphere). This is a further complication but one which the simple ideas discussed here could easily be extended to include.

Conclusions

The simple model of hydromagnetic wave behaviour in a finite non-uniform system discussed here has had some success in interpreting observations of geomagnetic micropulsations and the gross features of polarization sense and amplitude behaviour had already been noted (e.g. Southwood 1974a, b). Here we have emphasised the apparent limitations of the model and in addition suggested theoretical physical interpretation of the mathematics of the model. Finally the observational feature of tilts in the horizontal polarization ellipse have been described in the light of the model. Potentially these can tell us of the energy flow in the disturbance.

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