Comments on "Least Squares Restoration of Multichannel Images"

Michael K. Ng and Wilson C. Kwan

***Abstract*—In this correspondence, we give the correct matrix formulation arising from the constrained optimization of the least squares restoration of multichannel images in the above paper.[1]**

***Index Terms*—Image restoration, least squares method.**

The degradation of multichannel images can be modeled as solving

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \mathbf{n} \tag{1}$$

where

- $\mathbf{H}$ degradation matrix;
- $\mathbf{n}$ unknown Gaussian noise or measurement errors;
- $\mathbf{g}$ observed multichannel image;
- $\mathbf{f}$ original multichannel image.

For N channels of $M \times M$ pixels each, the observed and original images can be expressed as

$$\mathbf{g} = \begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \vdots \\ \mathbf{g}_N \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_N \end{pmatrix}$$

where each of the M^2-vectors $\mathbf{g}_i$ and $\mathbf{f}_i$ are the observed and the original images in each channel. The multichannel degradation operator $\mathbf{H}$ is given by

$$\mathbf{H} = \begin{pmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} & \cdots & \mathbf{H}_{1N} \\ \mathbf{H}_{21} & \mathbf{H}_{22} & \cdots & \mathbf{H}_{2N} \\ \vdots & \cdots & \ddots & \vdots \\ \mathbf{H}_{N1} & \mathbf{H}_{N2} & \cdots & \mathbf{H}_{NN} \end{pmatrix}.$$

Here, the operators $\mathbf{H}_{ii}$ and $\mathbf{H}_{ij}$ $(i \neq j)$ represent the within-channel and the cross-channel degradation operators, respectively.

As the system (1) is ill-conditioned and generally not positive definite, we solve it by using a constrained minimization technique. The solution $\hat{\mathbf{f}}$ is determined by minimizing $\|\mathbf{Q}\mathbf{f}\|_2^2$ subject to $\|\bar{\mathbf{H}}_i - \mathbf{g}_i\|_2^2 = \|\mathbf{n}_i\|_2^2$ for $1 \leq i \leq N$, where

$$\bar{\mathbf{H}}_i = (\mathbf{H}_{i1}, \mathbf{H}_{i2}, \ldots, \mathbf{H}_{iN}).$$

Using the method of Lagrange multipliers, the solution is obtained by minimizing

$$\Phi(\mathbf{f}, \lambda) = \sum_{i=1}^{N} \lambda_i \left(\|\bar{\mathbf{H}}_i \mathbf{f} - \mathbf{g}_i\|_2^2 - \|\mathbf{n}_i\|_2^2\right) + \|\mathbf{Q}\mathbf{f}\|_2^2 \tag{2}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)^t$ is the Lagrange multiplier vector. The necessary conditions for the solution of (2) are

$$\nabla_f \Phi(\mathbf{f}, \lambda) = \mathbf{0} \tag{3}$$

$$\nabla_\alpha \Phi(\mathbf{f}, \lambda) = \mathbf{0}. \tag{4}$$

From (3), we have

$$\left[\sum_{i=1}^{N} \lambda_i \bar{\mathbf{H}}_i^t \bar{\mathbf{H}}_i + \mathbf{Q}^\mathbf{t}\mathbf{Q}\right] \hat{\mathbf{f}} = \sum_{i=1}^{N} \lambda_i \bar{\mathbf{H}}_i^t \mathbf{g}_i.$$

Since

$$\sum_{i=1}^{N} \lambda_i \bar{\mathbf{H}}_i^t \bar{\mathbf{H}}_i = \mathbf{H}^t \mathbf{\Lambda} \mathbf{H}$$

where $\mathbf{\Lambda}$ is defined as

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 \mathbf{I} & 0 & \cdots & 0 \\ 0 & \lambda_2 \mathbf{I} & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \mathbf{I} \end{pmatrix}$$

the matrix system should be

$$[\mathbf{H}^\mathbf{t} \mathbf{\Lambda} \mathbf{H} + \mathbf{Q}^\mathbf{t}\mathbf{Q}]\hat{\mathbf{f}} = \mathbf{\Lambda}\mathbf{H}^\mathbf{t}\mathbf{g} \tag{5}$$

instead of

$$[\mathbf{\Lambda}\mathbf{H}^\mathbf{t} \mathbf{H} + \mathbf{Q}^\mathbf{t}\mathbf{Q}]\hat{\mathbf{f}} = \mathbf{\Lambda}\mathbf{H}^\mathbf{t}\mathbf{g} \tag{6}$$

which is stated in (2.14) of the paper. We remark that if there are only the within-channel blur and no cross-channel blur in the degradation, i.e., $H_{ij} = 0$, then both formulae (5) and (6) are identical.

Solving for $\hat{\mathbf{f}}$ in (5) yields

$$\hat{\mathbf{f}} = [\mathbf{H}^\mathbf{t} \mathbf{\Lambda} \mathbf{H} + \mathbf{Q}^\mathbf{t}\mathbf{Q}]^{-1}\mathbf{\Lambda}\mathbf{H}^\mathbf{t}\mathbf{g} \tag{7}$$

which is the solution of (2) with respect to chosen values in λ. We note in the paper that the incorrect solution is given by

$$\hat{\mathbf{f}} = [\mathbf{H}^\mathbf{t}\mathbf{H} + \mathbf{\Lambda}^{-1}\mathbf{Q}^\mathbf{t}\mathbf{Q}]^{-1}\mathbf{H}^\mathbf{t}\mathbf{g}, \tag{8}$$

[see (2.15) of the paper].

As the solution $\hat{\mathbf{f}}$ must satisfy the following functions:

$$Z_i(\lambda_1, \lambda_2, \ldots, \lambda_N) = \|\bar{\mathbf{H}}_i \hat{\mathbf{f}} - \mathbf{g}_i\|_2^2 - \|\mathbf{n}_i\|_2^2, \quad i = 1, 2, \ldots, N$$

finding the roots of the functions Z_i simultaneously yields the desired λ_i. Since the functions Z_i are nonlinear, Newton's method is therefore used to find these λ_i numerically. Since Newton's method for the estimate of λ_i is derived according to the form of $\hat{f}$, the correct formula of $\hat{f}$ is important in the estimation of λ_i and the restoration of multichannel images.

The (i, j)th element of the Jacobian matrix $\mathbf{J}$ of this nonlinear system of functions based on the matrix formulation (5) should be given by

$$[\mathbf{J}]_{ij} = \frac{\partial Z_i(\lambda_1, \lambda_2, \ldots, \lambda_N)}{\partial \lambda_j} = 2(\bar{\mathbf{H}}_i \hat{\mathbf{f}} - \mathbf{g}_i)^t \bar{\mathbf{H}}_i \frac{\partial \hat{\mathbf{f}}}{\partial \lambda_j}$$

for $1 \leq i, j \leq N$. By (7), we obtain

$$\frac{\partial \hat{\mathbf{f}}}{\partial \lambda_j} = [\mathbf{H}^\mathbf{t} \mathbf{\Lambda} \mathbf{H} + \mathbf{Q}^\mathbf{t}\mathbf{Q}]^{-1} \cdot \left(\mathbf{I}_{jj} - \bar{\mathbf{H}}_j^t \bar{\mathbf{H}}_j [\mathbf{H}^\mathbf{t} \mathbf{\Lambda} \mathbf{H} + \mathbf{Q}^\mathbf{t}\mathbf{Q}]^{-1}\mathbf{\Lambda}\right) \mathbf{H}^\mathbf{t}\mathbf{g}$$

where $\mathbf{I}_{jj}$ is a zero matrix, except that the (j, j)th diagonal block is an identity matrix. Hence, we have

$$[\mathbf{J}]_{ij} = 2(\bar{\mathbf{H}}_i \hat{\mathbf{f}} - \mathbf{g}_i)^t \bar{\mathbf{H}}_i [\mathbf{H}^\mathbf{t} \mathbf{\Lambda} \mathbf{H} + \mathbf{Q}^\mathbf{t}\mathbf{Q}]^{-1} \cdot \left(\mathbf{I}_{jj} - \bar{\mathbf{H}}_j^t \bar{\mathbf{H}}_j [\mathbf{H}^\mathbf{t} \mathbf{\Lambda} \mathbf{H} + \mathbf{Q}^\mathbf{t}\mathbf{Q}]^{-1}\mathbf{\Lambda}\right) \mathbf{H}^\mathbf{t}\mathbf{g}$$

which is the correct formula for the Jacobian matrix.

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The authors are with the Department of Mathematics, The University of Hong Kong, Hong Kong (e-mail: mng@maths.hku.hk).

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[1] N. Galatsanos, A. Katsaggelos, R. Chin, and A. Hillery, "Least squares restoration of multichannel images," *IEEE Trans. Signal Processing*, vol. 39, pp. 2222–2236, Oct. 1991.