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# FURTHER COMMENTS ON THE CONTINUITY OF DISTRIBUTION FUNCTIONS OBTAINED BY SUPERPOSITION

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ABSTRACT. Let $\{X(t)\}$ be a differential process with discontinuous distributions and $Y$ a nonnegative random variable independent of the process. The superposition $X(Y)$ has a continuous probability distribution if and only if the process has nonzero trend term and $Y$ has continuous distribution. The nature of discontinuities of the probability distribution of the superposition is indicated.

We continue the notation and terminology of [**1**] and [**3**]. Let $\{X(t)/t \in [0, \infty)\}$ be a *differential process* (homogeneous process) with discontinuous distributions. Then

$$X(t) = \tau_X t + X^*(t),$$

where

$$f_{X^*(t)}(u) = Ee^{iuX^*(t)} = \exp\left\{t\int_{-\infty}^{\infty}(e^{iux} - 1)\, dM_X(x)\right\}$$

and the Lévy spectral function satisfies

$$\int_{-\infty}^{\infty} dM_X(x) = \int_{-\infty}^{0} + \int_{0}^{\infty} dM_X(x) = \mu + \lambda < \infty.$$

$\tau_X$ is the *trend term* of the process. Let $Y \geqq 0$ be independent of the $\{X(t)\}$ process and consider the superposition $X(Y)$. We shall show that $X(Y)$ has continuous distribution if and only if the process has nonzero trend term and $Y$ has continuous distribution. The nature of discontinuities will be indicated.

LEMMA 1. *Let $\{X^*(t)\}$ be a differential process with discontinuous distributions and no trend term. Then*

$$\operatorname{Cont} F_{X^*(t)}(\cdot) = \operatorname{Cont} F_{X^*(1)}(\cdot), \qquad \forall t > 0.$$

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PROOF. Reviewing the argument of Theorem 2 in [1], we see that $F_{X^*(t)}(\cdot)$ is the distribution function of the random sum $Z(t)=X_1+\cdots+X_{Y(t)}$, where $X_1, X_2, \cdots$ are independent with common distribution

$$\begin{aligned} G(x) &= M_X(x)/(\mu+\lambda), && x<0, \\ &= \mu/(\mu+\lambda), && x=0, \\ &= (\mu+\lambda+M_X(x))/(\mu+\lambda), && x>0, \end{aligned}$$

and $\mathscr{L}(Y(t))=\mathscr{P}(t(\mu+\lambda))$. Thus $f_{X^*(t)}(u)=\sum_{k=0}^{\infty}(f_{X_1}(u))^k P[Y(t)=k]$. Let $j_{X^*(t)}(a)=F_{X^*(t)}(a)-F_{X^*(t)}(a-)$ be the jump of $F_{X^*(t)}(\cdot)$ at $a$. Applying Theorem 3.2.3 of [2], Fubini's Theorem, and the Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned} j_{X^*(t)}(a) &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{-iau} f_{X^*(t)}(u)\,du \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{-iau}\sum_{k=0}^{\infty}(f_{X_1}(u))^k P[Y(t)=k]\,du \\ &= \lim_{T\to\infty}\sum_{k=0}^{\infty}\frac{1}{2T}\int_{-T}^{T} e^{-iau}(f_{X_1}(u))^k\,du\; P[Y(t)=k] \\ &= \sum_{k=0}^{\infty}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{-iau}(f_{X_1}(u))^k\,du\; P[Y(t)=k] \\ &= e^{-t(\mu+\lambda)}+\sum_{k=1}^{\infty} j_{X_1+\cdots+X_k}(0)P[Y(t)=k], \qquad a=0, \\ &= \sum_{k=1}^{\infty} j_{X_1+\cdots+X_k}(a)P[Y(t)=k], \qquad a\neq 0. \end{aligned}$$

Thus $F_{X^*(t)}(\cdot)$ has a jump at $a$ if and only if some $F_{X_1+\cdots+X_k}(\cdot)$ has a jump at $a$; i.e.,

$$\widetilde{\text{Cont}}\, F_{X^*(t)}(\cdot) = \{0\} \cup \widetilde{\text{Cont}}\, F_{X_1}(\cdot) \cup \cdots, \qquad \forall t>0,$$

and the lemma is proved. □

Note that $j_{X^*(t)}(a)=\sum_{k=0}^{\infty} j_{X_1+\cdots+X_k}(a)P[Y(t)=k]$ and the Helly-Bray Theorem imply that $j_{X^*(\cdot)}(a)$ is continuous for $a$ fixed.

LEMMA 2. *Let $\{X(t)=\tau_X t+X^*(t)\}$ be a differential process with discontinuous distributions and nonzero trend term. Then for each fixed $a$, $\{t/j_{X(t)}(a)\neq 0\}$ is at most countable.*

PROOF. Applying Lemma 1, we note that $a\notin \text{Cont}\, F_{X(t)}(\cdot)$ if and only if $a-\tau_X t\notin \text{Cont}\, F_{X^*(t)}(\cdot)=\text{Cont}\, F_{X^*(1)}(\cdot)$. Thus $j_{X(t)}(a)\neq 0$ if and only if $t=(a-x)/\tau_X$ for some $x\notin \text{Cont}\, F_{X^*(1)}(\cdot)$. □

THEOREM 1. *Let $Y \geqq 0$ be independent of the differential process $\{X(t)\}$. Then for each fixed a*

$$j_{X(Y)}(a) = \int_0^\infty j_{X(t)}(a)\, dF_Y(t). \tag{I}$$

PROOF. Using the arguments of Lemma 1, we obtain

$$\begin{aligned} j_{X(Y)}(a) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-iau} f_{X(Y)}(u)\, du \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-iau} \int_0^\infty f_{X(t)}(u)\, dF_Y(t)\, du \\ &= \lim_{T\to\infty} \int_0^\infty \frac{1}{2T} \int_{-T}^{T} e^{-iau} f_{X(t)}(u)\, du\, dF_Y(t) \\ &= \int_0^\infty \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-iau} f_{X(t)}(u)\, du\, dF_Y(t) \\ &= \int_0^\infty j_{X(t)}(a)\, dF_Y(t). \quad \square \end{aligned}$$

COROLLARY 1. *Let $\{X(t)\}$ be a differential process with discontinuous distributions and nonzero trend term. Suppose $Y \geqq 0$ is independent of the process and has continuous distribution. Then the superposition $X(Y)$ has continuous distribution.*

PROOF. The integrand in (I) vanishes a.e. by Lemma 2 and $Y$ has no point masses.

COROLLARY 2. *Let $\{X(t)\}$ be a differential process with discontinuous distributions and nonzero trend term. Suppose $Y \geqq 0$ is independent of the process and has a discontinuous distribution. Then $X(Y)$ has a discontinuous distribution with jumps occurring at precisely those points of the form $a = t_0\tau_X + \alpha$, where $t_0 \notin$ Cont $F_Y(\cdot)$ and $\alpha \notin$ Cont $F_{X^*(1)}(\cdot)$.*

PROOF. The indicated points are precisely those where a positive value of the integrand in (I) coincides with a point mass of $Y$.

COROLLARY 3. *Let $\{X^*(t)\}$ be a differential process with discontinuous distributions and no trend term. Suppose $Y \geqq 0$ is independent of the process and $P[Y=0] < 1$. Then $X^*(Y)$ has discontinuous distribution and*

$$\text{Cont } F_{X^*(Y)}(\cdot) = \text{Cont } F_{X^*(1)}(\cdot).$$

PROOF. The integrand in (I) vanishes if $a \in \text{Cont } F_{X^*(t)}(\cdot)$. If $a \notin \text{Cont } F_{X^*(1)}(\cdot)$, then (I) and the observation that $j_{X^*(\cdot)}(a)$ is continuous and positive imply that $j_{X^*(Y)}(a) > 0$.

We also note that (I) immediately yields Corollary 1A of [**1**].

## References

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