Comments on the Interpretation of Instantaneous Frequency

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***Abstract*— Instantaneous frequency, taken as the derivative of the phase of the signal, is interpreted in the time–frequency literature as the average frequency of the signal at each time. We point out some difficulties with this interpretation, and show that for a generic two-component AM-FM signal, the interpretation holds only when the components are of equal strength. We conclude that instantaneous frequency and the average frequency at each time are generally two different quantities. One possible interpretation of the difference between these two quantities is suggested.**

I. Introduction

Many natural and man-made signals exhibit time-varying frequencies (e.g., sound of changing pitch, FM radio waves), which gives rise to the concept of instantaneous frequency. Commonly defined as the derivative of the phase, $\dot{\varphi}(t)$, of the signal $z(t) = A(t)e^{j\varphi(t)}$, where $z(t)$ is often taken to be the analytic signal (computed via the Hilbert transform of the given real signal) [6], [10], instantaneous frequency is interpreted in the time–frequency literature as the average frequency at each time in the signal [1]–[4], [9], [11]. This interpretation arises because an unlimited number of time–frequency distributions (TFD's) $P(t,\omega)$ of the signal $z(t)$ yield the derivative of the phase for the first conditional moment in frequency, [2], [3][1]

$$\begin{aligned}\langle\omega\rangle_t &= \int \omega P(\omega \mid t)d\omega = \int \omega P(t,\omega)d\omega \Big/ \int P(t,\omega)d\omega \\ &= \dot{\varphi}(t). \end{aligned} \tag{1}$$

This interpretation has resolved some apparent paradoxes associated with instantaneous frequency. For example, as an average, instantaneous frequency need not be a frequency that appears in the spectrum [3]. At the same time, however, other difficulties in interpretation remain. For example, the derivative of the phase can extend beyond the spectral range of the signal [8], yielding the paradox that the supposed average of a quantity exceeds the range of values of that quantity. It is this paradox that we explore in further detail here.

We give conditions on the amplitudes $A_1(t)$ and $A_2(t)$ of a two-component signal $A_1(t)e^{j\varphi_1(t)} + A_2(t)e^{j\varphi_2(t)} = A(t)e^{j\varphi(t)}$ such that the instantaneous frequency $\dot{\varphi}(t)$ lies between the individual instantaneous frequencies ($\dot{\varphi}_1(t)$ and $\dot{\varphi}_2(t)$) and is therefore amenable to interpretation as the average frequency at each time. We also show that the situations for which this interpretation holds are extremely limited, and thus conclude that instantaneous frequency of the signal and the average frequency at each time are generally different quantities. A reason for the origin of the paradox is suggested, and one possible interpretation of the difference between these two quantities is offered.

II. Results

Consider the two-tone signal $z(t) = A_1 e^{j\omega_1 t} + A_2 e^{j\omega_2 t} = A(t)e^{j\varphi}(t)$, where A_1 and A_2 are real. The instantaneous frequency is [3], [8]

$$\dot{\varphi}(t) = \frac{1}{2}(\omega_2 + \omega_1) + \frac{1}{2}(\omega_2 - \omega_1)\frac{A_2^2 - A_1^2}{A^2(t)} \tag{2}$$

where

$$A^2(t) = A_1^2 + A_2^2 + 2A_1A_2\cos((\omega_2 - \omega_1)t). \tag{3}$$

As Mandel noted [8], even though the signal is composed of two constant frequency tones, the instantaneous frequency is generally time-varying and exhibits asymmetrical deviations about the frequency $\frac{1}{2}(\omega_2 + \omega_1)$. We point out, though, that for equal strength tones, i.e., $|A_1| = |A_2|$, the instantaneous frequency is constant and consistent with the interpretation as "the average frequency at each time" of this multicomponent signal [see Fig. 1(a)].[2] However, that is the *only* case for which this interpretation holds, as we show next. For unequal strength tones, not only are there time-varying deviations in the instantaneous frequency, but these deviations always force the instantaneous frequency beyond the frequency range of the signal (i.e., ω_1 and ω_2). Accordingly, it can not be interpreted as the average frequency at each time. This result generalizes to an arbitrary two-component AM-FM signal, which we show following the case for two tones.

For the instantaneous frequency $\dot{\varphi}(t)$ to remain bounded by ω_1 and ω_2, we require, from (2) and (3)

$$\left|\frac{A_2^2 - A_1^2}{A_1^2 + A_2^2 + 2A_1A_2\cos((\omega_2 - \omega_1)t)}\right| \le 1 \tag{4}$$

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[1] We note that this result was derived from distributions $P(t,\omega)$ that can go negative. It was first derived by Ville for the Wigner distribution [11].

[2] Mandel did not consider the interpretation of instantaneous frequency as the average frequency at each time, but rather the relationship between instantaneous frequency, Fourier frequency, and their (global) averages and higher moments. While their mean values coincide (first shown by Ville [11]), their higher moments do not [8].

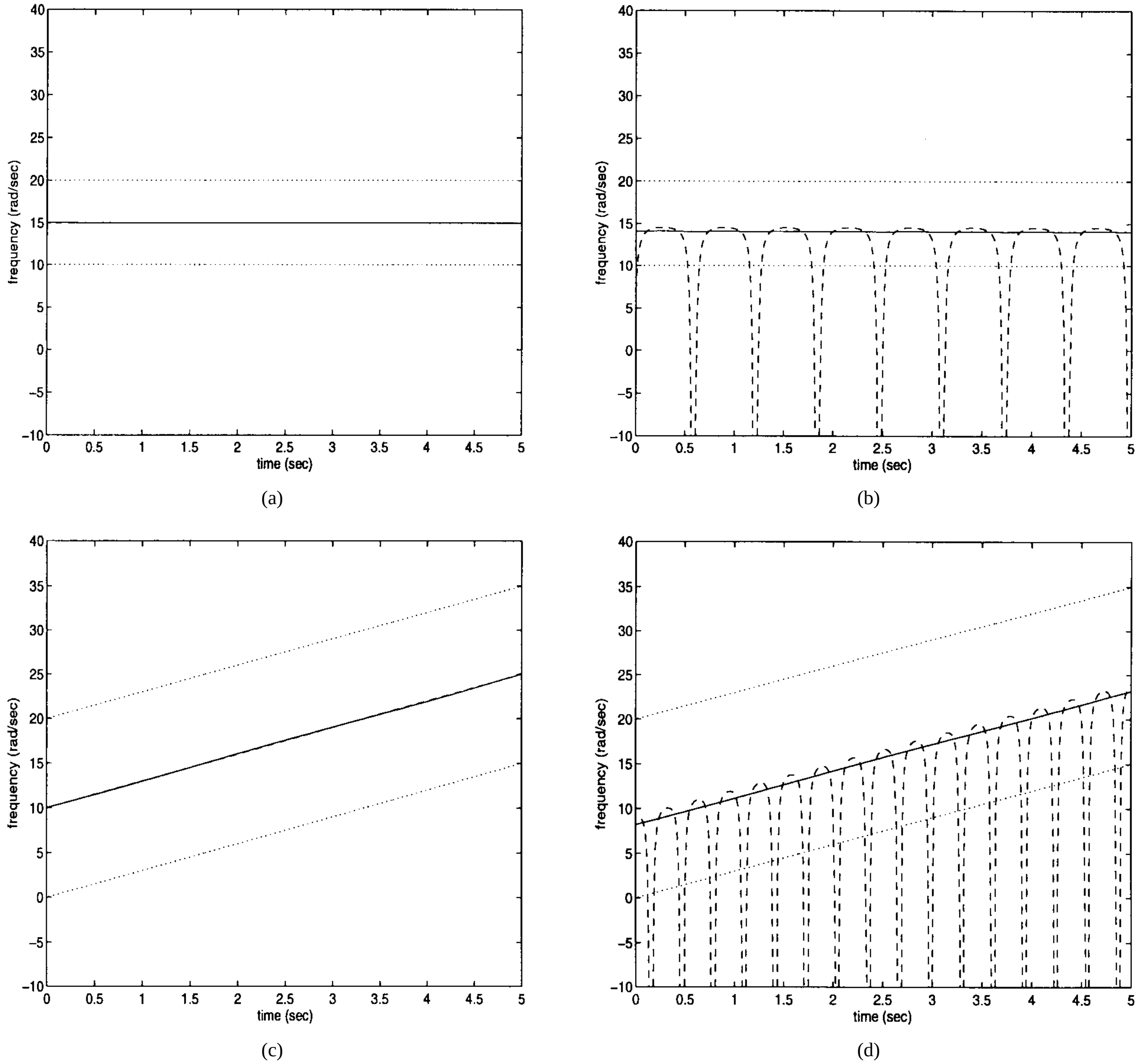

Fig. 1. The average frequency at each time [7] (solid line), the instantaneous frequency ($\dot{\varphi}(t)$, dashed line), and the individual instantaneous frequency of each component ($\dot{\varphi}_1(t)$ and, $\dot{\varphi}_2(t)$, dotted lines) for the sum of: (a) two tones of equal strength, (b) two tones of unequal strength, (c) two linear FM chirps of equal strength, and (d) two linear FM chirps of unequal strength. Only for equal strength components [(a) and (c)] is the instantaneous frequency $\dot{\varphi}(t)$ interpretable as the average frequency at each time.

or equivalently

$$A_1^2 + A_1 A_2 \cos((\omega_2 - \omega_1)t) \geq 0$$

and

$$A_2^2 + A_1 A_2 \cos((\omega_2 - \omega_1)t) \geq 0. \tag{5}$$

The case $A_1 A_2 > 0$ yields

$$\frac{A_1}{A_2} \geq -\cos((\omega_2 - \omega_1)t)$$

and

$$\frac{A_2}{A_1} \geq -\cos((\omega_2 - \omega_1)t). \tag{6}$$

The case $A_1 A_2 < 0$ yields

$$\frac{A_1}{A_2} \leq -\cos((\omega_2 - \omega_1)t)$$

and

$$\frac{A_2}{A_1} \leq -\cos((\omega_2 - \omega_1)t). \tag{7}$$

The only solution to both (6) and (7) is $|A_1| = |A_2|$. For unequal strength tones, $\dot{\varphi}(t)$ is time-varying and regularly extends beyond ω_1 and ω_2 [see Fig. 1(b)], and is, thus, not amenable to interpretation as the average frequency at each time.

Consider the more general case $z(t) = A_1(t)e^{j\varphi_1(t)} + A_2(t)e^{j\varphi_2(t)} = A(t)e^{j\varphi(t)}$. For $A_2(t) = cA_1(t)$ (c is a constant), the instantaneous frequency is [4]

$$\dot{\varphi}(t) = \frac{1}{2}(\dot{\varphi}_2(t) + \dot{\varphi}_1(t)) + \frac{1}{2}(\dot{\varphi}_2(t) - \dot{\varphi}_1(t)) \times \frac{A_2^2(t) - A_1^2(t)}{A^2(t)} \tag{8}$$

where

$$A^2(t) = A_1^2(t) + A_2^2(t) + 2A_1(t)A_2(t) \times \cos(\varphi_2(t) - \varphi_1(t)). \tag{9}$$

Proceeding as in the two-tone case, we obtain the following inequalities for the instantaneous frequency $\dot{\varphi}(t)$ to remain bounded by $\dot{\varphi}_1(t)$ and $\dot{\varphi}_2(t)$

$$\frac{A_1(t)}{A_2(t)} \geq -\cos((\varphi_2(t) - \varphi_1(t)))$$

and

$$\frac{A_2(t)}{A_1(t)} \geq -\cos((\varphi_2(t) - \varphi_1(t))) \tag{10}$$

$$\frac{A_1(t)}{A_2(t)} \leq -\cos((\varphi_2(t) - \varphi_1(t)))$$

and

$$\frac{A_2(t)}{A_1(t)} \leq -\cos((\varphi_2(t) - \varphi_1(t))). \tag{11}$$

Both (10) and (11) hold when $|A_1(t)| = |A_2(t)|$. For $|A_1(t)| \neq |A_2(t)|$, $\dot{\varphi}(t)$ generally exhibits large fluctuations and extends beyond the band defined by $\dot{\varphi}_1(t)$ and $\dot{\varphi}_2(t)$ [see Fig. 1(c) and (d)].

III. Conclusion

Although instantaneous frequency, defined as the derivative of the phase of the signal, is interpreted as the average frequency at each time in the time–frequency literature, we have shown that this interpretation often does not make sense, in that the instantaneous frequency often exceeds the minimum or maximum frequency in the signal. Indeed, as we showed, for a two-tone signal, the only case for which the instantaneous frequency can be interpreted as the average frequency at each time is when the tones are of equal strength. A similar result holds for an arbitrary two-component signal. It appears then that, in general, the instantaneous frequency of a signal and the average frequency at each time of the signal are different quantities. This issue is explored further in [7], where it is suggested that the difference between them can be interpreted as phase modulation within the signal.

In closing, we briefly remark on the origin of this interpretation of instantaneous frequency. While it is true that (1) holds for an unlimited number of time–frequency distributions, which gives rise to the interpretation, the derivation itself is suspect because it is based on calculations made from distributions that can go negative. Such distributions are not proper joint distribution functions. Hence, the conditional moments calculated from them may not always be interpretable in the usual sense (namely, as the average value of one quantity for a given value of the other). If we restrict our calculations of conditional averages to proper (time–frequency) distributions, the average frequency at each time $\langle \omega \rangle_t$ never exceeds the range of frequencies ω in the signal, and the equality $\langle \omega \rangle_t = \dot{\varphi}(t)$ does not always hold [5], [7]. Under what conditions it does hold for such distributions is an open question, for which we have given some insight here (namely, when the individual components of a two-component signal are of equal strength.)

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