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COMMON AND COINCIDENCE FIXED POINT THEOREMS FOR ASYMPTOTICALLY REGULAR MAPPINGS IN 2-BANACH SPACES

M. Pitchaimani[1] and D. Ramesh Kumar[2]

[1]Ramanujan Institute For Advanced Study In Mathematics
University of Madras, Chennai, India
e-mail: mpitchaimani@yahoo.com

[2]Ramanujan Institute For Advanced Study In Mathematics
University of Madras, Chennai, India
e-mail: ramesh.riasm@gmail.com

Abstract. In this article, we study existence of common and coincidence fixed points for two self-mappings satisfying many contractive conditions on a 2-Banach space. We also prove well-posedness of a common fixed point problem. The results we present, generalize several well known results in the literature.

1. Introduction

Considerable attention has been given to fixed points and fixed point theorems in metric and Banach spaces due to their tremendous applications to mathematics. Motivated by this work, several authors introduced similar concepts and proved analogous fixed point theorems in 2-metric and 2-Banach spaces as cited in the papers of the following authors. Some basic fixed point results in 2-metric and 2-Banach spaces are initially studied by Gähler [6, 7]. Many authors including Iseki [9], Rhoades [16], White [18] and Saluja [17] investigated different aspects of fixed point theory in 2-metric and 2-Banach spaces. Panja and Baisnab [11] studied asymptotically regularity and common fixed point theorems. In spite of the above work, study on 2-Banach space need more investigation.

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In this paper, we study some common and coincidence fixed point theorem for contraction mappings having the asymptotically regular property and also well-posedness of their fixed point problem in a 2-Banach space.

2. Preliminaries

Definition 2.1. Let X be a real linear space and $\|\cdot,\cdot\|$ be a non-negative real valued function defined on $X \times X$ satisfying the following conditions :

(ii) $\|x, y\| = \|y, x\|$, for all $x, y \in X$,
(iii) $\|x, ay\| = |a|\|x, y\|$, for all $x, y \in X$ and a being a real,
(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$, for all $x, y, z \in X$.

Then $\|\cdot,\cdot\|$ is called a 2-norm and the pair $(X, \|\cdot,\cdot\|)$ is called a linear 2-normed space.

Some of the basic properties of 2-norms are that they are non-negative satisfying $\|x, y + ax\| = \|x, y\|$, for all $x, y \in X$ and all real numbers a.

Definition 2.2. A sequence $\{x_n\}$ in a linear 2-normed space $(X, \|\cdot,\cdot\|)$ is called a Cauchy sequence if $\lim_{m,n\to\infty} \|x_m - x_n, y\| = 0$ for all $y \in X$.

Definition 2.3. A sequence $\{x_n\}$ in a linear 2-normed space $(X, \|\cdot,\cdot\|)$ is said to converge to a point $x \in X$ if $\lim_{n\to\infty} \|x_n - x, y\| = 0$ for all $y \in X$.

Definition 2.4. A linear 2-normed space $(X, \|\cdot,\cdot\|)$ in which every Cauchy sequence is convergent is called a 2-Banach space.

Definition 2.5. A self mapping T on a 2-Banach space is said to be asymptotically regular at a point $x \in X$ if $\lim_{n\to\infty} \|T^n x - T^{n+1} x, y\| = 0$ for all $y \in X$ where $T^n x$ denotes the n^{th} iterate of T at x.

Definition 2.6. A sequence $\{x_n\}$ in a 2-Banach space X is said to be asymptotically T-regular if $\lim_{n\to\infty} \|x_n - Tx_n, y\| = 0$ for all $y \in X$.

Definition 2.7. A pair of mappings (T, f) on a 2-Banach space X is said to be weakly compatible if $fTx = Tfx$ whenever $fx = Tx$.

A point $y \in X$ is called point of coincidence of two self mappings T and f on X if there exists a point $x \in X$ such that $y = Tx = fx$.

The following lemma was proved in metric space [4].

Lemma 2.8. *Let X be a non-empty set and the mappings $T, f : X \to X$ have a unique point of coincidence v in X. If the pair (T, f) is weakly compatible then T and f have a unique common fixed point.*

Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space, T and f be two self mappings on X with $T(X) \subset f(X)$ and $x_0 \in X$. Assume $x_1 \in X$ such that $fx_1 = Tx_0$ (since $T(X) \subset f(X)$). Proceeding in this way, we get $x_1, x_2, \ldots, x_n, x_{n+1}$ in X such that

$$fx_{n+1} = Tx_n, \;\; n = 0, 1, 2, \ldots.$$

The sequence $\{fx_n\}$ is called a T-sequence with initial point x_0.

Definition 2.9. Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space, T and f be two self mappings on X with $T(X) \subset f(X)$ and $x_0 \in X$. A mapping T is said to be asymptotically f-regular at x_0 if

$$\lim_{n\to\infty} \|fx_n - fx_{n+1}, y\| = 0, \quad for\ all\;\; y \in X.$$

3. Main results

We extend the reults studied by Abbas [1] and Rashwan [14] in 2-Banach Space setting.

Theorem 3.1. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space, S and T be two self mappings of X such that*

$$\begin{aligned} \|Sx - Ty, u\| \;\; &\leq \;\; \alpha\|x - Sx, u\| + \beta\|y - Ty, u\| + \gamma\|x - y, u\| \\ &\quad +\delta \min\left\{ \|x - Ty, u\|, \|y - Sx, u\| \right\}, \end{aligned} \tag{3.1}$$

for all $x, y, u \in X$ where α, β, γ and δ are non-negative reals with $\alpha+\beta+\gamma < 1$. Then S and T have a unique fixed point in X.

Proof. For $x_0 \in X$, we define a sequence $\{x_n\}$ as follows:

$$x_{2n+1} = Sx_{2n}, \;\; x_{2n+2} = Tx_{2n+1}, \;\; n = 0, 1, 2, \ldots.$$

Now, for all $u \in X$, from (3.1), we have

$$\begin{aligned} \|x_{2n+1} - x_{2n}, u\| &= \|Sx_{2n} - Tx_{2n-1}, u\| \\ &\leq \alpha\|x_{2n} - x_{2n+1}, u\| + \beta\|x_{2n-1} - x_{2n}, u\| + \gamma\|x_{2n} - x_{2n-1}, u\| \\ &\quad + \delta \min\left\{ \|x_{2n} - x_{2n}, u\|, \|x_{2n-1} - x_{2n+1}, u\| \right\}. \end{aligned}$$

Thus, we have

$$\|x_{2n+1} - x_{2n}, u\| \leq \frac{\beta + \gamma}{1 - \alpha} \|x_{2n} - x_{2n-1}, u\|.$$

Taking $k = \dfrac{\beta+\gamma}{1-\alpha} < 1$, we get

$$\|x_{2n+1} - x_{2n}, u\| \le k\|x_{2n} - x_{2n-1}, u\|.$$

Continuing in this fashion, we obtain

$$\|x_{2n+1} - x_{2n}, u\| \le k^{2n}\|x_1 - x_0, u\|, \quad n = 1, 2, 3, \ldots.$$

Also for $n > m$, we have

$$\begin{aligned}
&\|x_n - x_m, u\| \\
&\le \|x_n - x_{n-1}, u\| + \|x_{n-1} - x_{n-2}, u\| + \cdots + \|x_{m+1} - x_m, u\| \\
&\le (k^{n-1} + k^{n-2} + \cdots + k^m)\|x_1 - x_0, u\| \le \frac{k^m}{1-k}\|x_1 - x_0, u\|.
\end{aligned}$$

Thus $\|x_n - x_m, u\| \to 0$, as $n \to \infty$, since $\dfrac{k^m}{1-k} \to 0$, as $n \to \infty$. This shows that $\{x_n\}$ is a Cauchy sequence in X. Hence there exists a point $z \in X$ such that $x_n \to z$, as $n \to \infty$. Now further, we have

$$\begin{aligned}
&\|z - Tz, u\| \\
&\le \|z - x_{2n+1}, u\| + \|x_{2n+1} - Tz, u\| \\
&= \|z - x_{2n+1}, u\| + \|Sx_{2n} - Tz, u\| \\
&\le \|z - x_{2n+1}, u\| + \alpha\|x_{2n} - x_{2n+1}, u\| + \beta\|z - Tz, u\| + \gamma\|x_{2n} - z, u\| \\
&\quad + \delta \min\left\{\|x_{2n} - Tz, u\|, \|z - x_{2n+1}, u\|\right\}.
\end{aligned}$$

As $n \to \infty$, $x_{2n} \to z$, $x_{2n+1} \to z$ and $\{x_n\}$ is a Cauchy sequence, we obtain

$$\|z - Tz, u\| \le \beta\|z - Tz, u\|$$

which implies $z = Tz$, since $\beta < 1$.
Similarly, we get $z = Sz$. Thus z is a common fixed point of S and T.

Uniqueness: Let $v \in X$ be another common fixed point of S and T, that is, $Sv = Tv = v$. Then

$$\begin{aligned}
\|z - v, u\| &= \|Sz - Tv, u\| \\
&\le \alpha\|z - Sz, u\| + \beta\|v - Tv, u\| + \gamma\|z - v, u\| \\
&\quad + \delta \min\left\{\|z - Tv, u\|, \|v - Sz, u\|\right\} \\
&\le (\gamma + \delta)\|z - v, u\|.
\end{aligned}$$

Since $\gamma + \delta < 1, z = v$ for all $u \in X$. Thus z is a unique common fixed point of S and T. $\square$

Now we extend Theorem 3.1 to the case of pair of mappings S^p and T^q where p and q are some positive integers satisfying the condition (3.1).

Theorem 3.2. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space, S and T be two self mappings of X such that*

$$\begin{aligned}\|S^p x - T^q y, u\| &\le \alpha\|x - S^p x, u\| + \beta\|y - T^q y, u\| + \gamma\|x - y, u\| \\ &\quad + \delta \min\Big\{\|x - T^q y, u\|, \|y - S^p x, u\|\Big\},\end{aligned} \tag{3.2}$$

for all $x, y, u \in X$ where p and q are some positive integers and α, β, γ and δ are non-negative reals with $\gamma + \delta < 1$. Then S and T have a unique fixed point in X.

Proof. Note that S^p and T^q satisfy the conditions of Theorem 3.1. So S^p and T^q have a unique common fixed point. Let v be the common fixed point. Now

$$\begin{aligned}S^p v = v \quad &\Rightarrow \quad S(S^p v) = Sv, \\ S^p(Sv) \quad &= \quad Sv.\end{aligned}$$

If $Sv = x_0$ then $S^p(x_0) = x_0$. So, Sv is a fixed point of S^p. Similarly, $T^q(Tv) = Tv$. Now, we have

$$\begin{aligned}\|v - Tv, u\| &= \|S^p v - T^q(Tv), u\| \\ &\le \alpha\|v - S^p v, u\| + \beta\|Tv - T^q(Tv), u\| + \gamma\|v - Tv, u\| \\ &\quad + \delta \min\Big\{\|v - T^q(Tv), u\|, \|Tv - S^p v, u\|\Big\}, \\ &\le \gamma\|v - Tv, u\| + \delta \min\Big\{\|v - Tv, u\|, \|Tv - v, u\|\Big\} \\ &= (\gamma + \delta)\|v - Tv, u\|.\end{aligned}$$

Therefore $v = Tv$, since $\gamma + \delta < 1$. Similarly $Sv = v$. Hence v is a common fixed point of S and T.

For the uniqueness, let $w(\neq v)$ be another common fixed point of S and T. Then clearly w is also a common fixed point of S^p and T^q which implies $w = v$. Hence S and T have a unique common fixed point. □

Next we extend Theorem 3.1 to the case of a sequence of mappings satisfying the condition (3.1).

Theorem 3.3. *Let $(X, \|\cdot,\cdot\|$ be a 2-Banach space and $\{F_n\}$ be a sequence of mappings on X such that*

(i) *$\{F_n\}$ converging pointwise to F.*

(ii) $$\|F_nx - F_ny, u\| \leq \alpha\|x - F_nx, u\| + \beta\|y - F_ny, u\| + \gamma\|x - y, u\| + \delta \min\left\{\|x - F_ny, u\|, \|y - F_nx, u\|\right\},$$
for all $x, y, u \in X$ *where* α, β, γ *and* δ *are non-negative reals with* $\alpha + \beta + \gamma < 1$.

If $\{F_n\}$ *has a fixed point* v_n *and* F *has a fixed point* v*. Then the sequence* $\{v_n\}$ *converges to* v.

Proof. Note that $F_nv_n = v_n$ and $Fv = v$. Now consider

$$\begin{aligned}\|v - v_n, u\| &= \|Fv - F_nv_n, u\| \\ &\leq \|Fv - F_nv, u\| + \|F_nv - F_nv_n, u\| \\ &= \|Fv - F_nv, u\| + \alpha\|v - F_nv, u\| + \beta\|v_n - F_nv_n, u\| \\ &\quad + \gamma\|v - v_n, u\| + \delta \min\left\{\|v - F_nv_n, u\| + \|v_n - F_nv, u\|\right\}.\end{aligned}$$

By the fact that $F_nv \to Fv$ as $n \to \infty$, we get

$$\|v - v_n, u\| \leq (\gamma + \delta)\|v - v_n, u\|.$$

Therefore $v_n \to v$ as $n \to \infty$, since $\gamma + \delta < 1$. □

4. Coincidence fixed point theorems

Theorem 4.1. *Let* $(X, \|\cdot, \cdot\|)$ *be a 2-Banach space,* T *and* f *be two self mappings of* X *such that* $T(X) \subset f(X)$ *and*

$$\|Tx - Ty, u\| \leq \alpha \max\left\{\|fx - fy, u\|, \|fx - Tx, u\|, \|fy - Ty, u\|, \frac{\|fx - Ty, u\| + \|fy - Tx, u\|}{2}\right\},$$

for all $x, y, u \in X$ *where* $0 < \alpha < 1$*. If* $f(X)$ *or* $T(X)$ *is a complete subspace of* X *and* T *is asymptotically* f*-regular at some point* x_0 *in* X *then* T *and* f *have a point of coincidence.*

Proof. Let $\{fx_n\}$ be T-sequence in X. Then

$$\begin{aligned}
&\|fx_n - fx_m, u\| \\
&\leq \|fx_n - Tx_n, u\| + \|Tx_n - fx_m, u\| \\
&\leq \|fx_n - Tx_n, u\| + \|Tx_n - Tx_m, u\| + \|Tx_m - fx_m, u\| \\
&\leq \|fx_n - Tx_n, u\| + \|Tx_m - fx_m, u\| \\
&\quad + \alpha \max\Bigg\{\|fx_n - fx_m, u\|, \|fx_n - Tx_n, u\|, \|fx_m - Tx_m, u\|, \\
&\qquad\qquad \frac{\|fx_n - Tx_m, u\| + \|fx_m - Tx_n, u\|}{2}\Bigg\} \\
&\leq \|fx_n - fx_{n+1}, u\| + \|fx_{m+1} - fx_m, u\| \\
&\quad + \alpha \max\Bigg\{\|fx_n - fx_m, u\|, \|fx_n - fx_{n+1}, u\|, \|fx_m - fx_{m+1}, u\|, \\
&\qquad\qquad \frac{\|fx_n - fx_m, u\| + \|fx_m - fx_{m+1}, u\|}{2} \\
&\qquad\qquad + \frac{\|fx_m - fx_n, u\| + \|fx_n - fx_{n+1}, u\|}{2}\Bigg\}.
\end{aligned}$$

Since the sequence $\{fx_n\}$ is an asymptotically T-regular, taking limit as $n, m \to \infty$, we get

$$\|fx_n - fx_m, u\| \leq \alpha\|fx_n - fx_m, u\|, \ \ 0 < \alpha < 1.$$

Therefore $\|fx_n - fx_m, u\| \to 0$, as $n, m \to \infty$ which implies that $\{fx_n\}$ is a Cauchy sequence in X. By the completeness of $f(X)$, there exists $v \in X$ such that $fx_n \to v = fw$ (Its also true, by the completeness of $T(X)$ with $v \in T(X)$). We claim that w is a coincidence point of f and T. Suppose not, then $\|fw - Tw, u\| > 0$. Now,

$$\begin{aligned}
\|fw - Tw, u\| &\leq \|v - fx_{n+1}, u\| + \|fx_{n+1} - Tw, u\| \\
&= \|v - fx_{n+1}, u\| + \|Tx_n - Tw, u\| \\
&\leq \|v - fx_{n+1}, u\| + \alpha \max\Bigg\{\|fx_n - fw, u\|, \|fx_n - Tx_n, u\|, \\
&\qquad \|fw - Tw, u\|, \frac{\|fx_n - Tw, u\| + \|fw - Tx_n, u\|}{2}\Bigg\}.
\end{aligned}$$

Taking limit $n \to \infty$, we get

$$\|fw - Tw, u\| \leq \alpha\|fw - Tw, u\|,$$

which is a contradiction. Hence $v = fw = Tw$ is a point of coincidence of T and f. □

Theorem 4.2. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and T, f be two self mappings of X such that $T(X) \subset f(X)$ and*

$$\|Tx - Ty, u\| \leq \alpha \max\Big\{\|fx - fy, u\|, \|fx - Tx, u\|, \|fy - Ty, u\|, \frac{\|fx - Ty, u + \|fy - Tx, u\|}{2}\Big\}, \tag{4.1}$$

for all $x, y, u \in X$ where $0 < \alpha < 1$. Then T and f have atmost a unique point of coincidence.

Proof. Let $v_1, v_2 \in X$ be such that $v_1 = Tw_1 = fw_1$ and $v_2 = Tw_2 = fw_2$ for some $w_1, w_2 \in X$. Using (4.1), we get the following

$$\begin{aligned}\|v_1 - v_2, u\| &= \|Tw_1 - Tw_2, u\| \\ &\leq \alpha \max\Big\{\|fw_1 - fw_2, u\|, \|fw_1 - Tw_1, u\|, \|fw_2 - Tw_2, u\|, \\ &\qquad \frac{\|fw_1 - Tw_2, u\| + \|fw_2 - Tw_1, u\|}{2}\Big\} \\ &\leq \alpha \max\Big\{\|v_1 - v_2, u\|, 0, 0, \|v_1 - v_2, u\|\Big\} = \alpha\|v_1 - v_2, u\|.\end{aligned}$$

Thus $0 < \alpha < 1$ gives $v_1 = v_2$. □

Ćirić[4] studied necessary conditions to obtain a fixed point result of asymptotically regular mappings on complete metric spaces. Abbas and Aydi [1] extended the results of Ćirić [5] to the case of two mappings satifying a generalized contractive conditions in a metric space. Rashwan [14] proved the similar result in a Hilbert space. We extend the result in 2-Banach space as follows:

Theorem 4.3. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space, F_1 and F_2 be continuous at 0 and $T, f : X \to X$ be such that $T(X) \subset f(X)$ and*

$$\begin{aligned}\|Tx - Ty, u\| \leq\ & aF_1\Big[\min\{\|fx - Tx, u\|, \|fy - Ty, u\|\}\Big] \\ &+ bF_2\Big[\|fx - Tx, u\|\|fy - Ty, u\|\Big] \\ &+ c\|fx - fy, u\| + d\Big[\|fx - Tx, u\| + \|fy - Ty, u\|\Big] \\ &+ e\Big[\|fx - Ty, u\| + \|fy - Tx, u\|\Big],\end{aligned} \tag{4.2}$$

for all $x, y, u \in X$ where $a, b, c, d, e > 0$, $c + e < 1$ and $d + e < 1$. If $f(X)$ or $T(X)$ is a complete subspace of X and T is asymptotically f-regular at some point $x_0 \in X$ then T and f have a point of coincidence.

Proof. Let $\{f(x_n)\}$ be a T-sequence with initial point x_0. Note that

$$\|fx_n - fx_{n+1}, u\| \to 0 \text{ as } n \to \infty,$$

since T is asymptotically f-regular at $x_0 \in X$. For $n > m$, we have

$$\begin{aligned}
&\|fx_n - fx_m, u\| \\
&\leq \|fx_n - Tx_n, u\| + \|Tx_n - fx_m, u\| \\
&\leq \|fx_n - Tx_n, u\| + \|Tx_n - Tx_m, u\| + \|Tx_m - fx_m, u\| \\
&\leq \|fx_n - Tx_n, u\| + \|Tx_m - fx_m, u\| \\
&\quad + aF_1\Big[\min\{\|fx_n - Tx_n, u\|, \|fx_m - Tx_m, u\|\}\Big] \\
&\quad + F_2\Big[\|fx_n - Tx_n, u\|\|fx_m - Tx_m, u\|\Big] + c\|fx_n - fx_m, u\| \\
&\quad + d\Big[\|fx_n - Tx_n, u\| + \|fx_m - Tx_m, u\|\Big] \\
&\quad + e\Big[\|fx_n - Tx_m, u\| + \|fx_m - Tx_n, u\|\Big] \\
&\leq \|fx_n - fx_{n+1}, u\| + \|fx_{m+1} - fx_m, u\| \\
&\quad + aF_1\Big[\min\{\|fx_n - fx_{n+1}, u\|, \|fx_m - fx_{m+1}, u\|\}\Big] \\
&\quad + bF_2\Big[\|fx_n - fx_{n+1}, u\|\|fx_m - fx_{m+1}, u\|\Big] + c\|fx_n - fx_m, u\| \\
&\quad + d\Big[\|fx_n - fx_{n+1}, u\| + \|fx_m - fx_{m+1}, u\|\Big] \\
&\quad + e\Big[\|fx_n - fx_m, u\| + \|fx_m - fx_{m+1}, u\| + \|fx_m - fx_{n+1}, u\|\Big].
\end{aligned}$$

Then we get

$$\begin{aligned}
&(1 - e - c)\|fx_n - fx_m, u\| \\
&\leq \|fx_n - fx_{n+1}, u\| + \|fx_{m+1} - fx_m, u\| \\
&\quad + aF_1\Big[\min\{\|fx_n - fx_{n+1}, u\|, \|fx_m - fx_{m+1}, u\|\}\Big] \\
&\quad + bF_2\Big[\|fx_n - fx_{n+1}, u\|\|fx_m - fx_{m+1}, u\|\Big]
\end{aligned}$$

$$+ d\Big[\|fx_n - fx_{n+1}, u\| + \|fx_m - fx_{m+1}, u\|\Big] \\ + e\Big[\|fx_m - fx_{m+1}, u\| + \|fx_m - fx_{n+1}, u\|\Big]. \tag{4.3}$$

Since T is asymptotically f-regular and F_1, F_2 are continuous at 0, by (4.3), we get

$$\|fx_n - fx_m, u\| \to 0 \text{ as } n, m \to \infty.$$

Hence $\{fx_n\}$ is a Cauchy sequence in X. As $f(X)$ is a complete subspace of X, there exists $v, w \in X$ such that $fx_n \to v = fw$ (this holds, even if $T(X)$ is a complete subspace of X).

Next we claim that w is a coincidence point of T and f. If not, then $\|fw - Tw, u\| > 0$. Now

$$\begin{aligned} \|fw - Tw, u\| &\le \|v - fx_{n+1}, u\| + \|fx_{n+1} - Tw, u\| \\ &= \|v - fx_{n+1}, u\| + \|Tx_n - Tw, u\| \\ &\le \|v - fx_{n+1}, u\| + aF_1\Big[\min\{\|fx_n - Tx_n, u\|, \|fw - Tw, u\|\}\Big] \\ &\quad + bF_2\Big[\|fx_n - Tx_n, u\|\|fw - Tw, u\|\Big] + c\|fx_n - fw, u\| \\ &\quad + d\Big[\|fx_n - Tx_n, u\| + \|fw - Tw, u\|\Big] \\ &\quad + e\Big[\|fx_n - Tw, u\| + \|fw - Tx_n, u\|\Big] \\ &\le \|v - fx_{n+1}, u\| + aF_1\Big[\min\{\|fx_n - fx_{n+1}, u\|, \|fw - Tw, u\|\}\Big] \\ &\quad + bF_2\Big[\|fx_n - fx_{n+1}, u\|\|fw - Tw, u\|\Big] + c\|fx_n - fw, u\| \\ &\quad + d\Big[\|fx_n - fx_{n+1}, u\| + \|fw - Tw, u\|\Big] \\ &\quad + e\Big[\|fx_n - fw, u\| + \|fw - Tw, u\| + \|fw - fx_{n+1}, u\|\Big]. \end{aligned}$$

Finally, we obtain

$$\|fw - Tw, u\| \le (d + e)\|fw - Tw, u\| < \|fw - Tw, u\|,$$

which is a contradiction and hence $v = fw = Tw$ is a point of coincidence of T and f. □

Lemma 4.4. *Let $(X, \|\cdot, \cdot\|)$ be a 2-Banach space, F_1 and F_2 be continuous at 0 and $T, f : X \to X$ be such that $T(X) \subset f(X)$ and*

$$
\begin{aligned}
\|Tx - Ty, u\| \leq aF_1\Big[\min\{\|fx - Tx, u\|, \|fy - Ty, u\|\}\Big] \\
+ bF_2\Big[\|fx - Tx, u\|\|fy - Ty, u\|\Big] + c\|fy - fy, u\| \\
+ d\Big[\|fx - Tx, u\| + \|fy - Ty, u\|\Big] \\
+ e\Big[\|fx - Ty, u\| + \|fy - Tx, u\|\Big],
\end{aligned}
$$

for all $x, y, u \in X$ where $a, b, c, d, e > 0$ and $c + 2e < 1$. Then T and f have atmost a unique point of coincidence.

Combining Theorem 4.3 and Lemma 4.4 we get the following theorem.

Theorem 4.5. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space, F_1 and F_2 be continuous at 0 and $T, f : X \to X$ be such that $T(X) \subset f(X)$. Assume that T and f satisty the inequality (4.2) for all $x, y, u \in X$. If $f(X)$ or $T(X)$ is a complete subspace of X such thath (T, f) is weakly compatible and T is asymptotically f-regular at some point $x_0 \in X$ then T and f have a unique common fixed point.*

As a consequence of Theorem 4.3, Lemma 4.4 and Theorem 4.5 we get the following corollary.

Corollary 4.6. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space, F_1 and F_2 be continuous at 0 and $T : X \to X$ be such that*

$$
\begin{aligned}
\|Tx - Ty, u\| \leq aF_1\Big[\min\{\|x - Tx, u\|, \|y - Ty, u\|\}\Big] \\
+ bF_2\Big[\|x - Tx, u\|\|y - Ty, u\|\Big] \\
+ c\|x - y, u\| + d\Big[\|x - Tx, u\| + \|y - Ty, u\|\Big] \\
+ e\Big[\|x - Ty, u\| + \|y - Tx, u\|\Big],
\end{aligned}
$$

for all $x, y, u \in X$ where $a, b, c, d, e > 0$ and $c + e < 1$ and $d + e < 1$. If T is asymptotically f-regular at some point $x_0 \in X$ then T has a unique fixed point.

Taking $a = b = 0$ in the inequality (4.2), we obtain the following corollary.

Corollary 4.7. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and $T, f : X \to X$ be such that $T(X) \subset f(X)$ and*

$$\|Tx - Ty, u\| \leq c\|fx - fy, u\| + d\Big[\|fx - Tx, u\| + \|fy - Ty, u\|\Big] + e\Big[\|fx - Ty, u\| + \|fy - Tx, u\|\Big],$$

for all $x, y, u \in X$ where $a, b, c, d, e > 0$ and $d + e < 1$ and $c + e < 1$. If $f(X)$ or $T(X)$ is a complete subspace of X and T is asymptotically f-regular at some point $x_0 \in X$ then T and f have a unique point of coincidence.

Let $CF(T, f)$ denote the set of all common fixed points of T and f. Now we get the following result on the continuity on the set of common fixed points.

Theorem 4.8. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and $T, f : X \to X$ be such that T and f satisfy (4.2) for all $x, y, u \in X$. If $CF(T, f) \neq \emptyset$ then T is continuous at $v \in CF(T, f)$ whenever f is continuous at v.*

Proof. Let $v \in CF(T, f)$ and let $\{v_n\}$ be any sequence in X which converges to v. Putting $x = v$ and $y = v_n$ in (4.2), we get

$$\begin{aligned}\|Tv - Tv_n, u\| \leq{}& aF_1\Big[\min\{\|fv - Tv, u\|, \|fv_n - Tv_n, u\|\}\Big] \\ &+ bF_2\Big[\|fv - Tv, u\|\|fv_n - Tv_n, u\|\Big] \\ &+ c\|fv - fv_n, u\| + d\Big[\|fv - Tv, u\| + \|fv_n - Tv_n, u\|\Big] \\ &+ e\Big[\|fv - Tv_n, u\| + \|fv_n - Tv, u\|\Big].\end{aligned}$$

Using the fact $Tv = fv$, we get

$$\|Tv - Tv_n, u\| \leq c\|fv - fv_n, u\| + d[\|fv_n - Tv_n, u\|] + e\Big[\|fv - Tv_n, u\| + \|fv_n - Tv, u\|\Big].$$

Taking $n \to \infty$, we get

$$\limsup_{n\to\infty} \|Tv - Tv_n, u\| \leq (d + e) \limsup_{n\to\infty} \|Tv - Tv_n, u\|, \tag{4.4}$$

when f is continuous at v. The inequality (4.4) is true, only if

$$\limsup_{n\to\infty} \|Tv - Tv_n, u\| = 0.$$

Finally we get $Tv_n \to Tv$ as $n \to \infty$. □

5. Well-posedness

The notion of well-posedness of a fixed point problem has generated much interest to several mathematicians, for example [2, 3, 10, 12, 13, 15]. Here, we study well-posedness of a common fixed point problem.

Definition 5.1. Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space and f be a self mapping. The fixed point problem of f is said to be well-posed if

(i) f has a unique fixed point $x_0 \in X$,
(ii) for any sequence $\{x_n\} \subset X$, $\lim_{n\to\infty} \|x_n - fx_n, u\| = 0$ we have

$$\lim_{n\to\infty} \|x_n - x_0, u\| = 0.$$

Definition 5.2. A common fixed point problem of self mappings T and f on X, $CFP(T, f, X)$ is called well-posed if $CF(T, f)$ is singleton and for any sequence $\{x_n\}$ in X with

$$\tilde{x} \in CF(T, f) \text{ and } \lim_{n\to\infty} \|x_n - fx_n, u\| = \lim_{n\to\infty} \|x_n - Tx_n, u\| = 0$$

implies $\tilde{x} = \lim_{n\to\infty} x_n$.

Theorem 5.3. *Let $(X, \|\cdot,\cdot\|)$ be a 2-Banach space, T and f be self mappings on X as in* Theorem 4.3 *and* Lemma 4.4*. Then the common fixed problem of f and T is well posed.*

Proof. From Theorem 4.3 and Lemma 4.4, the mappings T and f have a unique common fixed point, say $v \in X$. Let $\{x_n\}$ be a sequence in X and $\lim_{n\to\infty} \|fx_n - x_n, u\| = \lim_{n\to\infty} \|Tx_n - x_n, u\| = 0$. Without loss of generality, assume that $v \neq x_n$ for any non-negative integer n. Using $fv = Tv$, we get

$$\begin{aligned}
\|v - x_n\| &\leq \|Tv - Tx_n, u\| + \|Tx_n - x_n, v\| \\
&\leq \|Tx_n - x_n, u\| + aF_1\Big[\min\{\|fv - Tv, u\|, \|fx_n - Tx_n, u\|\}\Big] \\
&\quad + bF_2\Big[\|fv - Tv, u\|\|fv_n - Tx_n, u\|\Big] \\
&\quad + c\|fv - fx_n, u\| + d\Big[\|fv - Tv, u\| + \|fx_n - Tx_n, u\|\Big] \\
&\quad + e\Big[\|fv - Tx_n, u\| + \|fx_n - Tv, u\|\Big] \\
&\leq \|Tx_n - x_n\| + c\Big[\|fv - x_n, u\| + \|x_n - fx_n, u\|\Big]
\end{aligned}$$

$$+ d\Big[\|fx_n - x_n, u\| + \|x_n - Tx_n, u\|\Big]$$
$$+ e\Big[\|fv - x_n, u\| + \|x_n - Tx_n, u\| + \|fx_n - x_n, u\| + \|x_n - Tv, u\|\Big].$$

Taking $n \to \infty$, we get $\limsup_{n\to\infty} \|v - x_n, u\| = 0$. Hence $x_n \to v$ as $n \to \infty$. □

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