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COMMON EXTENSION OF A FAMILY OF GROUP-VALUED, FINITELY ADDITIVE MEASURES

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We deal with the problem of finding common extensions of finitely additive measures ("charges") taking values in a group G. All groups will be assumed Abelian, and the usual additive notation for Abelian groups will be employed. Let X be a non-empty set and let $\mathcal{A}$ be a field of subsets of X. A function $\mu : \mathcal{A} \to G$ is a (G-valued) *charge* if $\mu(\emptyset) = 0$ and $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$ whenever A_1 and A_2 are disjoint sets in $\mathcal{A}$.

Now suppose that $\mathcal{A}$ and $\mathcal{B}$ are fields of subsets of X and that $\mu : \mathcal{A} \to G$ and $\nu : \mathcal{B} \to G$ are G-valued charges. We say that μ and ν are *consistent* if $\mu(C) = \nu(C)$ whenever $C \in \mathcal{A} \cap \mathcal{B}$. It is natural to ask when two such consistent charges have a common extension, i.e. a charge ϱ such that $\varrho(A) = \mu(A)$ if $A \in \mathcal{A}$ and $\varrho(B) = \nu(B)$ if $B \in \mathcal{B}$. The charge ϱ is to be defined on $\mathcal{A} \vee \mathcal{B}$, the field generated by $\mathcal{A} \cup \mathcal{B}$.

Say that a group G has the 2-*extension property* if every pair of consistent G-valued charges has a common extension. The following result is to be found in [1] and [3].

THEOREM 1. *A group has the* 2-*extension property if and only if it is a cotorsion group.*

A group G is said to be *cotorsion* if it is the homomorphic image of an algebraically compact group. Every divisible group (e.g. $\mathbb{R}$) is cotorsion. For information about these matters, see [2]. It is tempting to try an extension of this result in a naïve way. However, one might consider the following.

EXAMPLE. Put $X = \{x, y, z\}$ and let $\mathcal{A}$ be the field with atoms $\{x\}$ and $\{y, z\}$; let $\mathcal{B}$ be the field with atoms $\{y\}$ and $\{x, z\}$; let $\mathcal{C}$ be the field with atoms $\{z\}$ and $\{x, y\}$. Define real-valued charges μ, ν, τ on $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, respectively, so that

$$\begin{aligned} \mu\{x\} &= 1\,, & \nu\{y\} &= 1\,, & \tau\{z\} &= 1\,,\\ \mu\{y,z\} &= 0\,, & \nu\{x,z\} &= 0\,, & \tau\{x,y\} &= 0\,. \end{aligned}$$

Then the charges μ, ν, τ are *pairwise* consistent, but they have no common extension $\varrho : \mathcal{A} \vee \mathcal{B} \vee \mathcal{C} \to \mathbb{R}$.

The example illustrates the need for a stronger form of consistency in the case of more than two charges. Given a field $\mathcal{A}$ of subsets of X, let $S(\mathcal{A})$ be the set of all bounded functions $f : X \to \mathbb{Z}$ such that $f^{-1}(n) \in \mathcal{A}$ for each $n \in \mathbb{Z}$. We see that $S(\mathcal{A})$ is a group under pointwise addition of functions. Let $\mu : \mathcal{A} \to G$ be a charge. Given $A \in \mathcal{A}$, let I_A be its indicator function. Then the mapping $I_A \to \mu(A)$ extends uniquely to a homomorphism from $S(\mathcal{A})$ to G. The value of this homomorphism at $f \in S(\mathcal{A})$ will be denoted by $\int f\, d\mu$, the integral of f with respect to μ. Given fields $\mathcal{A}_1, \ldots, \mathcal{A}_k$ on a set X, we say that charges $\mu_1 : \mathcal{A}_1 \to G, \ldots, \mu_k : \mathcal{A}_k \to G$ are *consistent* if whenever $f_1 \in S(\mathcal{A}_1), \ldots, f_k \in S(\mathcal{A}_k)$ are such that $f_1 + \ldots + f_k = 0$, then $\int f_1\, d\mu_1 + \ldots + \int f_k\, d\mu_k = 0$. Clearly, consistency of the charges $\mu_1, \ldots, \mu_k$ is a condition necessary for the existence of a common extension $\varrho : \mathcal{A}_1 \vee \ldots \vee \mathcal{A}_k \to G$. For $k = 2$, it is not hard to verify that this definition of consistency agrees with the one given earlier. Say that a group G has the *k-extension property* if every set of k consistent charges $\mu_1, \ldots, \mu_k$ has a common extension $\varrho : \mathcal{A}_1 \vee \ldots \vee \mathcal{A}_k \to G$. Obviously, the $(k+1)$-extension property implies the k-extension property for each k.

THEOREM 2. *Let G be an Abelian group. The following conditions are equivalent*:

1) *G has the k-extension property for each k*;
2) *G has the 3-extension property*;
3) *G is divisible.*

Proof. The implication 1⇒2 is obvious. We demonstrate 2⇒3 by an induction argument. Assuming that G has the 3-extension property, we show that divisibility of every element of G by $n-1$ implies divisibility by n. (Note that divisibility by 1 is trivial in any group.) Take $X = \{u(i,j) : i = 1, 2;\ \ j = 1, \ldots, n\}$, a set of $2n$ elements. On X, we define fields $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. The field $\mathcal{A}$ has n atoms, each of the form $\{u(1,j), u(2,j)\}$ $(j = 1, \ldots, n)$; the field $\mathcal{B}$ has 2 atoms, of the form $\{u(i,1), u(i,2), \ldots, u(i,n)\}$ $(i = 1, 2)$; the field $\mathcal{C}$ has n atoms: $n-1$ of these are of the form $\{u(1,j), u(2,j-1)\}$ $(j = 2, \ldots, n)$, and the remaining atom is $\{u(1,1), u(2,n)\}$.

CLAIM 1. *The only functions in $(S(\mathcal{A}) + S(\mathcal{B})) \cap S(\mathcal{C})$ are constant.*

Proof of claim. Suppose that for $f \in S(\mathcal{A})$ and $g \in S(\mathcal{B})$, the function $h = f + g$ belongs to $S(\mathcal{C})$. Fix i with $1 \le i \le n-1$. Then

$$\begin{aligned} g(u(2,i)) - g(u(1,i)) &= h(u(2,i)) - h(u(1,i)) = h(u(1,i+1)) - h(u(1,i)) \\ &= f(u(1,i+1)) - f(u(1,i)), \end{aligned}$$

and

$$g(u(2,n)) - g(u(1,n)) = h(u(2,n)) - h(u(1,n)) = h(u(1,1)) - h(u(1,n))$$
$$= f(u(1,1)) - f(u(1,n))\,.$$

Since the quantity $g(u(2,i)) - g(u(1,i))$ is constant, i.e. independent of i, we see that f is a constant function. Thus $g \in S(\mathcal{B}) \cap S(\mathcal{C})$ is constant as well.

CLAIM 2. *A trio of charges* $\mu : \mathcal{A} \to G$, $\nu : \mathcal{B} \to G$, $\tau : \mathcal{C} \to G$ *is consistent so long as* $\mu(X) = \nu(X) = \tau(X)$.

Proof of claim. Suppose that $f + g + h = 0$ for $f \in S(\mathcal{A})$, $g \in S(\mathcal{B})$, $h \in S(\mathcal{C})$. Claim 1 and its proof imply that f, g, h are constant. Then $\int f\,d\mu + \int g\,d\nu + \int h\,d\tau = 0$, establishing the claim.

Given $a \in G$, use divisibility by $n-1$ to write $a = (n-1)b$ for some $b \in G$. Define G-valued charges μ, ν, τ on $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, respectively, as follows. For $j = 1, \ldots, n-1$, put $\mu(\{u(1,j), u(2,j)\}) = b$ and set $\mu(\{u(1,n), u(2,n)\}) = 0$. Let $\nu(\{u(i,1), \ldots, u(i,n)\})$ have the value $(n-1)b$ for $i = 1$ and the value 0 for $i = 2$. For $j = 2, \ldots, n$, put $\tau(\{u(1,j), u(2,j-1)\}) = b$ and set $\tau(\{u(1,1), u(2,n)\}) = 0$. Then $\mu(X) = \nu(X) = \tau(X) = (n-1)b = a$, so that (Claim 2) μ, ν, τ are consistent. If G has the 3-extension property, then these charges have a common extension to a charge $\varrho : \mathcal{A} \vee \mathcal{B} \vee \mathcal{C} \to G$. An elementary computation shows that $x = \varrho(u(1,j))$ is independent of j. Summing over j yields $nx = (n-1)b = a$. So we see that each $a \in G$ is divisible by n, as desired.

The implication 3⇒1 is easy, since any homomorphism into a divisible group can be extended. In particular, if $\mu_i : \mathcal{A} \to G$ $(i = 1, \ldots, k)$ are consistent charges on fields $\mathcal{A}_i$ over a set X, then the homomorphism

$$f_1 + \ldots + f_k \to \int f_1\,d\mu_1 + \ldots + \int f_k\,d\mu_k$$

from $S(\mathcal{A}_1) + \ldots + S(\mathcal{A}_k)$ to G extends to a homomorphism from $S(\mathcal{A}_1 \vee \ldots \vee \mathcal{A}_k)$ to G. Defining $\varrho(A)$ to be the value of this homomorphism at I_A yields the desired extended charge. ∎

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