COMMON FIXED POINT RESULTS FOR THREE MAPS IN G- METRIC SPACES

Mujahid Abbas, Talat Nazir and Pasquale Vetro

Abstract

In this paper, we use the setting of generalized metric spaces to obtain common fixed point results for three maps. These results generalize several well known comparable results in the literature.

1 Introduction and preliminaries

The study of fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity. Mustafa and Sims [8] generalized the concept of a metric space. Based on the notion of generalized metric spaces, Mustafa et al. [7, 9, 10, 11] obtained some fixed point theorems for mappings satisfying different contractive conditions. Abbas and Rhoades [1] initiated the study of a common fixed point theory in generalized metric spaces. Abbas et al. [2] obtained some periodic point results in generalized metric spaces. The coupled common fixed point results in two generalized metric spaces are obtained by [3]. For other references, the reader is referred to [5, 6, 13]. Shatanawi [14] proved some fixed point results for self mappings in a complete G-metric space under some contractive conditions related to a nondecreasing map $\phi : R^+ \to R^+$ with $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \geq 0$. Saadati et al. [12] proved some fixed point results for contractive mappings in partially ordered G-metric spaces. Recently, Abbas et at. [4] gives some common fixed points of $R-$ weakly commuting maps in generalized metric space.

The aim of this paper is to initiate the study of common fixed point for three mappings in complete G-metric space. It is worth mentioning that our results do not rely on the notion of continuity, weakly commuting or compatibility of mappings involved therein. Our results generalize Theorems 2.1 to 2.3 and 2.5 of Mustafa et. al [10] and Theorem 2.9 [9].

Consistent with Mustafa and Sims [8], the following definitions and results will be needed in the sequel.

Definition 1.1. Let X be a nonempty set. A function $G : X \times X \times X \to [0, +\infty[$ is called a G-metric if the following conditions are satisfied:

(a) $G(x, y, z) = 0$ if $x = y = z$,

(b) $0 < G(x, x, y)$ for all $x, y \in X$, with $x \neq y$,

(c) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $z \neq y$,

(d) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry),

(e) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

A G-metric is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.
If G is a G-metric on X, then the pair (X, G) is called a G-metric space.

Definition 1.2.   Let (X, G) be a G-metric space. We say that $\{x_n\} \subset X$ is:

(i) a G-*Cauchy* sequence if, for any $\varepsilon > 0$, there is a $n_0 \in \mathbb{N}$ (the set of all positive integers) such that for all $n, m, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$;

(ii) a G-*convergent* sequence if, for any $\varepsilon > 0$, there is a $x \in X$ and a $n_0 \in N$, such that for all $n, m \geq n_0$, $G(x, x_n, x_m) < \varepsilon$.

A G-metric space X is said to be complete if every G-Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ converges to $x \in X$ if and only if $G(x_m, x_n, x) \to 0$ as $n, m \to \infty$ and that $\{x_n\}$ is Cauchy if and only if $G(x_m, x_n, x_l) \to 0$ as $n, m, l \to \infty$.

Example 1.3.   Let (X, d) be a metric space. The function $G : X \times X \times X \to [0, +\infty[$, defined by

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$$

or

$$G(x, y, z) = d(x, y) + d(y, z) + d(z, x)$$

for all $x, y, z \in X$, is a G-metric on X.

Example 1.4.   Let (X, G) be a G-metric space. The function $G : X \times X \to [0, +\infty[$, defined by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \ \forall \ x, y \in X$$

is a metric on X.

Definition 1.5.   Let (X, G) and (X', G') be G-metric spaces. A map $f : (X, G) \to (X', G')$ is said to be G-continuous at a point $a \in X$ if and only if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$ and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \varepsilon$. A map f is G-continuous on X if and only if it is G-continuous at all $a \in X$.

Proposition 1.6.   Let (X, G) be a G-metric space. Then the following statements hold:

(i) G is jointly continuous in all three of its variables,

(ii) $G(x, y, y) \leq 2G(y, x, x)$.

2   Common fixed point theorems

In this section we obtain common fixed point theorems for three maps defined on a generalized metric space.

Theorem 2.1.   Let f, g and h be self maps on a complete G-metric space X satisfying

$$
\begin{aligned}
G(fx, gy, hz) \quad \leq \quad & \alpha G(x, y, z) + \beta[G(fx, x, x) + G(y, gy, y) + G(z, z, hz)] \\
& + \gamma[G(fx, y, z) + G(x, gy, z) + G(x, y, hz)]
\end{aligned}
\tag{1}
$$

for all $x, y, z \in X$, where $\alpha, \beta, \gamma > 0$ and $\alpha + 3\beta + 4\gamma < 1$. Then f, g and h have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and h and conversely.

Proof. We will proceed in two steps: first we prove that any fixed point of f is a fixed point of g and h. Assume that $p \in X$ is such that $fp = p$. Now, we prove that $p = gp = hp$. If it is not the case, then for $p \neq gp$ and $p \neq hp$, we get

$$
\begin{aligned}
& G(p, gp, hp) \\
= \quad & G(fp, gp, hp) \\
\leq \quad & \alpha G(p, p, p) + \beta[G(fp, p, p) + G(p, gp, p) + G(p, p, hp)] \\
& + \gamma[G(fp, p, p) + G(p, gp, p) + G(p, p, hp)] \\
= \quad & (\beta + \gamma)[G(p, gp, p) + G(p, p, hp)] \\
\leq \quad & (\beta + \gamma)[G(p, gp, hp) + G(p, gp, hp)] \\
= \quad & (2\beta + 2\gamma)G(p, gp, hp),
\end{aligned}
$$

that is a contradiction. Analogously, following the similar arguments to those given above, we obtain a contradiction for $p \neq gp$ and $p = hp$ or for $p \neq hp$ and $p = gp$. Hence in all the cases, we conclude that $p = gp = hp$. The same conclusion holds if $p = gp$ or $p = hp$.

Now, we prove that f, g and h have a unique common fixed point. Suppose x_0 is an arbitrary point in X. Define $\{x_n\}$ by $x_{3n+1} = fx_{3n}$, $x_{3n+2} = gx_{3n+1}$, $x_{3n+3} = hx_{3n+2}$, $n = 0, 1, 2, \ldots$. If $x_n = x_{n+1}$ for some n, with $n = 3m$, then $p = x_{3n}$ is a fixed point of f and, by the first step, p is a common fixed point for f, g and h. The same holds if $n = 3m + 1$ or $n = 3m + 2$. Now, we assume that

$x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then, we have

$$
\begin{aligned}
& G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\
=\ & G(fx_{3n}, gx_{3n+1}, hx_{3n+2}) \\
\leq\ & \alpha G(x_{3n}, x_{3n+1}, x_{3n+2}) + \beta[G(fx_{3n}, x_{3n}, x_{3n}) + G(x_{3n+1}, gx_{3n+1}, x_{3n+1}) \\
& + G(x_{3n+2}, x_{3n+2}, hx_{3n+2})] + \gamma[G(fx_{3n}, x_{3n+1}, x_{3n+2}) \\
& + G(x_{3n}, gx_{3n+1}, x_{3n+2}) + G(x_{3n}, x_{3n+1}, hx_{3n+2})] \\
=\ & \alpha G(x_{3n}, x_{3n+1}, x_{3n+2}) + \beta[G(x_{3n+1}, x_{3n}, x_{3n}) + G(x_{3n+1}, x_{3n+2}, x_{3n+1}) \\
& + G(x_{3n+2}, x_{3n+2}, x_{3n+3})] + \gamma[G(x_{3n+1}, x_{3n+1}, x_{3n+2}) \\
& + G(x_{3n}, x_{3n+2}, x_{3n+2}) + G(x_{3n}, x_{3n+1}, x_{3n+3})] \\
\leq\ & \alpha G(x_{3n}, x_{3n+1}, x_{3n+2}) + \beta[G(x_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n}, x_{3n+1}, x_{3n+2}) \\
& + G(x_{3n+1}, x_{3n+2}, x_{3n+3})] + \gamma[G(x_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n}, x_{3n+1}, x_{3n+2}) \\
& + \{G(x_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n+1}, x_{3n+2}, x_{3n+3})\}],
\end{aligned}
$$

that is

$$(1 - \beta - \gamma)G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq (\alpha + 2\beta + 3\gamma)G(x_{3n}, x_{3n+1}, x_{3n+2}).$$

Hence,

$$G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq \lambda G(x_{3n}, x_{3n+1}, x_{3n+2}),$$

where $\lambda = \dfrac{\alpha + 2\beta + 3\gamma}{1 - \beta - \gamma}$. Obviously $0 < \lambda < 1$. Similarly it can be shown that

$$G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \leq \lambda G(x_{3n+1}, x_{3n+2}, x_{3n+3})$$

and

$$G(x_{3n+3}, x_{3n+4}, x_{3n+5}) \leq \lambda G(x_{3n+2}, x_{3n+3}, x_{3n+4}).$$

Therefore, for all n,

$$
\begin{aligned}
G(x_{n+1}, x_{n+2}, x_{n+3}) &\leq \lambda G(x_n, x_{n+1}, x_{n+2}) \\
&\leq \cdots \leq \lambda^{n+1} G(x_0, x_1, x_2).
\end{aligned}
$$

Now, for any l, m, n with $l > m > n$,

$$
\begin{aligned}
G(x_n, x_m, x_l) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\
&\quad + \cdots + G(x_{l-1}, x_{l-1}, x_l) \\
&\leq G(x_n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+3}) \\
&\quad + \cdots + G(x_{l-2}, x_{l-1}, x_l) \\
&\leq [\lambda^n + \lambda^{n+1} + \cdots + \lambda^{l-2}]G(x_0, x_1, x_2) \\
&\leq \frac{\lambda^n}{1 - \lambda}G(x_0, x_1, x_2).
\end{aligned}
$$

The same holds if $l = m > n$ and if $l > m = n$ we have

$$G(x_n, x_m, x_l) \leq \frac{\lambda^{n-1}}{1-\lambda} G(x_0, x_1, x_2).$$

Consequently $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$. Hence $\{x_n\}$ is a G-Cauchy sequence. By G-completeness of X, there exists $u \in X$ such that $\{x_n\}$ converges to u as $n \to \infty$. We claim that $fu = u$. If not, then consider

$$
\begin{aligned}
& G(fu, x_{3n+2}, x_{3n+3}) \\
= \; & G(fu, gx_{3n+1}, hx_{3n+2}) \\
\leq \; & \alpha G(u, x_{3n+1}, x_{3n+2}) + \beta[G(fu, u, u) + G(x_{3n+1}, gx_{3n+1}, x_{3n+1}) \\
& + G(x_{3n+2}, x_{3n+2}, hx_{3n+2})] + \gamma[G(fu, x_{3n+1}, x_{3n+2}) \\
& + G(u, gx_{3n+1}, x_{3n+2}) + G(u, x_{3n+1}, hx_{3n+2})] \\
= \; & \alpha G(u, x_{3n+1}, x_{3n+2}) + \beta[G(fu, u, u) + G(x_{3n+1}, x_{3n+2}, x_{3n+1}) \\
& + G(x_{3n+2}, x_{3n+2}, x_{3n+3})] + \gamma[G(fu, x_{3n+1}, x_{3n+2}) \\
& + G(u, x_{3n+2}, x_{3n+2}) + G(u, x_{3n+1}, x_{3n+3})].
\end{aligned}
$$

On taking limit $n \to \infty$, we obtain that

$$G(fu, u, u) \leq (\beta + \gamma)G(fu, u, u),$$

a contradiction. Hence $fu = u$. Similarly it can be shown that $gu = u$ and $hu = u$.

To prove the uniqueness, suppose that v is another common fixed point of f, g and h, then

$$
\begin{aligned}
G(u, v, v) \; = \; & G(fu, gv, hv) \\
\leq \; & \alpha G(u, v, v) + \beta[G(fu, u, u) + G(v, gv, v) + G(v, v, hv)] \\
& + \gamma[G(fu, v, v) + G(u, gv, v) + G(u, v, hv)] \\
= \; & \alpha G(u, v, v) + \beta[G(u, u, u) + G(v, v, v) + G(v, v, v)] \\
& + \gamma[G(u, v, v) + G(u, v, v) + G(u, v, v)] \\
= \; & (\alpha + 3\gamma)G(u, v, v),
\end{aligned}
$$

which gives that $G(u, v, v) = 0$, and $u = v$. Hence u is a unique common fixed point of f, g and h. $\square$

Corollary 2.2. Let f, g and h be self maps on a complete G-metric space X satisfying

$$
\begin{aligned}
G(f^m x, g^m y, h^m z) \; \leq \; & \alpha G(x, y, z) + \beta[G(f^m x, x, x) + G(y, g^m y, y) + G(z, z, h^m z)] \\
& + \gamma[G(f^m x, y, z) + G(x, g^m y, z) + G(x, y, h^m z)] \quad (2)
\end{aligned}
$$

for all $x, y, z \in X$, where $\alpha, \beta, \gamma > 0$ and $\alpha + 3\beta + 4\gamma < 1$. Then f, g and h have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and h and conversely.

Proof. It follows from Theorem 2.1, that f^m, g^m and h^m have a unique common fixed point p. Now $f(p) = f(f^m(p)) = f^{m+1}(p) = f^m(f(p))$, $g(p) = g(g^m(p)) = g^{m+1}(p) = g^m(g(p))$ and $h(p) = h(h^m(p)) = h^{m+1}(p) = h^m(h(p))$ implies that fp, gp and hp are also fixed points for f^m, g^m and h^m. Since the common fixed point of f^m, g^m and h^m is unique, we deduce that $p = fp = gp = hp$. It is obvious that every fixed point of f is a fixed point of g and h and conversely.   □

Corollary 2.3. Let f, g and h be self maps on a complete G-metric space X satisfying (1). If $F(f)$ is a singleton, then $F(g)$ and $F(h)$ are singleton and conversely.

Proof. It follows from Theorem 2.1, since any fixed point of f is a fixed point of g and h and conversely.   □

The following Theorem generalizes the Theorems 2.1-2.3 in [10].

Theorem 2.4. Let f, g and h be self maps on a complete G-metric space X satisfying

$$G(fx, gy, hz) \leq a\,G(x, y, z) + b\,G(x, fx, fx) + c\,G(y, gy, gy) + d\,G(z, hz, hz) \quad (3)$$

for all $x, y, z \in X$, where $0 < a + b + c + d < 1$. Then f, g and h have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and h and conversely.

Proof. We will proceed in two steps: first we prove that any fixed point of f is a fixed point of g and h. Assume that $p \in X$ is such that $fp = p$. Now, we prove that $p = gp = hp$. If it is not then for $p \neq gp$ and $p \neq hp$, we obtain

$$
\begin{aligned}
G(p, gp, hp) &= G(fp, gp, hp) \\
&\leq a\,G(p, p, p) + b\,G(p, fp, fp) + c\,G(p, gp, gp) + d\,G(p, hp, hp) \\
&= a\,G(p, p, p) + b\,G(p, p, p) + c\,G(p, gp, gp) + d\,G(p, hp, hp) \\
&= c\,G(p, gp, gp) + d\,G(p, hp, hp) \\
&\leq (c + d)G(p, gp, hp),
\end{aligned}
$$

that is a contradiction. Analogously, for $p \neq gp$ and $p = hp$ or for $p \neq hp$ and $p = gp$, following the similar arguments to those given above, we obtain a contradiction. Hence in all the cases, we conclude that $p = gp = hp$. The same conclusion holds if $p = gp$ or $p = hp$.

Now, we prove that f, g and h have a unique common fixed point. Suppose x_0 is an arbitrary point in X. Define $\{x_n\}$ by $x_{3n+1} = fx_{3n}$, $x_{3n+2} = gx_{3n+1}$, $x_{3n+3} = hx_{3n+2}$, $n = 0, 1, 2, \ldots$. If $x_n = x_{n+1}$ for some n, with $n = 3m$, then $p = x_{3n}$ is a fixed point of f and, by the first step, p is a common fixed point for f, g and h. The same holds if $n = 3m + 1$ or $n = 3m + 2$. Now, we assume that

$x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then, we have

$$
\begin{aligned}
&G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\
={}& G(fx_{3n}, gx_{3n+1}, hx_{3n+2}) \\
\leq{}& a\,G(x_{3n}, x_{3n+1}, x_{3n+2}) + b\,G(x_{3n}, fx_{3n}, fx_{3n}) + c\,G(x_{3n+1}, gx_{3n+1}, gx_{3n+1}) \\
&+ d\,G(x_{3n+2}, hx_{3n+2}, hx_{3n+2}) \\
={}& a\,G(x_{3n}, x_{3n+1}, x_{3n+2}) + b\,G(x_{3n}, x_{3n+1}, x_{3n+1}) + c\,G(x_{3n+1}, x_{3n+2}, x_{3n+2}) \\
&+ d\,G(x_{3n+2}, x_{3n+3}, x_{3n+3}) \\
\leq{}& a\,G(x_{3n}, x_{3n+1}, x_{3n+2}) + b\,G(x_{3n}, x_{3n+1}, x_{3n+2}) + c\,G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\
&+ d\,G(x_{3n+1}, x_{3n+2}, x_{3n+3})
\end{aligned}
$$

which implies that

$$
(1 - c - d)G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq (a + b)G(x_{3n}, x_{3n+1}, x_{3n+2}).
$$

Hence,

$$
G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq \lambda\,G(x_{3n}, x_{3n+1}, x_{3n+2}),
$$

where $\lambda = \dfrac{a + b}{1 - c - d}$. Obviously $0 < \lambda < 1$. Similarly it can be shown that

$$
G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \leq \lambda\,G(x_{3n+1}, x_{3n+2}, x_{3n+3})
$$

and

$$
G(x_{3n+3}, x_{3n+4}, x_{3n+5}) \leq \lambda\,G(x_{3n+2}, x_{3n+3}, x_{3n+4}).
$$

Therefore, for all n,

$$
\begin{aligned}
G(x_{n+1}, x_{n+2}, x_{n+3}) &\leq \lambda\,G(x_n, x_{n+1}, x_{n+2}) \\
&\leq \cdots \leq \lambda^{n+1}\,G(x_0, x_1, x_2).
\end{aligned}
$$

Following similar arguments to those given in Theorem 2.1, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$. Hence $\{x_n\}$ is a G-Cauchy sequence. By G-completeness of X, there exists $u \in X$ such that $\{x_n\}$ converges to u as $n \to \infty$. We claim that $fu = u$. If not, then consider

$$
\begin{aligned}
G(fu, x_{3n+2}, x_{3n+3}) &= G(fu, gx_{3n+1}, hx_{3n+2}) \\
&\leq a\,G(u, x_{3n+1}, x_{3n+2}) + b\,G(u, fu, fu) \\
&\quad + c\,G(x_{3n+1}, gx_{3n+1}, gx_{3n+1}) + d\,G(x_{3n+2}, hx_{3n+2}, hx_{3n+2}) \\
&= a\,G(u, x_{3n+1}, x_{3n+2}) + b\,G(fu, u, x_{3n+1}) \\
&\quad + c\,G(x_{3n+1}, x_{3n+2}, x_{3n+2}) + d\,G(x_{3n+2}, x_{3n+3}, x_{3n+3}).
\end{aligned}
$$

On taking limit $n \to \infty$, we obtain that

$$
G(fu, u, u) \leq bG(fu, u, u),
$$

a contradiction. Hence $fu = u$. Similarly it can be shown that $gu = u$ and $hu = u$.

To prove the uniqueness, suppose that v is another common fixed point of f, g and h, then

$$
\begin{aligned}
G(u, v, v) &= G(fu, gv, hv) \\
&\leq a\,G(u, v, v) + b\,G(u, fu, fu) + c\,G(v, gv, gv) + d\,G(v, hv, hv) \\
&= a\,G(u, v, v) + b\,G(u, u, u) + c\,G(v, v, v) + d\,G(v, v, v) \\
&= a\,G(u, v, v),
\end{aligned}
$$

which gives that $G(u, v, v) = 0$ and hence $u = v$. Thus u is a unique common fixed point of f, g and h. $\qquad\square$

Example 2.5. Let $X = [0, 1]$ and $G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$ be a G-metric on X. Define $f, g, h : X \to X$ as

$$
f(x) = \begin{cases} x/12 & \text{for } x \in [0, \frac{1}{2}) \\ x/10 & \text{for } x \in [\frac{1}{2}, 1], \end{cases}
$$

$$
g(x) = \begin{cases} x/8 & \text{for } x \in [0, \frac{1}{2}) \\ x/6 & \text{for } x \in [\frac{1}{2}, 1], \end{cases}
$$

and

$$
h(x) = \begin{cases} x/5 & \text{for } x \in [0, \frac{1}{2}) \\ x/3 & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}
$$

Note that f, g and h are discontinuous maps. Also

$$
fg(\frac{1}{2}) = f(\frac{1}{12}) = \frac{1}{144}, \qquad gf(\frac{1}{2}) = g(\frac{1}{20}) = \frac{1}{160},
$$

$$
gh(\frac{1}{2}) = g(\frac{1}{6}) = \frac{1}{48}, \qquad hg(\frac{1}{2}) = h(\frac{1}{12}) = \frac{1}{60},
$$

and

$$
fh(\frac{1}{2}) = f(\frac{1}{6}) = \frac{1}{72}, \qquad hf(\frac{1}{2}) = h(\frac{1}{20}) = \frac{1}{100},
$$

which shows that f, g and h does not commute to each other.

Note that for $x, y, z \in [0, \frac{1}{2}]$,

$$
\begin{aligned}
G(x, y, z) &= \max\{|x - y|, |y - z|, |z - x|\}, \\
G(x, fx, fx) &= \max\{\left|x - \frac{x}{12}\right|, \left|\frac{x}{12} - \frac{x}{12}\right|, \left|\frac{x}{12} - x\right|\} = \frac{11x}{12}, \\
G(y, gy, gy) &= \max\{\left|y - \frac{y}{8}\right|, \left|\frac{y}{8} - \frac{y}{8}\right|, \left|\frac{y}{8} - y\right|\} = \frac{7y}{8}, \\
G(z, hz, hz) &= \max\{\left|z - \frac{z}{5}\right|, \left|\frac{z}{5} - \frac{z}{5}\right|, \left|\frac{z}{5} - z\right|\} = \frac{4z}{5}.
\end{aligned}
$$

Now

$$
\begin{aligned}
G(fx, gy, hz) &= \max\{\left|\frac{x}{12} - \frac{y}{8}\right|, \left|\frac{y}{8} - \frac{z}{5}\right|, \left|\frac{z}{5} - \frac{x}{12}\right|\} \\
&= \frac{1}{8}\max\{\left|\frac{2x}{3} - y\right|, \left|y - \frac{8z}{5}\right|, \left|\frac{8z}{5} - \frac{2x}{3}\right|\} \\
&\leq \frac{1}{8}[\max\{|x - y|, |y - z|, |z - x|\} + x + y + z] \\
&= \frac{1}{8}\max\{|x - y|, |y - z|, |z - x|\} + \frac{x}{8} + \frac{y}{8} + \frac{z}{8} \\
&= \frac{1}{8}\max\{|x - y|, |y - z|, |z - x|\} + \frac{3}{44}(\frac{11x}{12}) + \frac{1}{7}(\frac{7y}{8}) + \frac{5}{32}(\frac{4z}{5}) \\
&= a\,G(x, y, z) + b\,G(x, fx, fx) + c\,G(y, gy, gy) + d\,G(z, hz, hz).
\end{aligned}
$$

Thus (3) is satisfied for $0 < a + b + c + d = 0.49 < 1$.
For $x, y, z \in [\frac{1}{2}, 1]$,

$$
\begin{aligned}
G(x, y, z) &= \max\{|x - y|, |y - z|, |z - x|\}, \\
G(x, fx, fx) &= \max\{\left|x - \frac{x}{10}\right|, \left|\frac{x}{10} - \frac{x}{10}\right|, \left|\frac{x}{10} - x\right|\} = \frac{9x}{10}, \\
G(y, gy, gy) &= \max\{\left|y - \frac{y}{6}\right|, \left|\frac{y}{6} - \frac{y}{6}\right|, \left|\frac{y}{6} - y\right|\} = \frac{5y}{6}, \\
G(z, hz, hz) &= \max\{\left|z - \frac{z}{3}\right|, \left|\frac{z}{3} - \frac{z}{3}\right|, \left|\frac{z}{3} - z\right|\} = \frac{2z}{3}.
\end{aligned}
$$

Now

$$
\begin{aligned}
G(fx, gy, hz) &= \max\{\left|\frac{x}{10} - \frac{y}{6}\right|, \left|\frac{y}{6} - \frac{z}{3}\right|, \left|\frac{z}{3} - \frac{x}{10}\right|\} \\
&= \frac{1}{6}\max\{\left|\frac{3x}{5} - y\right|, |y - 2z|, \left|2z - \frac{3x}{5}\right|\} \\
&\leq \frac{1}{6}[\max\{|x - y|, |y - z|, |z - x|\} + x + y + z] \\
&= \frac{1}{6}\max\{|x - y|, |y - z|, |z - x|\} + \frac{x}{6} + \frac{y}{6} + \frac{z}{6} \\
&\leq \frac{1}{6}[\max\{|x - y|, |y - z|, |z - x|\} + \frac{5}{27}(\frac{9x}{10}) + \frac{1}{5}(\frac{5y}{6}) + \frac{1}{4}(\frac{2z}{3}) \\
&= a\,G(x, y, z) + b\,G(x, fx, fx) + c\,G(y, gy, gy) + d\,G(z, hz, hz).
\end{aligned}
$$

Thus (3) is satisfied for $0 < a + b + c + d = 0.8 < 1$.
Now for $x \in [0, \frac{1}{2})$, $y, z \in [\frac{1}{2}, 1]$,

$$
\begin{aligned}
G(x, y, z) &= \max\{|x - y|, |y - z|, |z - x|\}, \\
G(x, fx, fx) &= \max\{\left|x - \frac{x}{12}\right|, \left|\frac{x}{12} - \frac{x}{12}\right|, \left|\frac{x}{12} - x\right|\} = \frac{11x}{12}, \\
G(y, gy, gy) &= \max\{\left|y - \frac{y}{6}\right|, \left|\frac{y}{6} - \frac{y}{6}\right|, \left|\frac{y}{6} - y\right|\} = \frac{5y}{6}, \\
G(z, hz, hz) &= \max\{\left|z - \frac{z}{3}\right|, \left|\frac{z}{3} - \frac{z}{3}\right|, \left|\frac{z}{3} - z\right|\} = \frac{2z}{3}.
\end{aligned}
$$

Now

$$
\begin{aligned}
G(fx, gy, hz) &= \max\left\{\left|\frac{x}{12} - \frac{y}{6}\right|, \left|\frac{y}{6} - \frac{z}{3}\right|, \left|\frac{z}{3} - \frac{x}{12}\right|\right\} \\
&= \frac{1}{6}\max\left\{\left|y - \frac{x}{2}\right|, |y - 2z|, \left|2z - \frac{x}{2}\right|\right\} \\
&\leq \frac{1}{6}[\max\{|x - y|, |y - z|, |z - x|\} + x + y + z] \\
&= \frac{1}{6}\max\{|x - y|, |y - z|, |z - x|\} + \frac{x}{6} + \frac{y}{6} + \frac{z}{6} \\
&= \frac{1}{6}[\max\{|x - y|, |y - z|, |z - x|\} + \frac{2}{11}(\frac{11x}{12}) + \frac{1}{5}(\frac{5y}{6}) + \frac{1}{4}(\frac{2z}{3}) \\
&= a\,G(x, y, z) + b\,G(x, fx, fx) + c\,G(y, gy, gy) + d\,G(z, hz, hz).
\end{aligned}
$$

Thus (3) is satisfied for $0 < a + b + c + d = 0.8 < 1$.
Finally for $x, y \in [0, \frac{1}{2})$, $z \in [\frac{1}{2}, 1]$,

$$
\begin{aligned}
G(x, y, z) &= \max\{|x - y|, |y - z|, |z - x|\}, \\
G(x, fx, fx) &= \max\left\{\left|x - \frac{x}{12}\right|, \left|\frac{x}{12} - \frac{x}{12}\right|, \left|\frac{x}{12} - x\right|\right\} = \frac{11x}{12}, \\
G(y, gy, gy) &= \max\left\{\left|y - \frac{y}{8}\right|, \left|\frac{y}{8} - \frac{y}{8}\right|, \left|\frac{y}{8} - y\right|\right\} = \frac{7y}{8}, \\
G(z, hz, hz) &= \max\left\{\left|z - \frac{z}{3}\right|, \left|\frac{z}{3} - \frac{z}{3}\right|, \left|\frac{z}{3} - z\right|\right\} = \frac{2z}{3}.
\end{aligned}
$$

Now

$$
\begin{aligned}
G(fx, gy, hz) &= \max\left\{\left|\frac{x}{12} - \frac{y}{8}\right|, \left|\frac{y}{8} - \frac{z}{3}\right|, \left|\frac{z}{3} - \frac{x}{12}\right|\right\} \\
&= \frac{1}{6}\max\left\{\left|\frac{3y}{4} - \frac{x}{2}\right|, \left|2z - \frac{3y}{4}\right|, \left|2z - \frac{x}{2}\right|\right\} \\
&\leq \frac{1}{6}[\max\{|x - y|, |y - z|, |z - x|\} + x + y + z] \\
&= \frac{1}{6}\max\{|x - y|, |y - z|, |z - x|\} + \frac{x}{6} + \frac{y}{6} + \frac{z}{6} \\
&= \frac{1}{6}[\max\{|x - y|, |y - z|, |z - x|\} + \frac{2}{11}(\frac{11x}{12}) + \frac{4}{21}(\frac{7y}{8}) + \frac{1}{4}(\frac{2z}{3}) \\
&= a\,G(x, y, z) + b\,G(x, fx, fx) + c\,G(y, gy, gy) + d\,G(z, hz, hz).
\end{aligned}
$$

Thus (3) is satisfied for $0 < a + b + c + d = 0.8 < 1$ and so the condition of Theorem 2 is satisfied for all $x, y, z \in X$. 0 is the unique common fixed point of f, g and h. Also any fixed point of f is a fixed point of g and h and conversely.

Example 2.6. Let $X = [0, 1]$ and $G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$ be a G-metric on X. Define $f, g, h : X \to X$ by $f(x) = \frac{x}{8}$, $g(x) = \frac{x}{4}$ and $h(x) = \frac{x}{2}$. Without loss of generality, we assume that $x \leq y \leq z$.

Now

$$
\begin{aligned}
G(x,y,z) &= \max\{|x-y|,|y-z|,|z-x|\} = z - x, \\
G(x,fx,fx) &= \max\{\left|x-\frac{x}{8}\right|,\left|\frac{x}{8}-\frac{x}{8}\right|,\left|\frac{x}{8}-x\right|\} = \frac{7x}{8}, \\
G(y,gy,gy) &= \max\{\left|y-\frac{y}{4}\right|,\left|\frac{y}{4}-\frac{y}{4}\right|,\left|\frac{y}{4}-y\right|\} = \frac{3y}{4}, \\
G(z,hz,hz) &= \max\{\left|z-\frac{z}{2}\right|,\left|\frac{z}{2}-\frac{z}{2}\right|,\left|\frac{z}{2}-z\right|\} = \frac{z}{2}.
\end{aligned}
$$

$$
\begin{aligned}
G(fx,gy,hz) &= \max\{\left|\frac{x}{8}-\frac{y}{4}\right|,\left|\frac{y}{4}-\frac{z}{2}\right|,\left|\frac{z}{2}-\frac{x}{8}\right|\} \\
&= \frac{z}{2}-\frac{x}{8} \\
&= \frac{z}{4}-\frac{x}{4}+\frac{x}{8}+\frac{z}{4} \\
&= \frac{1}{4}(z-x)+\frac{x}{16}+\frac{x}{16}+\frac{z}{4} \\
&= \frac{1}{4}(z-x)+\frac{1}{14}(\frac{7x}{8})+\frac{1}{12}(\frac{3x}{4})+\frac{z}{4} \\
&\leq \frac{1}{4}(z-x)+\frac{1}{14}(\frac{7x}{8})+\frac{1}{12}(\frac{3y}{4})+\frac{1}{2}(\frac{z}{2}) \\
&= a\,G(x,y,z)+b\,G(x,fx,fx)+c\,G(y,gy,gy)+d\,G(z,hz,hz).
\end{aligned}
$$

Therefore (3) is satisfied for all $x,y,z \in X$, where $a+b+c+d = \dfrac{76}{84} < 1$. 0 is the unique common fixed point of f,g and h. Also any fixed point of f is a fixed point of g and h and conversely.

From Theorem 2.4, we deduce the following corollary.

Corollary 2.7. Let f,g and h be self maps on a complete G-metric space X satisfying

$$
\begin{aligned}
G(f^m x, g^m y, h^m z) &\leq\; a\,G(x,y,z)+b\,G(x,f^m x, f^m x) \\
&\quad +c\,G(y,g^m y, g^m y)+d\,G(z,h^m z, h^m z) \tag{4}
\end{aligned}
$$

for all $x,y,z \in X$, where $0 < a+b+c+d < 1$. Then f, g and h have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and h and conversely.

Theorem 2.8. Let f,g and h be self maps on a complete G-metric space X satisfying

$$
G(fx,gy,hz) \leq a[G(y,fx,fx)+G(z,gy,gy)+G(x,hz,hz)] \tag{5}
$$

for all $x,y,z \in X$, where $0 \leq a < \dfrac{1}{2}$. Then f,g and h have a common fixed point in X. Moreover, any fixed point of f is a fixed point of g and h and conversely.

Proof. We will proceed in two steps: first we prove that any fixed point of f is a fixed point of g and h. Assume that p is a fixed point of f and we prove that $p = gp = hp$. If it is not the case, then for $p \neq gp$ and $p \neq hp$, we get

$$
\begin{aligned}
G(p, gp, hp) &= G(fp, gp, hp) \\
&\leq a[G(p, fp, fp) + G(p, gp, gp) + G(p, hp, hp)] \\
&= a[G(p, p, p) + G(p, gp, gp) + G(p, hp, hp)] \\
&= a[G(p, gp, gp) + G(p, hp, hp)] \\
&\leq 2aG(p, gp, hp),
\end{aligned}
$$

that is a contradiction. Analogously, for $p \neq gp$ and $p = hp$ or for $p \neq hp$ and $p = gp$, following similar arguments as those given above, we obtain a contradiction. Hence in all the cases, we conclude that $p = gp = hp$. The same conclusion holds if $p = gp$ or $p = hp$.

Now, we prove that f, g and h have a common fixed point. Suppose x_0 is an arbitrary point in X. Define $\{x_n\}$ by $x_{3n+1} = fx_{3n}$, $x_{3n+2} = gx_{3n+1}$, $x_{3n+3} = hx_{3n+2}$, $n = 0, 1, 2, \ldots$. If $x_n = x_{n+1}$ for some n, with $n = 3m$, then $p = x_{3n}$ is a fixed point of f and, by the first step, p is a common fixed point for f, g and h. The same holds if $n = 3m + 1$ or $n = 3m + 2$. Now, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then, we have

$$
\begin{aligned}
&G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\
&= G(fx_{3n}, gx_{3n+1}, hx_{3n+2}) \\
&\leq a[G(x_{3n+1}, fx_{3n}, fx_{3n}) + G(x_{3n+2}, gx_{3n+1}, gx_{3n+1}) + G(x_{3n}, hx_{3n+2}, hx_{3n+2})] \\
&= a[G(x_{3n+1}, x_{3n+1}, x_{3n+1}) + G(x_{3n+2}, x_{3n+2}, x_{3n+2}) + G(x_{3n}, x_{3n+3}, x_{3n+3})] \\
&\leq a[G(x_{3n}, x_{3n+1}, x_{3n+1}) + G(x_{3n+1}, x_{3n+3}, x_{3n+3})] \\
&\leq a[G(x_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n+1}, x_{3n+2}, x_{3n+3})]
\end{aligned}
$$

implies that

$$
(1 - a)G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq aG(x_{3n}, x_{3n+1}, x_{3n+2}).
$$

Hence,

$$
G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq hG(x_{3n}, x_{3n+1}, x_{3n+2}),
$$

where $h = \dfrac{a}{1 - a}$. Obviously $0 < h < 1$. Similarly it can be shown that

$$
G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \leq hG(x_{3n+1}, x_{3n+2}, x_{3n+3})
$$

and

$$
G(x_{3n+3}, x_{3n+4}, x_{3n+5}) \leq hG(x_{3n+2}, x_{3n+3}, x_{3n+4}).
$$

Therefore, for all n,

$$
\begin{aligned}
G(x_{n+1}, x_{n+2}, x_{n+3}) &\leq hG(x_n, x_{n+1}, x_{n+2}) \\
&\leq \cdots \leq h^{n+1}G(x_0, x_1, x_2).
\end{aligned}
$$

Following similar arguments as those given in Theorem 2.1, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$. Hence $\{x_n\}$ is a G-Cauchy sequence. By G-completeness of X, there exists $u \in X$ such that $\{x_n\}$ converges to u as $n \to \infty$. We claim that $fu = u$. If not, then consider

$$
\begin{aligned}
& G(fu, x_{3n+2}, x_{3n+3}) \\
= \ & G(fu, gx_{3n+1}, hx_{3n+2}) \\
\leq \ & a[G(x_{3n+1}, fu, fu) + G(x_{3n+2}, gx_{3n+1}, gx_{3n+1}) + G(u, hx_{3n+2}, hx_{3n+2})] \\
= \ & a[G(x_{3n+1}, fu, fu) + G(x_{3n+2}, x_{3n+2}, x_{3n+2}) + G(u, x_{3n+3}, x_{3n+3})].
\end{aligned}
$$

On taking limit $n \to \infty$, we obtain that

$$
G(fu, u, u) \leq a\, G(fu, fu, u) \leq 2a\, G(fu, u, u),
$$

a contradiction. Hence $fu = u$. Similarly it can be shown that $gu = u$ and $hu = u$ and so u is a common fixed point for f, g and h. $\qquad\square$

Remark 2.9. Let f, g and h be self maps on a complete G-metric space X satisfying (5). Then f, g and h have a unique common fixed point in X provided that $0 \leq a < \dfrac{1}{3}$.

Proof. Existence of common fixed points of f, g and h follows from Theorem 2.8. To prove the uniqueness, suppose that v is another common fixed point of f, g and h, then

$$
\begin{aligned}
G(u, v, v) \ & = \ G(fu, gv, hv) \\
& \leq \ a[G(v, fu, fu) + G(v, gv, gv) + G(u, hv, hv)] \\
& = \ a[G(v, u, u) + G(v, v, v) + G(u, v, v)] \\
& \leq \ 3aG(u, v, v),
\end{aligned}
$$

which gives that $G(u, v, v) = 0$, and so $u = v$. Hence u is a unique common fixed point of f, g and h. $\qquad\square$

The following Theorem generalizes the Theorem 2.5 in [10].

Corollary 2.10. Let f, g and h be self maps on a complete G-metric space X satisfying

$$
G(f^m x, g^m y, h^m z) \leq a[G(y, f^m x, f^m x) + G(z, g^m y, g^m y) + G(x, h^m z, h^m z)] \quad (6)
$$

for all $x, y, z \in X$, where $0 \leq a < \dfrac{1}{3}$. Then f, g and h have a common fixed point in X. Moreover, any fixed point of f is a fixed point of g and h and conversely.

Proof. It follows from Theorem 2.8 and Remark 2.9, that f^m, g^m and h^m have a unique common fixed point p. Now $f(p) = f(f^m(p)) = f^{m+1}(p) = f^m(f(p))$, $g(p) = g(g^m(p)) = g^{m+1}(p) = g^m(g(p))$ and $h(p) = h(h^m(p)) = h^{m+1}(p) = h^m(h(p))$ implies that fp, gp and hp are also fixed points for f^m, g^m and h^m. Since the common fixed point of f^m, g^m and h^m is unique, we deduce that $p = fp = gp = hp$. It is obvious that every fixed point of f is a fixed point of g and h and conversely. $\quad\square$

Theorem 2.11.   Let f, g and h be self maps on a complete G-metric space X satisfying

$$G(fx, gy, hz) \leq k[G(x, fx, fx) + G(y, gy, gy) + G(z, hz, hz)] \tag{7}$$

for all $x, y, z \in X$, where $0 < k < \dfrac{1}{3}$. Then f, g and h have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and h and conversely.

Proof. We will proceed in two steps: first we prove that any fixed point of f is a fixed point of g and h. Assume that p is a fixed point of f and we prove that $p = gp = hp$. If it is not the case, then for $p \neq gp$ and $p \neq hp$, we obtain

$$
\begin{aligned}
G(p, gp, hp) &= G(fp, gp, hp) \\
&\leq k[G(fp, fp, p) + G(p, gp, gp) + G(p, hp, hp)] \\
&= k[G(p, p, p) + G(p, gp, gp) + G(p, hp, hp)] \\
&= k[G(p, gp, gp) + G(p, hp, hp)] \\
&\leq k[G(p, gp, hp) + G(p, gp, hp)] \\
&= 2kG(p, gp, hp),
\end{aligned}
$$

that is a contradiction. Analogously, for $p \neq gp$ and $p = hp$ or for $p \neq hp$ and $p = gp$, following similar arguments as those given above, we obtain a contradiction. Hence in all the cases, we conclude that $p = gp = hp$. The same conclusion holds if $p = gp$ or $p = hp$.

    Now, we prove that f, g and h have a unique common fixed point. Suppose x_0 is an arbitrary point in X. Define $\{x_n\}$ by $x_{3n+1} = fx_{3n}$, $x_{3n+2} = gx_{3n+1}$, $x_{3n+3} = hx_{3n+2}$, $n = 0, 1, 2, \ldots$. If $x_n = x_{n+1}$ for some n, with $n = 3m$, then $p = x_{3n}$ is a fixed point of f and, by the first step, p is a common fixed point for f, g and h. The same holds if $n = 3m + 1$ or $n = 3m + 2$. Now, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then, we have

$$
\begin{aligned}
&G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\
&= G(fx_{3n}, gx_{3n+1}, hx_{3n+2}) \\
&\leq k[G(fx_{3n}, fx_{3n}, x_{3n}) + G(x_{3n+1}, gx_{3n+1}, gx_{3n+1}) + G(x_{3n+2}, hx_{3n+2}, hx_{3n+2})] \\
&= k[G(x_{3n+1}, x_{3n+1}, x_{3n}) + G(x_{3n+1}, x_{3n+2}, x_{3n+2}) + G(x_{3n+2}, x_{3n+3}, x_{3n+3})] \\
&\leq k[G(x_{3n}, x_{3n+1}, x_{3n+2}) + 2G(x_{3n+1}, x_{3n+2}, x_{3n+3})]
\end{aligned}
$$

implies that

$$(1 - 2k)G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq kG(x_{3n}, x_{3n+1}, x_{3n+2}).$$

Hence,

$$G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq hG(x_{3n}, x_{3n+1}, x_{3n+2}),$$

where $h = \dfrac{k}{1 - 2k}$. Obviously $0 < h < 1$. Similarly it can be shown that

$$G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \leq hG(x_{3n+1}, x_{3n+2}, x_{3n+3})$$

and

$$G(x_{3n+3}, x_{3n+4}, x_{3n+5}) \leq hG(x_{3n+2}, x_{3n+3}, x_{3n+4}).$$

Therefore, for all n,

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+3}) &\leq h\, G(x_n, x_{n+1}, x_{n+2}) \\ &\leq \cdots \leq h^{n+1} G(x_0, x_1, x_2). \end{aligned}$$

Following similar arguments as those given in Theorem 2.1, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$. Hence $\{x_n\}$ is a G-Cauchy sequence. By G-completeness of X, there exists $u \in X$ such that $\{x_n\}$ converges to u as $n \to \infty$. We claim that $fu = u$. If not, then consider

$$\begin{aligned} &G(fu, x_{3n+2}, x_{3n+3}) \\ =\ & G(fu, gx_{3n+1}, hx_{3n+2}) \\ \leq\ & k[G(fu, fu, u) + G(x_{3n+1}, gx_{3n+1}, gx_{3n+1}) + G(x_{3n+2}, hx_{3n+2}, hx_{3n+2})] \\ =\ & k[G(fu, fu, u) + G(x_{3n+1}, x_{3n+2}, x_{3n+2}) + G(x_{3n+2}, x_{3n+3}, x_{3n+3})]. \end{aligned}$$

On taking limit $n \to \infty$, we obtain that

$$G(fu, u, u) \leq 2kG(fu, u, u),$$

a contradiction. Hence $fu = u$. Similarly it can be shown that $gu = u$ and $hu = u$.
The uniqueness is a consequence of condition (5). □

Corollary 2.12. Let f, g and h be self maps on a complete G-metric space X satisfying

$$G(f^m x, g^m y, h^m z) \leq k[G(f^m x, f^m x, x) + G(y, g^m y, g^m y) + G(z, h^m z, h^m z)] \quad (8)$$

for all $x, y, z \in X$, where $0 < k < \dfrac{1}{3}$. Then f, g and h have a unique common fixed point in X. Moreover, any fixed point of f is a fixed point of g and h and conversely.

Proof. It follows from Theorem 2.11, that f^m, g^m and h^m have a unique common fixed point p. Now $f(p) = f(f^m(p)) = f^{m+1}(p) = f^m(f(p))$, $g(p) = g(g^m(p)) = g^{m+1}(p) = g^m(g(p))$ and $h(p) = h(h^m(p)) = h^{m+1}(p) = h^m(h(p))$ implies that fp, $g(p)$ and hp are also fixed points for f^m, g^m and h^m. Since the common fixed point of f^m, g^m and h^m is unique, we deduce that $p = fp = gp = hp$. It is obvious that every fixed point of f is a fixed point of g and h and conversely. □

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Author 1: Mujahid Abbas
Department of Mathematics, Lahore University of Management Sciences, 54792-Lahore, PAKISTAN.

E-mail: `mujahid@lums.edu.pk`

Author 2: Talat Nazir

Department of Mathematics, Lahore University of Management Sciences, 54792-Lahore, PAKISTAN.

E-mail: `talat@lums.edu.pk`

Author 3: Pasquale Vetro

Università degli Studi di Palermo, Dipartimento di Matematica e Informatica, Via Archirafi, 34 - 90123 Palermo (Italy)

E-mail: `vetro@math.unipa.it`