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Common fixed point results for two families of multivalued $A-$dominated contractive mappings on closed ball with applications

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Abstract: The purpose of this paper is to find common fixed point results for two families of multivalued mappings fulfilling generalized rational type A–dominated contractive conditions on a closed ball in complete dislocated b-metric spaces. Some new fixed point results with graphic contractions on a closed ball for two families of multi-graph dominated mappings on dislocated b-metric space have been established. An application to the unique common solution of two families of nonlinear integral equations is presented to show the novelty of our results.

Keywords: fixed point, closed ball, two families of multivalued mapping, dislocated b–metric space, application to the system of integral equations

MSC: 47H10, 54H25

1 Introduction and preliminaries

Fixed point theory plays a fundamental role in functional analysis. Nadler [1], started the investigation of fixed point results for the set-valued functions. Due to its significance, a large number of authors have proved many interesting multiplications of his result (see [2 – 14]).

Nazir et al. [2] showed common fixed point results for the family of generalized multivalued F-contraction mappings in ordered metric spaces. Recently Shoaib et al. [4] discussed some theorems for a family of set-valued functions. Rasham et al. [11] proved multivalued fixed point theorems for new F-contractive functions on dislocated metric spaces.

In this paper, we have obtained fixed point results of two families of multivalued mappings satisfying conditions only on a sequence. We have used a more weaker class of strictly increasing mappings A rather than class of mappings F used in [15 – 22]. An example is given to demonstrate the variety of our results. Moreover, we investigate our results in a more better framework of dislocated b-metric space (see [23]). New results in ordered spaces, partial b-metric space, dislocated metric space, partial metric space, b-metric space and metric space can be obtained as corollaries of our results. We give the following concepts which will be helpful in this paper.

Definition 1.1. [23] Let M be a nonempty set and $d_b : M \times M \to [0, \infty)$ be a function. If, for any $x, y, z \in M$, the following conditions hold:

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(i) $d_b(x,y) \le b[d_b(x,z)+d_b(z,y)]$, (where $b \ge 1$);
(ii) $d_b(x,y)=0$ implies $x=y$;
(iii) $d_b(x,y)=d_b(y,x)$.

Then d_b is called a dislocated b-metric with coefficient b (or simply d_b-metric) and the pair (M,d_b) is called a dislocated b-metric space (or simply DBM space). It should be noted that every dislocated metric is a dislocated b-metric with $b=1$. Also, if $x=y$, then $d_b(x,y)$ may not be 0. For $x \in M$ and $\varepsilon > 0$, $\overline{B(x,\varepsilon)} = \{y \in M : d_b(x,y) \le \varepsilon\}$ is a closed ball in M.

Definition 1.2. [23] Let (M,d_b) be a $D.B.M$ space.

(i) A sequence $\{x_n\}$ in (M,d_b) is called Cauchy sequence if given $\varepsilon > 0$, there corresponds $n_0 \in N$ such that for all $n,m \ge n_0$ we have $d_b(x_m,x_n) < \varepsilon$ or $\lim_{n,m\to\infty} d_b(x_n,x_m)=0$.

(ii) A sequence $\{x_n\}$ dislocated b-converges (for short d_b-converges) to x if $\lim_{n\to\infty} d_b(x_n,x)=0$. In this case x is called a d_b-limit of $\{x_n\}$.

(iii) (M,d_b) is called complete if every Cauchy sequence in M converges to a point $x \in M$ such that $d_b(x,x)=0$.

Definition 1.3. Let K be a nonempty subset of $D.B.M$ space of M and let $x \in M$. An element $y_0 \in K$ is called a best approximation in K if

$$d_b(x,K) = d_b(x,y_0), \text{ where } d_b(x,K) = \inf_{y\in K} d_b(x,y).$$

We denote $P(M)$ be the set of all closed proximinal subsets of M.

Definition 1.4. [12] The function $H_{d_b} : P(M) \times P(M) \to R^+$, defined by

$$H_{d_b}(N,R) = \max\{\sup_{n\in N} d_b(n,R),\ \sup_{r\in R} d_b(N,r)\}$$

is called dislocated Hausdorff b–metric on $P(M)$.

Definition 1.5. Let (M,d_b) be a $D.B.M$ space. Let $S : M \to P(M)$ be multivalued mapping, $\alpha : M\times M \to [0,+\infty)$ and $\alpha_*(i,Si) = \inf\{\alpha(i,l) : l \in Si\}$. Let $H \subseteq M$, then S is said to be α_*-dominated on H, whenever $\alpha_*(i,Si) \ge 1$ for all $i \in H$. If $H = M$, then we say that the S is α_*-dominated. If $S : M \to M$ is a self mapping, then S is α-dominated on H, whenever $\alpha(i,Si) \ge 1$ for all $i \in H$.

Lemma 1.6. [13] Let (M,d_b) be a $D.B.M$ space and $(P(M),H_{d_b})$ be a dislocated Hausdorff b-metric space. For all G,H in $P(M)$ and for any $g \in G$ such that $d_b(g,H) = d_b(g,h_g)$, where $h_g \in H$. Then $H_{d_b}(G,H) \ge d_b(g,h_g)$ holds.

2 Main result

Let (M,d_b) be a $D.B.M$ space, $c_0 \in M$, let $\{S_\sigma : \sigma \in \Omega\}$ and $\{T_\beta : \beta \in \Phi\}$ be two families of multifunctions from M to $P(M)$. Let $c_1 \in S_a c_0$ be an element such that $d_b(c_0,S_a c_0) = d_b(c_0,c_1)$. Let $c_2 \in T_b c_1$ be such that $d_b(c_1,T_b c_1) = d_b(c_1,c_2)$. Let $c_3 \in S_c c_2$ be such that $d_b(c_2,S_c c_2) = d_b(c_2,c_3)$. In this way, we get a sequence $\{T_\beta S_\sigma(c_n)\}$ in M, where $c_{2n+1} \in S_i c_{2n}$, $c_{2n+2} \in T_j c_{2n+1}$, $n \in \mathbb{N}$, $i \in \Omega$ and $j \in \Phi$. Also $d_b(c_{2n},S_i c_{2n}) = d_b(c_{2n},c_{2n+1})$, $d_b(c_{2n+1},T_j c_{2n+1}) = d_b(c_{2n+1},c_{2n+2})$. $\{T_\beta S_\sigma(c_n)\}$ is said to be a sequence in M generated by c_0. If $\{S_\sigma : \sigma \in \Omega\} = \{T_\beta : \beta \in \Phi\}$, then we say $\{S_\sigma(c_n)\}$ instead of $\{T_\beta S_\sigma(c_n)\}$.

Theorem 2.1. Let (M,d_b) be a complete $D.B.M$ space with constant $b \ge 1$. Let $r > 0$, $c_0 \in \overline{B_{d_b}(c_0,r)} \subseteq M$, $\alpha : M\times M \to [0,\infty)$ and $\{S_\sigma : \sigma \in \Omega\}$, $\{T_\beta : \beta \in \Phi\}$ be two families of α_*-dominated multivalued mappings from M to $P(M)$ on $\overline{B_{d_b}(c_0,r)}$. Suppose that the following are satisfied:

(i) There exist $\tau,\mu_1,\mu_2,\mu_3,\mu_4 > 0$ satisfying $b\mu_1 + b\mu_2 + (1+b)b\mu_3 + \mu_4 < 1$ and a strictly increasing mapping A such that

$$\tau + A(H_{d_b}(S_\sigma e, T_\beta y)) \le A\left(\mu_1 d_b(e,y) + \mu_2 d_b(e,S_\sigma e) + \mu_3 d_b(e,T_\beta y) + \mu_4 \frac{d_b(e,S_\sigma e).d_b(y,T_\beta y)}{1+d_b(e,y)}\right), \quad (2.1)$$

whenever $e, y \in \overline{B_{d_b}(c_0, r)} \cap \{T_\beta S_\sigma(c_n)\}$, $\alpha(e, y) \geq 1$, $\sigma \in \Omega$, $\beta \in \Phi$ and $H_{d_b}(S_\sigma e, T_\beta y) > 0$.

(ii) If $\eta = \frac{\mu_1+\mu_2+b\mu_3}{1-b\mu_3-\mu_4}$, then

$$d_b(c_0, S_a c_0) \leq \eta(1 - b\eta)r. \tag{2.2}$$

Then $\{T_\beta S_\sigma(c_n)\}$ is a sequence in $\overline{B_{d_b}(c_0, r)}$, $\alpha(c_n, c_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{T_\beta S_\sigma(c_n)\} \to u \in \overline{B_{d_b}(c_0, r)}$. Also, if u satisfies (2.1) and either $\alpha(c_n, u) \geq 1$ or $\alpha(u, c_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then S_σ and T_β have common fixed point u in $\overline{B_{d_b}(c_0, r)}$ for all $\sigma \in \Omega$ and $\beta \in \Phi$.

Proof. Consider a sequence $\{T_\beta S_\sigma(c_n)\}$. From (2.2), we get

$$d_b(c_0, c_1) = d_b(c_0, S_a c_0) \leq \eta(1 - b\eta)r < r.$$

It follows that,

$$c_1 \in \overline{B_{d_b}(c_0, r)}.$$

Let $c_2, \cdots, c_j \in \overline{B_{d_b}(c_0, r)}$ for some $j \in \mathbb{N}$. If j is odd, then $j = 2\imath + 1$ for some $\imath \in \mathbb{N}$. Since $\{S_\sigma : \sigma \in \Omega\}$ and $\{T_\beta : \beta \in \Phi\}$ are two families of $\alpha_\star$-dominated multivalued mappings on $\overline{B_{d_b}(c_0, r)}$, so $\alpha_\star(c_{2\imath}, S_\sigma c_{2\imath}) \geq 1$ and $\alpha_\star(c_{2\imath+1}, T_\beta c_{2\imath+1}) \geq 1$ for all $\sigma \in \Omega$ and $\beta \in \Phi$. As $\alpha_\star(c_{2\imath}, S_\sigma c_{2\imath}) \geq 1$, this implies $\inf\{\alpha(c_{2\imath}, b) : b \in S_\sigma c_{2\imath}\} \geq 1$. Also $c_{2\imath+1} \in S_f c_{2\imath}$ for some $f \in \Omega$, so $\alpha(c_{2\imath}, c_{2\imath+1}) \geq 1$. Also $c_{2\imath+2} \in T_g c_{2\imath+1}$ for some $g \in \Phi$. Now by using Lemma 1.6, we have

$$\begin{aligned}
\tau + A(d_b(c_{2\imath+1}, c_{2\imath+2})) &\leq \tau + A(H_{d_b}(S_f c_{2\imath}, T_g c_{2\imath+1})) \\
&\leq A\Bigg(\mu_1 d_b(c_{2\imath}, c_{2\imath+1}) + \mu_2 d_b\left(c_{2\imath}, S_f c_{2\imath}\right) + \mu_3 d_b(c_{2\imath}, T_g c_{2\imath+1}) \\
&\quad + \mu_4 \frac{d_b\left(c_{2\imath}, S_f c_{2\imath}\right) . d_b(c_{2\imath+1}, T_g c_{2\imath+1})}{1 + d_b(c_{2\imath}, c_{2\imath+1})}\Bigg) \\
&\leq A\Bigg(\mu_1 d_b(c_{2\imath}, c_{2\imath+1}) + \mu_2 d_b(c_{2\imath}, c_{2\imath+1}) + b\mu_3 d_b(c_{2\imath}, c_{2\imath+1}) \\
&\quad + b\mu_3 d_b(c_{2\imath+1}, c_{2\imath+2}) + \mu_4 \frac{d_b(c_{2\imath}, c_{2\imath+1}) . d_b(c_{2\imath+1}, c_{2\imath+2})}{1 + d_b(c_{2\imath}, c_{2\imath+1})}\Bigg) \\
&\leq A((\mu_1 + \mu_2 + b\mu_3) d_b(c_{2\imath}, c_{2\imath+1}) + (b\mu_3 + \mu_4) d_b(c_{2\imath+1}, c_{2\imath+2})).
\end{aligned}$$

This implies

$$A(d_b(c_{2\imath+1}, c_{2\imath+2})) < A((\mu_1 + \mu_2 + b\mu_3) d_b(c_{2\imath}, c_{2\imath+1}) + (b\mu_3 + \mu_4) d_b(c_{2\imath+1}, c_{2\imath+2})).$$

As A is strictly increasing, we obtain

$$\begin{aligned}
d_b(c_{2\imath+1}, c_{2\imath+2}) &< (\mu_1 + \mu_2 + b\mu_3) d_b(c_{2\imath}, c_{2\imath+1}) \\
&\quad + (b\mu_3 + \mu_4) d_b(c_{2\imath+1}, c_{2\imath+2}).
\end{aligned}$$

Which implies

$$\begin{aligned}
(1 - b\mu_3 - \mu_4) d_b(c_{2\imath+1}, c_{2\imath+2}) &< (\mu_1 + \mu_2 + b\mu_3) d_b(c_{2\imath}, c_{2\imath+1}) \\
d_b(c_{2\imath+1}, c_{2\imath+2}) &< \left(\frac{\mu_1 + \mu_2 + b\mu_3}{1 - b\mu_3 - \mu_4}\right) d_b(c_{2\imath}, c_{2\imath+1}).
\end{aligned}$$

By assumptions $\eta = \frac{\mu_1+\mu_2+b\mu_3}{1-b\mu_3-\mu_4} < 1$. Hence

$$d_b(c_{2\imath+1}, c_{2\imath+2}) < \eta d_b(c_{2\imath}, c_{2\imath+1}) < \eta^2 d_b(c_{2\imath-1}, c_{2\imath}) < \cdots < \eta^{2\imath+1} d_b(c_0, c_1).$$

Similarly, if j is even, we have

$$d_b(c_{2\imath+2}, c_{2\imath+3}) < \eta^{2\imath+2} d_b(c_0, c_1).$$

Summing up, we have

$$d_b(c_j, c_{j+1}) < \eta^j d_b(c_0, c_1) \text{ for some } j \in \mathbb{N}. \tag{2.3}$$

It follows,

$$d_b(c_0, c_{j+1}) \le bd_b(c_0, c_1) + b^2 d_b(c_1, c_2) + \cdots + b^{j+1} d_b(c_j, c_{j+1})$$
$$\le bd_b(c_0, c_1) + b^2 \eta(d_b(c_0, c_1)) + \cdots + b^{j+1}\eta^j(d_b(c_0, c_1)), \qquad \text{(by (2.3))}$$
$$d_b(c_0, c_{j+1}) \le \left(\frac{b(1-(b\eta)^{j+1})}{1-b\eta}\right)\eta(1-b\eta)r < r.$$

As $\mu_1, \mu_2, \mu_3, \mu_4 > 0$, $b \ge 1$ and $b\mu_1 + b\mu_2 + (1+b)b\mu_3 + \mu_4 < 1$, so $|b\eta| < 1$. Then, we have

$$d_b(c_0, c_{j+1}) \le \left(\frac{b(1-(b\eta)^{j+1})}{1-b\eta}\right)\eta(1-b\eta)r < r,$$

the last inequality following by $b\eta < 1$, that is the assumption (i). So $c_{j+1} \in \overline{B_{d_b}(c_0, r)}$. Hence, by induction $c_n \in \overline{B_{d_b}(c_0, r)}$ for all $n \in N$. Also $\alpha(c_n, c_{n+1}) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$. Now,

$$d_b(c_n, c_{n+1}) < \eta^n d_b(c_0, c_1) \text{ for all } n \in \mathbb{N}. \tag{2.4}$$

Hence, for any positive integers m, n $(n > m)$, we have

$$d_b(c_m, c_n) \le b(d_b(c_m, c_{m+1})) + b^2(d_b(c_{m+1}, c_{m+2})) + \cdots + b^{n-m}(d_b(c_{n-1}, c_n)),$$
$$< b\eta^m d_b(c_0, c_1) + b^2\eta^{m+1} d_b(c_0, c_1) + \cdots + b^{n-m}\eta^{n-1} d_b(c_0, c_1), \qquad \text{(by (2.4))}$$
$$< b\eta^m(1 + b\eta + \cdots)d_b(c_0, c_1)$$
$$< \left(\frac{b\eta^m}{1-b\eta}\right) d_b(c_0, c_1) \to 0 \text{ as } m \to \infty.$$

Hence $\{T_\beta S_\sigma(c_n)\}$ is a Cauchy sequence in $\overline{B_{d_b}(c_0, r)}$. Since $(\overline{B_{d_b}(c_0, r)}, d_b)$ is a complete metric space, so there exists $u \in \overline{B_{d_b}(c_0, r)}$ such that $\{T_\beta S_\sigma(c_n)\} \to u$ as $n \to \infty$, then

$$\lim_{n\to\infty} d_b(c_n, u) = 0, \tag{2.5}$$

by assumption, $\alpha(c_n, u) \ge 1$. Suppose that $d_b(u, T_\beta u) > 0$, then there exists a positive integer k such that $d_b(c_n, T_\beta u) > 0$ for all $n \ge k$. For $n \ge k$, we have

$$d_b(u, T_\beta u) \le bd_b(u, c_{2n+1}) + bd_b(c_{2n+1}, T_\beta u).$$

Now, there exists some $e \in \Omega$ such that $c_{2n+1} \in S_e c_{2n}$ and $d_b(c_{2n}, S_e c_{2n}) = d_b(c_{2n}, c_{2n+1})$. By using Lemma 1.6 and inequality (2.1), we have

$$d_b(u, T_\beta u) \le bd_b(u, c_{2n+1}) + bH_{d_b}(S_e c_{2n}, T_\beta u), \text{ for some } \beta \in \Phi$$
$$< bd_b(u, c_{2n+1}) + b\mu_1 d_b(c_{2n}, u) + b\mu_2 d_b(c_{2n}, S_e c_{2n})$$
$$+ b\mu_3 d_b(c_{2n}, T_\beta u) + b\mu_4 \frac{d_b(c_{2n}, S_e c_{2n}).d_b(u, T_\beta u)}{1 + d_b(c_{2n}, u)}.$$

Letting $n \to \infty$, and by using (2.5) we get

$$d_b(u, T_\beta u) < b\mu_3 d_b(u, T_\beta u) < d_b(u, T_\beta u),$$

which is a contradiction. So our supposition is wrong. Hence $d_b(u, T_\beta u) = 0$ or $u \in T_\beta u$ for all $\beta \in \Phi$. Similarly, by using Lemma 1.6 and inequality (2.1), we can show that $d_b(u, S_\sigma u) = 0$ or $u \in S_\sigma u$ for all $\sigma \in \Omega$. Hence the S_σ and T_β have a common fixed point u in $\overline{B_{d_b}(c_0, r)}$ for all $\sigma \in \Omega$ and $\beta \in \Phi$. Now,

$$d_b(u, u) \le bd_b(u, T_\beta u) + bd_b(T_\beta u, u) \le 0.$$

This implies that $d_b(u, u) = 0$.

Example 2.2. Let $M = Q^+ \cup \{0\}$ and let $d_b : M \times M \to M$ be the complete $D.B.M$ space defined by

$$d_b(i, j) = (i + j)^2 \text{ for all } i, j \in M,$$

with $b = 2$. Define, two families of multivalued mappings $S_\sigma, T_\beta : M \times M \to P(M)$ by

$$S_m x = \begin{cases} \left[\frac{x}{3m}, \frac{2x}{3m}\right] \text{ if } x \in [0,14] \cap M \\ [xm, 2mx] \text{ if } x \in (14, \infty) \cap M \end{cases} \quad \text{where } m = 1, 2, 3, \cdots$$

and

$$T_n x = \begin{cases} \left[\frac{x}{4n}, \frac{3x}{4n}\right] \text{ if } x \in [0,14] \cap M \\ [2nx, 3nx] \text{ if } x \in (14, \infty) \cap M. \end{cases} \quad \text{where } n = 1, 2, 3, \cdots$$

Suppose that, $x_0 = 1$, $r = 225$, then $\overline{B_{d_b}(x_0, r)} = [0, 14] \cap M$. Now, $d_b(x_0, S_1 x_0) = d_b(1, S_1 1) = d_b(1, \frac{1}{3})$. So $x_1 = \frac{1}{3}$. Now, $d_b(x_1, T_1 x_1) = d_b(\frac{1}{3}, T_1 \frac{1}{3}) = d_b(\frac{1}{3}, \frac{1}{12})$. So $x_2 = \frac{1}{12}$. Now, $d_b(x_2, S_2 x_2) = d_b(\frac{1}{12}, S_2 \frac{1}{12}) = d_b(\frac{1}{12}, \frac{1}{72})$. So $x_3 = \frac{1}{72}$. Continuing in this way, we have $\{T_n S_m(x_n)\} = \{1, \frac{1}{3}, \frac{1}{12}, \frac{1}{72} \ldots .\}$. Take $\mu_1 = \frac{1}{10}$, $\mu_2 = \frac{1}{20}$, $\mu_3 = \frac{1}{60}$, $\mu_4 = \frac{1}{30}$, then $b\mu_1 + b\mu_2 + (1 + b)b\mu_3 + \mu_4 < 1$ and $\eta = \frac{11}{56}$. Now

$$d_b(x_0, S_1 x_0) = \frac{16}{9} < \frac{11}{56}\left(1 - \frac{22}{56}\right) 225 = \eta(1 - b\eta)r.$$

Consider the mapping $\alpha : M \times M \to [0, \infty)$ by

$$\alpha(j,k) = \begin{Bmatrix} 1 & \text{if } j > k \\ \frac{1}{2} & \text{otherwise} \end{Bmatrix}.$$

Now, if $x, y \in \overline{B_{d_b}(x_0, r)} \cap \{T_\beta S_\sigma(x_n)\}$ with $\alpha(x, y) \geq 1$, we have

$$\begin{aligned} H_{d_b}(S_m x, T_n y) &= \max\{\sup_{a \in S_m x} d_b(a, T_n y), \sup_{b \in T_n y} d_b(S_m x, b)\} \\ &= \max\left\{\sup_{a \in S_m x} d_b\left(a, \left[\frac{y}{4n}, \frac{3y}{4n}\right]\right), \sup_{b \in T_n y} d_b\left(\left[\frac{x}{3m}, \frac{2x}{3m}\right], b\right)\right\} \\ &= \max\left\{d_b\left(\frac{2x}{3m}, \left[\frac{y}{4n}, \frac{3y}{4n}\right]\right), d_b\left(\left[\frac{x}{3m}, \frac{2x}{3m}\right], \frac{3y}{4n}\right)\right\} \\ &= \max\left\{d_b\left(\frac{2x}{3m}, \frac{y}{4n}\right), d_b\left(\frac{x}{3m}, \frac{3y}{4n}\right)\right\} \\ &= \max\left\{\left(\frac{2x}{3m} + \frac{y}{4n}\right)^2, \left(\frac{x}{3m} + \frac{3y}{4n}\right)^2\right\} \\ &< \frac{1}{10}(x+y)^2 + \frac{1}{20}\left(x + \frac{x}{3m}\right)^2 + \frac{1}{60}\left(x + \frac{y}{4n}\right)^2 + \frac{1}{30}\frac{\left(x + \frac{x}{3m}\right)^4 . \left(y + \frac{y}{4n}\right)^2}{\{1 + (x+y)^4\}} \\ &= \frac{1}{10} d_b(x,y) + \frac{1}{20} d_b\left(x, \left[\frac{x}{3m}, \frac{2}{3m}x\right]\right) + \frac{1}{60} d_b\left(x, \left[\frac{y}{4n}, \frac{3}{4n}y\right]\right) \\ &\quad + \frac{1}{30}\frac{d_b\left(x, \left[\frac{x}{3m}, \frac{2}{3m}x\right]\right) . d_b\left(y, \left[\frac{y}{4n}, \frac{3}{4n}y\right]\right)}{1 + d_b(x,y)}. \end{aligned}$$

Thus,

$$H_{d_b}(S_m x, T_n y) < \mu_1 d_b(x,y) + \mu_2 d_b(x, S_m x) + \mu_3 d_b(x, T_n y) + \mu_4 \frac{d_b(x, S_m x).d_b(y, T_n y)}{1 + d_b(x,y)},$$

which implies that, for any $\tau \in (0, \frac{12}{95}]$ and for a strictly increasing mapping $A(s) = \ln s$, we have

$$\tau + A(H_{d_b}(S_m x, T_n y)) \leq A\left(\mu_1 d_b(x,y) + \mu_2 d_b(x, S_m x) + \mu_3 d_b(x, T_n y) + \mu_4 \frac{d_b(x, S_m x).d_b(y, T_n y)}{1 + d_b(x,y)}\right).$$

Note that, for $16, 15 \in M$, then $\alpha(16, 15) \geq 1$. But, we have

$$\tau + A(H_{d_b}(S_2 16, T_1 15)) > A\Big(\mu_1 d_b(16, 15) + \mu_2 d_b(16, S_2 16) + \mu_3 d_b(16, T_1 15)$$

$$+\mu_4 \frac{d_b(16, S_2 16).(15, T_1 15)}{1 + d_b(16, 15)}\Bigg).$$

So condition (2.1) does not holds on all M but holds only on $\overline{B_{d_b}(1, 225)}$. Thus all the conditions of Theorem 2.1 are satisfied. Hence S_σ and T_β have a common fixed point for all $\sigma \in \Omega$ and $\beta \in \Phi$.

If, we take $\{S_\sigma : \sigma \in \Omega\} = \{T_\beta : \beta \in \Phi\}$ in Theorem 2.1, then we have the following result.

Corollary 2.3. Let (M, d_b) be a complete $D.B.M$ space with constant $b \geq 1$. Let $r > 0$, $c_0 \in \overline{B_{d_b}(c_0, r)} \subseteq M$, $\alpha : M \times M \to [0, \infty)$ and $\{S_\sigma : \sigma \in \Omega\}$ be a family of $\alpha_\star$-dominated multivalued mappings from M to $P(M)$ on $\overline{B_{d_b}(c_0, r)}$. Suppose that the following satisfy:

(i) There exist $\tau, \mu_1, \mu_2, \mu_3, \mu_4 > 0$ satisfying $b\mu_1 + b\mu_2 + (1 + b)b\mu_3 + \mu_4 < 1$ and a strictly increasing mapping A such that

$$\tau + A(H_{d_b}(S_\sigma e, S_\beta y)) \leq A\left(\mu_1 d_b(e, y) + \mu_2 d_b(e, S_\sigma e) + \mu_3 d_b(e, S_\beta y) + \mu_4 \frac{d_b(e, S_\sigma e).d_b(y, S_\beta y)}{1 + d_b(x, y)}\right), \quad (2.6)$$

whenever $e, y \in \overline{B_{d_b}(c_0, r)} \cap \{S_\sigma(c_n)\}$, $\alpha(e, y) \geq 1$, $\sigma, \beta \in \Omega$ and $H_{d_b}(S_\sigma e, S_\beta y) > 0$.

(ii) If $\eta = \frac{\mu_1 + \mu_2 + b\mu_3}{1 - b\mu_3 - \mu_4}$, then

$$d_b(c_0, S_\sigma c_0) \leq \eta(1 - b\eta)r.$$

Then $\{MS_\sigma(c_n)\}$ is a sequence in $\overline{B_{d_b}(c_0, r)}$, $\alpha(c_n, c_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{S_\sigma(c_n)\} \to u \in \overline{B_{d_b}(c_0, r)}$. Also, if u satisfies (2.6) and either $\alpha(c_n, u) \geq 1$ or $\alpha(u, c_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then $\{S_\sigma : \sigma \in \Omega\}$ have common fixed point u in $\overline{B_{d_b}(c_0, r)}$.

3 Results for families of multi-graph dominated mappings

In this section we present an application of Theorem 2.1 in graph theory. Jachymski, [24], proved the result concerning contraction mappings on metric space with a graph. Hussain et al. [25], introduced the fixed points theorem for graphic contraction and gave an application.

Definition 3.1. Let X be a nonempty set and $G = (V(G), E(G))$ be a graph such that $V(G) = X$, $A \subseteq X$. A mapping $F : X \to P(X)$ is said to be multi graph dominated on A if $(x, y) \in E(G)$, for all $y \in Fx$ and $x \in A$.

Theorem 3.2. Let (M, d_b) be a complete $D.B.M$ space endowed with a graph G with constant $b \geq 1$. Let $r > 0$, $c_0 \in \overline{B_{d_b}(c_0, r)}$ and $\{S_\sigma : \sigma \in \Omega\}$, $\{T_\beta : \beta \in \Phi\}$ be two families of multivalued mappings from M to $P(M)$. Suppose that the following are satisfied:

(i) $\{S_\sigma : \sigma \in \Omega\}$, $\{T_\beta : \beta \in \Phi\}$ are two families of multi graph dominated on $\overline{B_{d_b}(c_0, r)} \cap \{T_\beta S_\sigma(c_n)\}$.

(ii) There exist $\tau, \mu_1, \mu_2, \mu_3, \mu_4 > 0$ satisfying $b\mu_1 + b\mu_2 + (1 + b)b\mu_3 + \mu_4 < 1$ and a strictly increasing mapping A such that

$$\tau + A(H_{d_b}(S_\sigma e, T_\beta y)) \leq A\left(\mu_1 d_b(e, y) + \mu_2 d_b(e, S_\sigma e) + \mu_3 d_b(e, T_\beta y) + \mu_4 \frac{d_b(e, S_\sigma e).d_b(y, T_\beta y)}{1 + d_b(e, y)}\right), \quad (3.1)$$

whenever $e, y \in \overline{B_{d_b}(c_0, r)} \cap \{T_\beta S_\sigma(c_n)\}$, $(e,y) \in E(G)$, $\sigma \in \Omega$, $\beta \in \Phi$ and $H_{d_b}(S_\sigma e, T_\beta y) > 0$.

(iii) $d_b(c_0, S_\sigma c_0) \leq \eta(1 - b\eta)r$, where $\eta = \frac{\mu_1 + \mu_2 + b\mu_3}{1 - b\mu_3 - \mu_4}$.

Then, $\{T_\beta S_\sigma(c_n)\}$ is a sequence in $\overline{B_{d_b}(c_0, r)}$, $(c_n, c_{n+1}) \in E(G)$ and $\{T_\beta S_\sigma(c_n)\} \to m^\star$. Also, if $m^\star$ satisfies (3.1) and $(c_n, m^\star) \in E(G)$ or $(m^\star, c_n) \in E(G)$ for all $n \in N \cup \{0\}$, then S_σ and T_β have common fixed point $m^\star$ in $\overline{B_{d_b}(c_0, r)}$ for all $\sigma \in \Omega$ and $\beta \in \Phi$.

Proof. Define $\alpha : M \times M \to [0, \infty)$ by

$$\alpha(e, y) = \begin{cases} 1, & \text{if } e \in \overline{B_{d_b}(c_0, r)},\ (e, y) \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

As S_σ and T_β are two families of graph dominated on $\overline{B_{d_b}(c_0, r)}$, then for $e \in\leq$, $(e, y) \in E(G)$ for all $y \in S_\sigma e$ and $(e, y) \in E(G)$ for all $y \in T_\beta e$. So, $\alpha(e, y) = 1$ for all $y \in S_\sigma e$ and $\alpha(e, y) = 1$ for all $y \in T_\beta e$. This implies that $\inf\{\alpha(e,y) : y \in S_\sigma e\} = 1$ and $\inf\{\alpha(e,y) : y \in T_\beta e\} = 1$. Hence $\alpha_\star(e, S_\sigma e) = 1$, $\alpha_\star(e, T_\beta e) = 1$ for all $e \in \overline{B_{d_b}(c_0, r)}$. So, $S_\sigma, T_\beta : M \to P(M)$ are two families of $\alpha_\star$-dominated mappings on $\overline{B_{d_b}(c_0, r)}$. Moreover, inequality (3.1) can be written as

$$\tau + A(H_{d_b}(S_\sigma e, T_\beta y)) \le A\left(\mu_1 d_b(e,y) + \mu_2 d_b(e, S_\sigma e) + \mu_3 d_b(e, T_\beta y) + \mu_4 \frac{d_b(e, S_\sigma e).d_b(y, T_\beta y)}{1 + d_b(e,y)}\right),$$

whenever $e, y \in \overline{B_{d_b}(c_0, r)} \cap \{T_\beta S_\sigma(c_n)\}$, $\alpha(e,y) \ge 1$ and $H_{d_b}(S_\sigma e, T_\beta y) > 0$. Also, (iii) holds. Then, by Theorem 2.1, we have $\{T_\beta S_\sigma(c_n)\}$ is a sequence in $\overline{B_{d_b}(c_0, r)}$ and $\{T_\beta S_\sigma(c_n)\} \to m^* \in \overline{B_{d_b}(c_0, r)}$. Now, $c_n, m^* \in \overline{B_{d_b}(c_0, r)}$ and either $(c_n, m^*) \in E(G)$ or $(m^*, c_n) \in E(G)$ implies that either $\alpha(c_n, m^*) \ge 1$ or $\alpha(m^*, c_n) \ge 1$. So, all the conditions of Theorem 2.1 are satisfied. Hence, by Theorem 2.1, S_σ and T_β have a common fixed point m^* in $\overline{B_{d_b}(c_0, r)}$ and $d_b(m^*, m^*) = 0$.

4 Fixed point results for single valued mapping

In this section, we discussed some new fixed point results for single valued mapping in complete $D.B.M$ space. Let (M, d_b) be a $D.B.M$ space, $c_0 \in M$ and $S_\sigma, T_\beta : M \to M$ be two families of mappings. Let $c_1 = S_\sigma c_0$, $c_2 = T_\beta c_1$, $c_3 = S_\sigma c_2$. Continuing in this way, we get a sequence c_n of points in M such that $c_{2n+1} = S_\sigma c_{2n}$ and $c_{2n+2} = T_\beta c_{2n+1}$, where $n = 0, 1, 2, \ldots$. We denote this iterative sequence by $\{T_\beta S_\sigma(c_n)\}$. We say that $\{T_\beta S_\sigma(c_n)\}$ is a sequence in M generated by c_0. If $\{S_\sigma : \sigma \in \Omega\} = \{T_\beta : \beta \in \Phi\}$, then we say $\{MS_\sigma(c_n)\}$ instead of $\{T_\beta S_\sigma(c_n)\}$.

Theorem 4.1. Let (M, d_b) be a complete $D.B.M$ space with constant $b \ge 1$. Let $r > 0$, $c_0 \in \overline{B_{d_b}(c_0, r)} \subseteq M$, $\alpha : M \times M \to [0, \infty)$ and $\{S_\sigma : \sigma \in \Omega\}$, $\{T_\beta : \beta \in \Phi\}$ be two families of α-dominated mappings from M to M on $\overline{B_{d_b}(c_0, r)}$. Suppose that the following are satisfied:

(i) There exist $\tau, \mu_1, \mu_2, \mu_3, \mu_4 > 0$ satisfying $b\mu_1 + b\mu_2 + (1+b)b\mu_3 + \mu_4 < 1$ and a strictly increasing mapping A such that

$$\tau + A(H_{d_b}(S_\sigma e, T_\beta y)) \le A\left(\mu_1 d_b(e,y) + \mu_2 d_b(e, S_\sigma e) + \mu_3 d_b(e, T_\beta y) + \mu_4 \frac{d_b(e, S_\sigma e).d_b(y, T_\beta y)}{1 + d_b(e,y)}\right), \quad (4.1)$$

whenever $e, y \in \overline{B_{d_b}(c_0, r)} \cap \{T_\beta S_\sigma(c_n)\}$, $\alpha(e,y) \ge 1$, $\sigma \in \Omega$, $\beta \in \Phi$ and $d_b(S_\sigma e, T_\beta y) > 0$.

(ii) If $\eta = \frac{\mu_1 + \mu_2 + b\mu_3}{1 - b\mu_3 - \mu_4}$, then

$$d_b(c_0, S_\sigma c_0) \le \eta(1 - b\eta)r.$$

Then $\{T_\beta S_\sigma(c_n)\}$ is a sequence in $\overline{B_{d_b}(c_0, r)}$, $\alpha(c_n, c_{n+1}) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{T_\beta S_\sigma(c_n)\} \to u \in \overline{B_{d_b}(c_0, r)}$. Also, if u satisfies (4.1) and either $\alpha(c_n, u) \ge 1$ or $\alpha(u, c_n) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$, then S_σ and T_β have common fixed point u in $\overline{B_{d_b}(c_0, r)}$ for all $\sigma \in \Omega$ and $\beta \in \Phi$.

Proof. The proof of the above Theorem is similar to Theorem 2.1.

If, we take $\{S_\sigma : \sigma \in \Omega\} = \{T_\beta : \beta \in \Phi\}$ in Theorem 4.1, then we have the following result.

Corollary 4.2. Let (M, d_b) be a complete $D.B.M$ space with constant $b \ge 1$. Let $r > 0$, $c_0 \in \overline{B_{d_b}(c_0, r)} \subseteq M$, $\alpha : M \times M \to [0, \infty)$ and $\{S_\sigma : \sigma \in \Omega\}$ be a family of α-dominated mappings from M to M on $\overline{B_{d_b}(c_0, r)}$. Suppose that the following satisfy:

(i) There exist $\tau, \mu_1, \mu_2, \mu_3, \mu_4 > 0$ satisfying $b\mu_1 + b\mu_2 + (1+b)b\mu_3 + \mu_4 < 1$ and a strictly increasing mapping A such that

$$\tau + A(H_{d_b}(S_\sigma e, S_\beta y)) \le A\left(\mu_1 d_b(e,y) + \mu_2 d_b(e, S_\sigma e) + \mu_3 d_b(e, S_\beta y) + \mu_4 \frac{d_b(e, S_\sigma e).d_b(y, S_\beta y)}{1 + d_b(x,y)}\right), \quad (4.2)$$

whenever $e, y \in \overline{B_{d_b}(c_0, r)} \cap \{MS_\sigma(c_n)\}$, $\alpha(e,y) \ge 1$, $\sigma, \beta \in \Omega$, and $d_b(S_\sigma e, S_\sigma y) > 0$.

(ii) If $\eta = \frac{\mu_1 + \mu_2 + b\mu_3}{1 - b\mu_3 - \mu_4}$, then

$$d_b(c_0, S_\sigma c_0) \le \eta(1 - b\eta)r.$$

Then $\{MS_\sigma(c_n)\}$ is a sequence in $\overline{B_{d_b}(c_0, r)}$, $\alpha(c_n, c_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{MS_\sigma(c_n)\} \to u \in \overline{B_{d_b}(c_0, r)}$. Also, if u satisfies (4.2) and either $\alpha(c_n, u) \geq 1$ or $\alpha(u, c_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then S_σ has a fixed point u in $\overline{B_{d_b}(c_0, r)}$ for all $\sigma \in \Omega$.

5 Application to the systems of integral equations

Theorem 5.1. Let (M, d_b) be a complete $D.B.M$ space with coefficient $b \geq 1$. Let $c_0 \in M$ and $\{S_\sigma : \sigma \in \Omega\}$, $\{T_\beta : \beta \in \Phi\}$ be two families of mappings from M to M. Assume that there exist $\tau, \mu_1, \mu_2, \mu_3, \mu_4 > 0$ satisfying $b\mu_1 + b\mu_2 + (1+b)b\mu_3 + \mu_4 < 1$ and $A : \mathbb{R}_+ \to \mathbb{R}$ is a strictly increasing mapping such that the following holds:

$$\tau + A(H_{d_b}(S_\sigma e, T_\beta y)) \leq A\left(\mu_1 d_b(e, y) + \mu_2 d_b(e, S_\sigma e) + \mu_3 d_b(e, T_\beta y) + \mu_4 \frac{d_b(e, S_\sigma e).d_b(y, T_\beta y)}{1 + d_b(e, y)}\right), \quad (5.1)$$

whenever $e, y \in \{T_\beta S_\sigma(c_n)\}$, $\sigma \in \Omega$, $\beta \in \Phi$ and $d_b(S_\sigma e, T_\beta y) > 0$. Then $\{T_\beta S_\sigma(c_n)\} \to u \in M$. Also, if inequality (5.1) holds for $e, y \in \{u\}$, then S_σ and T_β have unique common fixed point u in M for all $\sigma \in \Omega$ and $\beta \in \Phi$.

Proof. The proof of this theorem is similar to Theorem 2.1. We have to prove the uniqueness only. Let v be another common fixed point of S_σ and T_β. Suppose $d_b(S_\sigma u, T_\beta v) > 0$. Then, we have

$$\tau + A(d_b(S_\sigma u, T_\beta v)) \leq A\left(\mu_1 d_b(u, v) + \mu_2 d_b(u, S_\sigma u) + \mu_3 d_b(u, T_\beta v) + \mu_4 \frac{d_b(u, S_\sigma u).d_b(v, T_\beta v)}{1 + d_b(u, v)}\right).$$

This implies that

$$d_b(u, v) < \mu_1 d_b(u, v) + \mu_3 d_b(u, v) < d_b(u, v),$$

which is a contradiction. So $d_b(S_\sigma u, T_\beta v) = 0$. Hence $u = v$.

In this section, we discuss the application of fixed point Theorem 5.1 in the form of a unique solution of two families Volterra type integral equations given below:

$$u(k) = \int_0^k H_\sigma(k, h, u(h))dh, \quad (5.2)$$

$$c(k) = \int_0^k G_\beta(k, h, c(h))dh \quad (5.3)$$

for all $k \in [0, 1]$, $\sigma \in \Omega$, $\beta \in \Phi$ and H_σ, G_β be the mappings from $[0, 1] \times [0, 1] \times C([0, 1], \mathbb{R}_+)$ to $\mathbb{R}$. We find the solution of (5.2) and (5.3). Let $M = C([0, 1], \mathbb{R}_+)$ be the set of all continuous functions on $[0, 1]$, endowed with the complete dislocated b–metric. For $u \in C([0, 1], \mathbb{R}_+)$, define supremum norm as: $\|u\|_\tau = \sup_{k\in[0,1]} \{|u(k)|\, e^{-\tau k}\}$, where $\tau > 0$ is taken arbitrarily. Then define

$$d_\tau(u, c) = \left[\sup_{k\in[0,1]} \{|u(k) + c(k)|\, e^{-\tau k}\}\right]^2 = \|u + c\|_\tau^2$$

for all $u, c \in C([0, 1], \mathbb{R}_+)$, with these settings, $(C([0, 1], \mathbb{R}_+), d_\tau)$ becomes a complete $D.B.M$ space.

Now we prove the following theorem to ensure the existence of solution of integral equations.

Theorem 5.2. Assume the following conditions are satisfied:

(i) $\{H_\sigma, \sigma \in \Omega\}$, $\{G_\beta, \beta \in \Phi\}$ be two families of mappings from $[0, 1] \times [0, 1] \times C([0, 1], \mathbb{R}_+)$ to $\mathbb{R}$;

(ii) Define

$$(S_\sigma u)(k) = \int_0^k H_\sigma(k, h, u(h))dh,$$

$$(T_\beta c)(k) = \int_0^k G_\beta(k, h, c(h))dh.$$

Suppose there exists $\tau > 0$, such that

$$\left|H_\sigma(k, h, u) + G_\beta(k, h, c)\right| \leq \frac{\tau N_{(\sigma,\beta)}(u, c)}{\tau N_{(\sigma,\beta)}(u, c) + 1}$$

for all $k, h \in [0, 1]$ and $u, c \in C([0, 1], \mathbb{R})$, where

$$N_{(\sigma,\beta)}(u, c) = \mu_1 \|u + c\|_\tau^2 + \mu_2 \|u + S_\sigma u\|_\tau^2 + \mu_3 \left\|u + T_\beta c\right\|_\tau^2 + \mu_4 \frac{\|u + S_\sigma u\|_\tau^2 . \left\|u + T_\beta c\right\|_\tau^2}{1 + \|u + c\|_\tau^2},$$

where $\mu_1, \mu_2, \mu_3\ \mu_4 \geq 0$, and $\mu_1 + \mu_2 + 2b\mu_3 + \mu_4 < 1$. Then integral equations (5.2) and (5.3) have a unique solution.

Proof: By assumption (ii)

$$\begin{aligned}
\left|S_\sigma u + T_\beta c\right| &= \int_0^k \left|H_\sigma(k, h, u) + G_\beta(k, h, c)\right| dh, \\
&\leq \int_0^k \frac{\tau N_{(\sigma,\beta)}(u, c)}{\tau N_{(\sigma,\beta)}(u, c) + 1} e^{\tau h} dh \\
&\leq \frac{\tau N_{(\sigma,\beta)}(u, c)}{\tau N_{(\sigma,\beta)}(u, c) + 1} \int_0^k e^{\tau h} dh \\
&\leq \frac{N_{(\sigma,\beta)}(u, c)}{\tau N_{(\sigma,\beta)}(u, c) + 1} e^{\tau k}.
\end{aligned}$$

This implies

$$\begin{aligned}
\left|S_\sigma u + T_\beta c\right| e^{-\tau k} &\leq \frac{N_{(\sigma,\beta)}(u, c)}{\tau N_{(\sigma,\beta)}(u, c) + 1}, \\
\|S_\sigma u + T_\beta c\|_\tau &\leq \frac{N_{(\sigma,\beta)}(u, c)}{\tau N_{(\sigma,\beta)}(u, c) + 1}, \\
\frac{\tau N_{(\sigma,\beta)}(u, c) + 1}{N_{(\sigma,\beta)}(u, c)} &\leq \frac{1}{\|S_\sigma u + T_\beta c\|_\tau}, \\
\tau + \frac{1}{N_{(\sigma,\beta)}(u, c)} &\leq \frac{1}{\|S_\sigma u + T_\beta c\|_\tau},
\end{aligned}$$

which further implies

$$\tau - \frac{1}{\|S_\sigma u(k) + T_\beta c(k)\|_\tau} \leq \frac{-1}{N_{(\sigma,\beta)}(u, c)}.$$

So all the conditions of Theorem 5.1 are satisfied for $A(c) = \frac{-1}{\sqrt{c}}$; $c > 0$ and $d_\tau(u, c) = \|u + c\|_\tau^2$. Hence two families of integral equations given in (5.2) and (5.3) have a unique common solution.

6 Conclusion

In the present paper, we have achieved fixed point results for a pair of families of multivalued generalized $A-$ dominated contractive mappings on an intersection of a closed ball and a sequence for a more general class of $\alpha_\star$-dominated mappings rather than $\alpha_\star$-admissible mappings and for a more weaker class of strictly increasing mappings A rather than class of mappings F used by Wardowski [17]. The notion of multi graph dominated mapping is introduced. Fixed point results with graphic contractions on a closed ball for such mappings are established. Examples are given to demonstrate the variety of our results. An application is given to approximate the unique common solution of two families of nonlinear integral equations. Moreover, we investigate our results in a new, better framework. New results in ordered spaces, partial b-metric space, dislocated metric space, partial metric space, b-metric space and metric space can be obtained as corollaries of our results.

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