

## COMMON FIXED POINT RESULTS FOR WEAKLY ISOTONE INCREASING MAPPINGS IN PARTIALLY ORDERED PARTIAL METRIC SPACES

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In this paper we prove some common fixed point results for pair of weakly isotone increasing mappings in the context of partially ordered partial metric spaces. Thus our results in the new context generalize, extend, unify, enrich and complement fixed point theorems of contractive mappings in several aspects. We provide examples to illustrate the usability of our results.

### 1. Introduction and preliminaries

Matthews [23] generalized the concept of a metric space introducing partial metric spaces. Based on the notion of partial metric spaces, Matthews [22], [23], Oltra and Valero [30], Ilić et al. [19], [20] obtained very interesting fixed point theorems for mappings satisfying different contractive conditions. Recently, Abdeljawad et al. [6], proved one fixed point result for generalized contraction principle with control functions on partial metric spaces. For more recent results on partial metric spaces see [1]-[2], [4]- [10], [12]-[14], [16], [18]-[21], [26], [32], [33], [35].

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Fixed point results in ordered metric spaces has been initiated in 2004 by Ran and Reurings [31], and further studied by Nieto and Lopez [28]. For other results on ordered (partial) metric spaces see for example [1], [12], [16], [33] and [34]. The aim of this paper is to investigate existence of fixed point of mappings satisfying  $T$ -Hardy-Rogers conditions in the context of partially ordered partial metric spaces. Our results generalize, extend, unify, enrich and complement various known results in fixed point theory.

Consistent with Matthews [22], [23] and O'Neill [27], [29] the following definitions and results will be needed in the sequel.

**Definition 1.1.** A partial metric on a nonempty set  $X$  is a function  $p : X \times X \rightarrow \mathbb{R}^+$  such that for all  $x, y, z \in X$  :

$$(p_1) \quad x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y),$$

$$(p_2) \quad p(x, x) \leq p(x, y),$$

$$(p_3) \quad p(x, y) = p(y, x),$$

$$(p_4) \quad p(x, z) \leq p(x, y) + p(y, z) - p(y, y).$$

A partial metric space is a pair  $(X, p)$  such that  $X$  is a nonempty set and  $p$  is a partial metric on  $X$ .

For a partial metric  $p$  on  $X$ , the function  $p^s : X \times X \rightarrow \mathbb{R}^+$  given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \quad (1)$$

is a (usual) metric on  $X$ . Each partial metric  $p$  on  $X$  generates a  $T_0$  topology  $\tau_p$  on  $X$  with a base of the family of open  $p$ -balls  $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ , where  $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$  for all  $x \in X$  and  $\varepsilon > 0$ .

**Definition 1.2.** [23] A sequence  $\{x_n\}$  in a partial metric space  $(X, p)$  converges to  $x \in X$  if and only if  $p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x)$ ;

- (i) a sequence  $\{x_n\}$  in a partial metric space  $(X, p)$  is called Cauchy if and only if  $\lim_{n, m \rightarrow \infty} p(x_n, x_m)$  exists (and finite);
- (ii) a partial metric space  $(X, p)$  is said to be complete if every Cauchy sequence  $\{x_n\}$  in  $X$  converges, with respect to  $\tau_p$ , to a point  $x \in X$  such that  $p(x, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m)$ ;
- (iii) A mapping  $f : X \rightarrow X$  is said to be continuous at  $x_0 \in X$ , if for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $f(B_p(x_0, \delta)) \subset B_p(fx_0, \varepsilon)$ .

**Lemma 1.3.** [[23]] Let  $(X, p)$  be a partial metric space. Then:

- (1) The sequence  $\{x_n\}$  is a Cauchy in a partial metric space  $(X, p)$  if and only if  $\{x_n\}$  is a Cauchy in a metric space  $(X, p^s)$ ;
- (2) A partial metric space  $(X, p)$  is complete if and only if a metric space  $(X, p^s)$  is complete; Moreover,  $\lim_{n \rightarrow \infty} p^s(x_n, x) = 0$  if and only if

$$p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m) \quad (2)$$

**Remark 1.4.** (1) [25] Clearly, a limit of a sequence in a partial metric space need not be unique. Moreover, the function  $p(\cdot, \cdot)$  need not be continuous in the sense that  $x_n \rightarrow x$  and  $y_n \rightarrow y$  implies  $p(x_n, y_n) \rightarrow p(x, y)$ . For example, if  $X = [0, +\infty)$  and  $p(x, y) = \max\{x, y\}$  for  $x, y \in X$ , then for  $\{x_n\} = \{1\}$ ,  $p(x_n, x) = x = p(x, x)$  for each  $x \geq 1$  and so, e.g.,  $x_n \rightarrow 2$  and  $x_n \rightarrow 3$  when  $n \rightarrow \infty$ .

- (2) [6] However, if  $p(x_n, x) \rightarrow p(x, x) = 0$  then  $p(x_n, y) \rightarrow p(x, y)$  for all  $y \in X$ .

**Remark 1.5.** It is worth noting that the notions  $p$ -continuous and  $p^s$ -continuous of any function in the context of partial metric spaces are incomparable, in general. Indeed, if  $X = [0, +\infty)$ ,  $p(x, y) = \max\{x, y\}$ ,  $p^s(x, y) = |x - y|$ ,  $f0 = 1$  and  $fx = x^2$  for all  $x > 0$ ,  $gx = |\sin x|$ , then  $f$  is a  $p$ -continuous and  $p^s$ -discontinuous at point  $x = 0$ ; while  $g$  is a  $p$ -discontinuous and  $p^s$ -continuous at  $x = \pi$  (for some details see [29], [33]). Therefore, in this paper, according to [29] we take that  $T : X \rightarrow X$  is continuous if both  $T : (X, p) \rightarrow (X, p)$  and  $T : (X, p^s) \rightarrow (X, p^s)$  are continuous.

**Definition 1.6.** Let  $(X, \preceq)$  be a partially ordered set. Then:

- (a) elements  $x, y \in X$  are called comparable if  $x \preceq y$  or  $y \preceq x$  holds;
- (b) a subset  $A$  of  $X$  is said to be well ordered if every two elements of  $A$  are comparable;
- (c) a mapping  $f : X \rightarrow X$  is called nondecreasing w.r.t.  $\preceq$  if  $x \preceq y$  implies  $fx \preceq fy$ .

**Definition 1.7.** Let  $X$  be a nonempty set. Then  $(X, p, \preceq)$  is called a partially ordered partial metric space if:

- (i)  $(X, p)$  is a partial metric space, and
- (ii)  $(X, \preceq)$  is a partially ordered set.

**Definition 1.8.** Let  $(X, p, \preceq)$  be a partially ordered partial metric space. We say that  $X$  is regular if the following hypothesis holds: if  $\{z_n\}$  is a non decreasing (resp. non increasing) sequence in  $X$  with respect to  $\preceq$  such that  $z_n p^s \rightarrow z \in X$  as  $n \rightarrow \infty$ , then  $z_n \preceq z$  (resp.  $z \preceq z_n$ ) for all  $n \in \mathbb{N}$ .

**Definition 1.9.** [11] Let  $(X, \preceq)$  be a partially ordered set. A pair  $(f, g)$  of self-maps of  $X$  is said to be weakly increasing if  $fx \preceq gfx$  and  $gx \preceq fgx$  for all  $x \in X$ . Hence, a pair  $(f, f)$  is said to be weakly increasing if  $fx \preceq ffx$  for all  $x \in X$ .

Note that two weakly increasing mappings need not be nondecreasing. There exist some examples to illustrate this fact in [11].

**Definition 1.10.** [25] Let  $(X, \preceq)$  be a partially ordered set and let  $f, g : X \rightarrow X$  be two mappings. The pair  $(f, g)$  is weakly isotone increasing if for all  $x \in X$  we have  $fx \preceq gfx \preceq fgfx$ .

**Remark 1.11.** If  $f, g : X \rightarrow X$  are weakly increasing, then the pair  $(f, g)$  is weakly isotone increasing.

**Definition 1.12.** Let  $(X, p)$  be a partial metric space. A mapping  $T : X \rightarrow X$  is said to be:

- (i) a sequentially convergent if for any sequence  $\{y_n\}$  in  $X$  such that  $\{Ty_n\}$  is convergent in  $(X, p^s)$  implies that  $\{y_n\}$  is convergent in  $(X, p^s)$ ,
- (ii) a subsequentially convergent if for any sequence  $\{y_n\}$  in  $X$  such that  $\{Ty_n\}$  is convergent in  $(X, p^s)$  implies that  $\{y_n\}$  has a convergent subsequence in  $(X, p^s)$ .

Motivated by the work of [15], we give following definition.

**Definition 1.13.** Let  $(X, p)$  be a partial metric space and  $T : X \rightarrow X$ . A pair  $(f, g)$  of selfmaps on  $X$  is said to be a T-Hardy-Rogers pair on  $X$  if there exist  $a_i \geq 0, i = 1, \dots, 5$  with  $\sum_{i=1}^5 a_i < 1$  such that for all  $x, y \in X$

$$p(Tfx, Tgy) \leq a_1p(Tx, Ty) + a_2p(Tx, Tfx) + a_3p(Ty, Tgy) + a_4p(Tx, Tgy) + a_5p(Ty, Tfx). \quad (3)$$

If  $a_1 = a_4 = a_5 = 0$  and  $a_2 = a_3 \neq 0$ , (resp.  $a_1 = a_2 = a_3 = 0$  and  $a_4 = a_5 \neq 0$ ;  $a_4 = a_5 = 0$  and  $a_1, a_2, a_3 \neq 0$ ) then  $(f, g)$  is called T-Kannan (resp. T-Chatterjea; T-Reich) pair from [24].

## 2. Fixed point results

In this section, we obtain some fixed point results of T-Hardy-Rogers pair defined on a partially ordered partial metric space which is complete. We begin with the following result.

**Theorem 2.1.** *Let  $(X, \preceq)$  be a partially ordered set and suppose that there exists a partial metric  $p$  on  $X$  such that  $(X, p)$  is a complete partial metric space and  $f, g : X \rightarrow X$ . Let  $T : X \rightarrow X$  be a continuous, injective mapping and  $(f, g)$  be a  $T$ -Hardy-Rogers pair on the set of all comparable elements of  $X$ . If one of the following two conditions is satisfied:*

(a)  *$f$  or  $g$  is continuous;*

(b)  *$(X, p, \preceq)$  is regular;*

*then  $f$  and  $g$  have a common fixed point provided that  $T$  is subsequentially convergent and pair  $(f, g)$  is weakly isotone increasing. Moreover, the set of common fixed point of  $f$  and  $g$  is well ordered if and only if  $f$  and  $g$  have a unique common fixed point.*

*Proof.* Let  $x_0$  be arbitrary point in  $X$ . If  $x_0 = fx_0$  or  $x_0 = gx_0$  the proof can be easily finished using contractive condition (3). Indeed, let  $x_0 = fx_0$  then we have

$$p(Tfx_0, Tgx_0) \leq a_1p(Tx_0, Tx_0) + a_2p(Tx_0, Tfx_0) + a_3p(Tx_0, Tgx_0) \\ + a_4p(Tx_0, Tgx_0) + a_5p(Tx_0, Tfx_0),$$

or

$$p(Tx_0, Tgx_0) \leq a_1p(Tx_0, Tx_0) + a_2p(Tx_0, Tx_0) + a_3p(Tx_0, Tgx_0) \\ + a_4p(Tx_0, Tgx_0) + a_5p(Tx_0, Tx_0) \\ = (a_1 + a_2 + a_5)p(Tx_0, Tx_0) + (a_3 + a_4)p(Tx_0, Tgx_0) \\ \leq (a_1 + a_2 + a_5 + a_3 + a_4)p(Tx_0, Tgx_0) \\ < p(Tx_0, Tgx_0).$$

Hence,  $Tx_0 = Tgx_0$ , that is.,  $gx_0 = x_0$ .

Similarly, if  $x_0 = gx_0$  we obtain that  $x_0 = fx_0$ .

So we assume that  $x_0 \neq fx_0$  and  $x_0 \neq gx_0$ . We can define a sequence  $\{x_n\}$  in  $X$ , as follows:

$$x_{2n+1} = fx_{2n} \quad \text{and} \quad x_{2n+2} = gx_{2n+1} \quad \text{for } n = 0, 1, 2, \dots$$

We can also suppose that the successive terms of  $\{x_n\}$  are different. Otherwise we have again finished. Since pair  $(f, g)$  is weakly isotone increasing, we have

$$x_1 = fx_0 \preceq gfx_0 = gx_1 = x_2 \preceq fgfx_0 = fgx_1 = fx_2 = x_3, \\ x_3 = fx_2 \preceq gfx_2 = gx_3 = x_4 \preceq fgfx_2 = fgx_3 = gx_4 = x_5,$$

and continuing this process we get

$$x_1 \preceq x_2 \preceq \dots \preceq x_n \preceq x_{n+1} \preceq \dots \quad (4)$$

Now we claim that for all  $n \in \mathbb{N}$

$$p(x_n, x_{n+1}) \leq kp(x_{n-1}, x_n),$$

where  $k = \frac{2a_1+a_2+a_3+a_4+a_5}{2-a_2-a_3-a_4-a_5} \in [0, 1)$ .

Since the successive terms of  $\{x_n\}$  are comparable, therefore by replacing  $x$  by  $x_{2n}$  and  $y$  by  $x_{2n+1}$  in (3), we have

$$\begin{aligned} p(Tx_{2n+1}, Tx_{2n+2}) &= p(Tfx_{2n}, Tgx_{2n+1}) \\ &\leq a_1p(Tx_{2n}, Tx_{2n+1}) + a_2p(Tx_{2n}, Tfx_{2n}) + a_3p(Tx_{2n+1}, Tgx_{2n+1}) \\ &\quad + a_4p(Tx_{2n}, Tgx_{2n+1}) + a_5p(Tx_{2n+1}, Tfx_{2n}) \\ &= a_1p(Tx_{2n}, Tx_{2n+1}) + a_2p(Tx_{2n}, Tx_{2n+1}) + a_3p(Tx_{2n+1}, Tx_{2n+2}) \\ &\quad + a_4p(Tx_{2n}, Tx_{2n+2}) + a_5p(Tx_{2n+1}, Tx_{2n+1}) \\ &\leq a_1p(Tx_{2n}, Tx_{2n+1}) + a_2p(Tx_{2n}, Tx_{2n+1}) + a_3p(Tx_{2n+1}, Tx_{2n+2}) \\ &\quad + a_4p(Tx_{2n}, Tx_{2n+1}) + a_4p(Tx_{2n+1}, Tx_{2n+2}) - a_4p(Tx_{2n+1}, Tx_{2n+1}) \\ &\quad + a_5p(Tx_{2n+1}, Tx_{2n+1}) \\ &= (a_1 + a_2 + a_4)p(Tx_{2n}, Tx_{2n+1}) + a_4p(Tx_{2n+1}, Tx_{2n+2}) \\ &\quad + (a_5 - a_4)p(Tx_{2n+1}, Tx_{2n+1}), \end{aligned}$$

or equivalently,

$$\begin{aligned} (1 - a_3 - a_4)p(Tx_{2n+1}, Tx_{2n+2}) \\ \leq (a_1 + a_2 + a_4)p(Tx_{2n}, Tx_{2n+1}) + (a_5 - a_4)p(Tx_{2n+1}, Tx_{2n+1}). \quad (5) \end{aligned}$$

Similarly, replacing  $x$  by  $x_{2n+1}$  and  $y$  by  $x_{2n}$  in (3), we obtain

$$\begin{aligned} (1 - a_2 - a_5)p(Tx_{2n+1}, Tx_{2n+2}) \\ \leq (a_1 + a_3 + a_5)p(Tx_{2n}, Tx_{2n+1}) + (a_4 - a_5)p(Tx_{2n+1}, Tx_{2n+1}). \quad (6) \end{aligned}$$

Summing (5) and (6), we obtain  $p(Tx_{2n+1}, Tx_{2n+2}) \leq kp(Tx_{2n}, Tx_{2n+1})$ , where  $k = \frac{2a_1+a_2+a_3+a_4+a_5}{2-a_2-a_3-a_4-a_5}$ . Obviously  $0 \leq k < 1$ . Similarly, it can be shown that  $p(Tx_{2n+1}, Tx_{2n}) \leq kp(Tx_{2n}, Tx_{2n-1})$ . Therefore, for all  $n \geq 1$ ,

$$p(Tx_n, Tx_{n+1}) \leq kp(Tx_{n-1}, Tx_n) \leq \dots \leq k^n p(Tx_0, Tx_1).$$

Now, for any  $m \in \mathbb{N}$  with  $m > n$ , we have

$$\begin{aligned} p(Tx_n, Tx_m) &\leq p(Tx_n, Tx_{n+1}) + p(Tx_{n+1}, Tx_{n+2}) + \dots + p(Tx_{m-1}, Tx_m) \\ &\leq (k^n + k^{n+1} + \dots + k^{m-1})p(Tx_0, Tx_1) \leq \frac{k^n}{1-k}p(Tx_0, Tx_1), \end{aligned}$$

which implies that  $p(Tx_n, Tx_m) \rightarrow 0$  as  $n, m \rightarrow \infty$ . Hence  $\{Tx_n\}$  is a Cauchy sequence in  $(X, p)$  and in  $(X, p^s)$ . Since  $(X, p)$  is complete, therefore from Lemma 1.3 (2)  $(X, p^s)$  is a complete metric space. Hence  $\{Tx_n\}$  converges to some  $v \in X$  with respect to the metric  $p^s$ , that is,

$$\lim_{n \rightarrow \infty} p^s(Tx_n, v) = 0, \quad (7)$$

or equivalently,

$$p(v, v) = \lim_{n \rightarrow \infty} p(Tx_n, v) = \lim_{n, m \rightarrow \infty} p(Tx_n, Tx_m) = 0. \quad (8)$$

Suppose now that  $T$  is subsequentially convergent. Therefore convergence of  $\{Tx_n\}$  in  $(X, p^s)$  implies that  $\{x_n\}$  has a convergent subsequence  $\{x_{n_i}\}$  in  $(X, p^s)$ . So

$$\lim_{n \rightarrow \infty} p^s(x_{n_i}, u) = 0, \quad (9)$$

for some  $u \in X$ . As  $T$  is a continuous, (9) and Remark 1.5 imply that

$$\lim_{n \rightarrow \infty} p^s(Tx_{n_i}, Tu) = 0.$$

From (7) and by the uniqueness of the limit in metric space  $(X, p^s)$ , we obtain  $Tu = v$ . Consequently,

$$0 = p(Tu, Tu) = \lim_{i \rightarrow \infty} p(Tx_{n_i}, Tu) = \lim_{i, j \rightarrow \infty} p(p x_{n_i}, p x_{n_j}). \quad (10)$$

(a) If  $f$  is a continuous self map on  $X$ , then  $f x_{2n_i} \rightarrow fu$  and  $T f x_{2n_i} \rightarrow T fu$  as  $i \rightarrow \infty$ . Since  $T x_{2n_i} \rightarrow Tu$  as  $i \rightarrow \infty$ , we obtain that  $Tu = T fu$ . As  $T$  is injective, so we have  $fu = u$ . Also, because  $u \preceq u$ , we obtain

$$\begin{aligned} p(Tu, Tgu) &= p(Tfu, Tgu) \leq a_1 p(Tu, Tu) + a_2 p(Tu, Tfu) \\ &+ a_3 p(Tu, Tgu) + a_4 p(Tu, Tgu) + a_5 p(Tu, Tu) \\ &= a_1 p(Tu, Tu) + a_2 p(Tu, Tu) + a_3 p(Tu, Tgu) + a_4 p(Tu, Tgu) + a_5 p(Tu, Tu) \\ &= (a_3 + a_4) p(Tu, Tgu) \leq (a_1 + a_2 + a_3 + a_4 + a_5) p(Tu, Tgu) \\ &< p(Tu, Tgu), \end{aligned}$$

and  $Tu = Tgu$  and again using injectiveness of  $T$ , we have  $u = gu$ . Similarly, the result follows when  $g$  is continuous.

(b) Further, if  $f$  and  $g$  are not continuous then by given assumption we have  $x_n \preceq u$  for all  $n \in \mathbb{N}$ . Thus for a subsequences  $\{x_{2n_i}\}$  and  $\{x_{2n_i+1}\}$  of  $\{x_n\}$  we

have  $x_{2n_i} \preceq u$  and  $x_{2n_i+1} \preceq u$ . Therefore, we have

$$\begin{aligned} p(Tfu, Tu) &\leq p(Tfu, Tgx_{2n_i+1}) + p(Tgx_{2n_i+1}, Tu) - p(Tgx_{2n_i+1}Tgx_{2n_i+1}) \\ &\leq a_1p(Tu, Tx_{2n_i+1}) + a_2p(Tu, Tfu) + a_3p(Tx_{2n_i+1}, Tgx_{2n_i+1}) \\ &\quad + a_4p(Tx_{2n_i+1}, Tfu) + a_5p(Tu, Tgx_{2n_i+1}) + p(Tgx_{2n_i+1}, Tu) \\ &\leq a_1p(Tu, Tx_{2n_i+1}) + a_2p(Tu, Tfu) + a_3p(Tx_{2n_i+1}, Tx_{2n_i+2}) \\ &\quad + a_4p(Tx_{2n_i+1}, Tfu) + a_5p(Tu, Tx_{2n_i+2}) + p(Tx_{2n_i+1}, Tu). \end{aligned}$$

On taking limit as  $i \rightarrow \infty$  and applying Remark 1.4 (2) we have

$$\begin{aligned} p(Tfu, Tu) &\leq (a_2 + a_4) p(Tu, Tfu) \\ &\leq (a_1 + a_2 + a_3 + a_4 + a_5) p(Tu, Tfu) < p(Tu, Tfu), \quad (11) \end{aligned}$$

which implies that  $p(Tu, Tfu) = 0$ , that is.,  $Tu = Tfu$ . Now injectivity of  $T$  gives  $u = fu$ . Also, because  $u \preceq u$  we obtain

$$\begin{aligned} p(Tu, Tgu) &= p(Tfu, Tgu) \\ &\leq a_1p(Tu, Tu) + a_2p(Tu, Tfu) + a_3p(Tu, Tgu) \\ &\quad + a_4p(Tu, Tgu) + a_5p(Tu, Tu) \\ &= (a_3 + a_4) p(Tu, Tgu) \\ &\leq (a_1 + a_2 + a_3 + a_4 + a_5) p(Tu, Tgu) \\ &< p(Tu, Tgu) \end{aligned}$$

and  $Tu = Tgu$ , that is.,  $u = gu$ . Hence  $u$  is the common fixed point of  $f$  and  $g$ .

Since sequentially convergent maps are subsequentially convergent then the result holds when  $T$  is sequentially convergent.

Now suppose that set of common fixed points of  $f$  and  $g$  is well ordered. Then common fixed point of  $f$  and  $g$  is unique. Assume on contrary that,  $fu = gu = u$  and  $fv = gv = v$  but  $u \neq v$ . Now, by (3) we have

$$\begin{aligned} p(Tfu, Tgv) &\leq a_1p(Tu, Tv) + a_2p(Tu, Tfu) + a_3p(Tv, Tgv) \\ &\quad + a_4p(Tu, Tgv) + a_5p(Tv, Tfu) \\ &= (a_1 + a_4 + a_5) p(Tu, Tv) \\ &\leq (a_1 + a_2 + a_3 + a_4 + a_5) p(Tu, Tv) \\ &< p(Tu, Tv), \end{aligned}$$

which is a contradiction. Hence,  $u = v$ . Conversely, if  $f$  and  $g$  have only one common fixed point then the set of common fixed point of  $f$  and  $g$  being singleton is well ordered.  $\square$



If we take  $f = g$  in Theorem 2.1, then we have the following corollary (similar to Theorem 2.1 of Filipović et al. [15]).

**Corollary 2.2.** *Let  $(X, \preceq)$  be a partially ordered set and suppose that there exists a partial metric  $p$  on  $X$  such that  $(X, p)$  is a complete partial metric space and  $f : X \rightarrow X$ . Let  $T : X \rightarrow X$  be a continuous, injective mapping and  $(f, f)$  be a  $T$ -Hardy-Rogers pair on the set of all comparable elements of  $X$ . If one of the following two conditions is satisfied:*

- (a)  $f$  is continuous;
- (b)  $(X, p, \preceq)$  is regular;

*then  $f$  have a fixed point provided that  $T$  is subsequentially convergent and pair  $(f, f)$  is weakly isotone increasing. Moreover, the set of fixed point of  $f$  is well ordered if and only if  $f$  has a unique fixed point.*

Taking  $Tx = x$  in Theorem 2.1, we get the Hardy-Rogers type [17] (and so Kannan, Chatterjea and Reich) fixed point theorem for two weakly isotone increasing maps on partially ordered partial metric spaces.

**Corollary 2.3.** *Let  $(X, \preceq)$  be a partially ordered set and suppose that there exists a partial metric  $p$  on  $X$  such that  $(X, p)$  is a complete partial metric space and  $f, g : X \rightarrow X$  such that*

$$p(fx, gy) \leq a_1 p(x, y) + a_2 p(x, fx) + a_3 p(y, fy) + a_4 p(x, gy) + a_5 p(y, fx), \quad (12)$$

*for all comparable  $x, y \in X$ , where  $a_i \geq 0, i = 1, \dots, 5$  and  $\sum_{i=1}^5 a_i < 1$ . If one of the following two conditions is satisfied:*

- (a)  $f$  or  $g$  is continuous;
- (b)  $(X, p, \preceq)$  is regular;

*then  $f$  and  $g$  have a common fixed point provided that pair  $(f, g)$  is weakly isotone increasing. Moreover, the set of common fixed point of  $f$  and  $g$  is well ordered if and only if  $f$  and  $g$  have a unique common fixed point.*

Taking  $g = f$  in Theorem 2.1, we get the following.

**Corollary 2.4.** [2] *Let  $(X, \preceq)$  be a partially ordered set and suppose that there exists a partial metric  $p$  on  $X$  such that  $(X, p)$  is a complete partial metric space and  $f : X \rightarrow X$ . Let  $T : X \rightarrow X$  be a continuous, injective mapping and  $(f, f)$  be a  $T$ -Hardy-Rogers pair on the set of all comparable elements of  $X$ . If one of the following two conditions is satisfied:*

(a)  $f$  is continuous;

(b)  $(X, p, \preceq)$  is regular,

then  $f$  have a fixed point provided that  $T$  is subsequentially convergent and pair  $(f, f)$  is weakly isotone increasing. Moreover, the set of fixed point of  $f$  is well ordered if and only if  $f$  a unique fixed point.

### 3. Examples

We demonstrate the use of Theorem 2.1 and Corollaries 2.2-2.4 with the help of the following example. It will show also that this theorem is more general than that some other known fixed point results ([2], [15], [17], [24]).

(a) Let  $X = [0, 1]$  be endowed with the following relation:  $x \preceq y$  if and only if  $x \geq y$  where  $\geq$  is usual order on  $X$ . Then,  $(X, \preceq)$  is a partially ordered set. Let  $p : X \times X \rightarrow \mathbb{R}^+$  be defined by  $p(x, y) = \max\{x, y\}$ . The partial metric space  $(X, p)$  is complete because  $(X, p^s)$  is complete. Indeed, for any  $x, y \in X$ ,

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) = 2\max\{x, y\} - (x + y) = |x - y|.$$

Thus,  $(X, p^s) = ([0, 1], |\cdot|)$  is the usual metric space which is complete.

Define  $T, f, g : X \rightarrow X$  as  $Tx = x^2, fx = \frac{1}{3}x$  and  $gx = \frac{1}{2}x$ . It is clear that pair  $(f, g)$  is weakly isotone increasing. All the conditions of Theorem 2.1 are fulfilled with  $a_1 = a_2 = a_4 = a_5 = 0, a_3 = \frac{1}{3}$ . Indeed, for all  $x, y \in X, x < y$  we have

$$p(Tfx, Tgy) = \max\left\{\frac{1}{9}x^2, \frac{1}{4}y^2\right\} = \frac{1}{4}y^2,$$

and

$$\begin{aligned} & a_1p(Tx, Ty) + a_2p(Tx, Tfx) + a_3p(Ty, Tgy) \\ & + a_4p(Tx, Tgy) + a_5p(+yTy, Tfx) = \frac{1}{3}(p(Tx, Tfx) + p(Ty, Tgy)) \\ & = \frac{1}{3}\left(\max\left\{x^2, \frac{x^2}{9}\right\} + \max\left\{y^2, \frac{y^2}{4}\right\}\right) = \frac{1}{3}(x^2 + y^2). \end{aligned}$$

Since  $\frac{1}{4}y^2 \leq \frac{1}{3}(x^2 + y^2)$  for all  $x, y \in X, x < y$ , we obtain that (3) holds. Hence, by Theorem 2.1  $f$  and  $g$  have a common fixed point. Here 0 is the common fixed point of  $f$  and  $g$ .

(b) Taking  $a_2 = a_3 = a_4 = 0, a_1 = \frac{1}{7}, a_5 = \frac{1}{9}$  we obtain that (3) holds. Indeed, for all  $x, y \in X, x < y$  we have

$$p(Tfx, Tgy) = \max\left\{\frac{1}{9}x^2, \frac{1}{4}y^2\right\} = \frac{1}{4}y^2,$$

and

$$\begin{aligned} & a_1p(Tx, Ty) + a_2p(Tx, Tfx) + a_3p(Ty, Tgy) \\ & + a_4p(Tx, Tgy) + a_5p(Ty, Tfx) = \frac{1}{7}p(Tx, Ty) + \frac{1}{9}p(Ty, Tfx) \\ & = \frac{1}{7} \max\{x^2, y^2\} + \frac{1}{9} \max\left\{y^2, \frac{1}{9}x^2\right\} = \frac{1}{7}y^2 + \frac{1}{9}y^2. \end{aligned}$$

Since  $\frac{1}{4}y^2 \leq \frac{1}{7}y^2 + \frac{1}{9}y^2$  for all  $x, y \in X, x < y$ , we obtain that (3) holds.

On the other hand, consider the same problem in the standard metric  $d(x, y)$  and take  $x = y = 1$ . Then  $d(Tf1, Tg1) = \left|\frac{1}{9} - \frac{1}{4}\right| = \frac{5}{36}$  and

$$\frac{1}{7}d(T1, T1) + \frac{1}{9}d(T1, Tf1) = \frac{1}{9} \left|1 - \frac{1}{9}\right| = \frac{8}{81} < \frac{5}{36} = d(Tf1, Tg1).$$

Hence, the condition (3) if  $p = d$  does not hold and existence of a common fixed point of  $f$  and  $g$  cannot be obtained from the known results in standard metric spaces (see, e.g., ([15], [24]).

(c) The following example shows that the existence of order may be crucial. Taking

$$p(x, y) = \begin{cases} \max\{x, y\}, & x \in [0, \frac{1}{2}) \vee y \in [0, \frac{1}{2}) \\ |x - y|, & x, y \in [\frac{1}{2}, 1], \end{cases}$$

instead  $p(x, y) = \max\{x, y\}$  in (a) and (b). The order  $\preceq$  is given by

$$x \preceq y \Leftrightarrow \left(x, y \in [0, \frac{1}{2}) \wedge x \geq y\right) \vee (x = y).$$

Take  $a_i, i = 1, \dots, 5; T, f$  and  $g$  as in (b). It is easy to check that the contractive condition (3) holds. All the conditions of Theorem 2.1 are fulfilled and mappings  $f$  and  $g$  have a common fixed point ( $x = 0$ ).

On the other hand, consider the same problem, but without order. Then the condition (3) does not hold and the conclusion about the common fixed point cannot be obtained in this way. Indeed, take  $x = y = 1$ . Then  $p(Tf1, Tg1) = \left|\frac{1}{9} - \frac{1}{4}\right| = \frac{5}{36}$  and

$$\frac{1}{7}p(T1, T1) + \frac{1}{9}p(T1, Tf1) = \frac{1}{9} \left|1 - \frac{1}{9}\right| = \frac{8}{81} < \frac{5}{36} = p(Tf1, Tg1).$$

## REFERENCES

- [1] M. Abbas - T. Nazir, *Fixed point of generalized weakly contractive mappings in ordered partial metric spaces*, Fixed Point Theory Appl. (1) 2012.
- [2] M. Abbas - H. Aydi - S. Radenović, *Fixed Point of T-Hardy-Rogers Contractive Mappings in Partially Ordered Partial Metric Spaces*, Internat. J. Math. Math. Sci. Volume 2012, Article ID 313675.
- [3] M. Abbas - T. Nazir - S. Radenović, *Common fixed points of four maps in partially ordered metric spaces*, Appl. Math. Lett. 24 (2011), 1520–1526.
- [4] T. Abdeljawad, *Fixed points for generalized weakly contractive mappings in partial metric spaces*, Math. Comput. Model. 54 (11-12) (2011), 2923–2927
- [5] T. Abdeljawad - E. Karapinar - K. Tas, *A generalized contraction principle with control functions on partial metric spaces*, Comput. Math. Appl. 6 (3) (2012), 716–719.
- [6] T. Abdeljawad - E. Karapinar - K. Tas, *Existence and uniqueness of a common fixed point on partial metric spaces*, Appl. Math. Lett. 24 (2011), 1900–1904.
- [7] T. Abdeljawad, *Coupled fixed point theorems for partially contractive mappings*, Fixed Point Theory Appl. 148 (2012).
- [8] T. Abdeljawad - J. O. Alzabut - E. Mukheimer - Y. Zaidan, *Banach contraction principle for cyclic mappings on partial metric spaces*, Fixed Point Theory Appl. 2012, 2012:154
- [9] T. Abdeljawad - J. O. Alzabut - E. Mukheimer - Y. Zaidan, *Best proximity points for cyclic contraction mappings with boundedly compact decompositions in partial metric spaces*, Journal of Computational Analysis and Applications (2012), in press.
- [10] M.Á. Alghamdi - N. Shahzad - O. Valero, *On fixed point theory in partial metric spaces*, Fixed Point Theorey Appl. 2012:175.
- [11] I. Altun - B. Damjanović - D. Đorić, *Fixed point and common fixed point theorems on ordered cone metric spaces*, Appl. Math. Lett. 23 (3) (2010), 310–316.
- [12] I. Altun - A. Erduran, *Fixed point theorems for monotone mappings on partial metric spaces*, Fixed Point Theory Appl. Volume 2011, Article ID 508730.
- [13] C. Di Bari - M. Milojević - S. Radenović - P. Vetro, *Common fixed points for self-mappings on partial metric spaces*, Fixed Point Theory Appl. 2012:140.
- [14] D. Đukić - Z. Kadelburg - S. Radenović, *Fixed points of Geraghty-type mappings in various generalized metric spaces*, Abstract and Appl. Anal. Volume 2011, Article ID 561245.
- [15] M. Filipović - Lj. Paunović - S. Radenović - M. Rajović, *Remarks on "Cone metric spaces and fixed point theorems of T-Kannan and T-Chatterjea contractive mappings"*, Math. Comput. Model. 54 (2011), 1467–1472.
- [16] Z. Golubović - Z. Kadelburg - S. Radenović, *Coupled coincidence points of mappings in ordered partial metric paces*, Abstract and Appl. Anal. Volume 2012, Article ID 192581.

- [17] G. E. Hardy - T. D. Rogers, *A generalization of a fixed point theorem of Reich*, *Canad. Math. Bull.* 16 (1973), 201–206.
- [18] Z. Kadelburg - H.K.Nashine - S. Radenović, *Fixed point results under various contractive conditions in partial metric spaces*, RASCAM DOI: 10.1007/s13398-012-0066-6.
- [19] D. Ilić - V. Pavlović - V. Rakočević, *Some new extensions of Banach's contractions principle in partial metric spaces*, *Appl. Math. Lett.* 24 (2011), 1326–1330.
- [20] D. Ilić - V. Pavlović - V. Rakočević, *Extensions of Zamfirescu theorem to partial metric spaces*, *Math. Comput. Model.* 55 (2012), 801–809.
- [21] E. Karapinar - Inci M. Erhan, *Fixed point theorems for operators on partial metric spaces*, *Appl. Math. Lett.* 24 (2011), 1894–1899.
- [22] S. G. Matthews, *Partial metric topology*, Research Report 212. Dept. of Computer Science, University of Warwick, 1992.
- [23] S. G. Matthews, *Partial metric topology*, Proc. 8th Summer Conference on General Topology and Applications, *Ann. New York Acad. Sci.* 728 (1994), 183–197
- [24] José R. Morales - E. Rojas, *Cone metric spaces and fixed point theorems of T-Kannan contractive mappings*, *Int. J. Math. Anal.*, 4 (4) (2010), 175–184.
- [25] H. K. Nashine - Z. Kadelburg - S. Radenović, *Common fixed point theorems for weakly isotone increasing mappings in ordered partial metric spaces*, *Mathematical and Computer Modelling* (2011), doi:10.1016/j.mcm.2011.12.019.
- [26] H. K. Nashine - Z. Kadelburg - S. Radenović - J. K. Kim, *Fixed point theorems under Hardy-Rogers contractive conditions on 0-complete ordered partial metric spaces*, *Fixed Point Theory Appl.* 2012:180.
- [27] S. J. O'Neill, *Partial metrics, valuations and domain theory*, Proc. 11th Summer Conference on General Topology and Applications, *Ann. New York Acad. Sci.* 806 (1996) 304–315.
- [28] J. J. Nieto - R. L. Pouso - R. Rodriguez-Lopez, *Fixed point theorems in partially ordered sets*, *Proc. Am. Math. Soc.* 132 (2007), 2505–2517.
- [29] S. J. O'Neill, *Two topologies are better than one*, Tech report, University of Warwick, Coventry, UK. <http://www.dcs.warwick.ac.uk/reports/283.html>.1995.
- [30] S. Oltra - O. Valer, *Banach's fixed point theorem for partial metric spaces*, *Rend. Istit. Math. Univ. Trieste* 36 (2004), 17–26.
- [31] A. C. M. Ran - M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, *Proc. Am. Math. Soc.* 132, (2004), 1435–1443.
- [32] S. Romaguera, *Fixed point theorems for generalized contractions on partial metric spaces*, *Topology Appl.* 159 (2011), 194–199.
- [33] B. Samet - M. Rajović - R. Lazović - R. Stojiljković, *Common fixed point results for nonlinear contractions in ordered partial metric spaces*, *Fixed Point Theory Appl.* 2011:71.

- [34] W. Shatanawi - Z. Mustafa - N. Tahat, *Some Coincidence point theorems for Non-linear Contraction in Ordered Metric Spaces*, Fixed Point Theory Appl. 2012, 2011:68.
- [35] F. Vetro - S. Radenović, *Nonlinear  $\psi$ -quasi-contractions of Ćirić-type in partial metric spaces*, Appl. Math. Comput. 219 (4) (2012), 1594–1600.

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