



## Open Mathematics

### Research Article

Xiao-lan Liu\*, Mi Zhou\*, Lakshmi Narayan Mishra, Vishnu Narayan Mishra, and Boško Damjanović

# Common fixed point theorem of six self-mappings in Menger spaces using $(CLR_{ST})$ property

<https://doi.org/10.1515/math-2018-0120>

Received June 3, 2018; accepted October 1, 2018.

**Abstract:** In this paper, we study the existence and uniqueness of common fixed point of six self-mappings in Menger spaces by using the common limit range property (denoted by  $(CLR_{ST})$ ) of two pairs. Our results improve, extend, complement and generalize several existing results in the literature. Also, some examples are provided to illustrate the usability of our results.

**Keywords:** Fixed point, Menger space,  $(CLR_{ST})$  property, Property  $(E.A)$ , Weakly compatible

**MSC:** 47H10, 54H25, 54E50

## 1 Introduction and Preliminaries

The famous Banach Contraction Principle in metric spaces was proposed in 1922. From then on, there were so many generalizations of metric space, one of which was the probabilistic metric space. Menger first introduced the notion of probabilistic metric space in 1942 [1]. Sequentially, in 1960, Schweizer and Sklar investigated and obtained some results with relevance to this space [2]. In 1972, Sehgal and Bharucha-Reid [3] generalized the Banach Contraction Principle to complete Menger spaces, which was a milestone in the development of fixed point theory in Menger space. In 1982, Sessa [4] introduced the notion of weakly commuting mappings in metric spaces. In sequel, in 1986, Jungck [5] weakened weakly commuting mappings to compatible mappings in metric spaces. In 1991, Mishra [6] introduced compatible mappings in Menger spaces. In 1998, Jungck and Rhoades [7] proposed the notion of weak compatibility if they commute at their coincidence points, and proved that compatible mappings are weak compatible but the reverse does not hold. In 2002, Aamri and Moutawakil [8] introduced the property  $(E.A)$  of one pair and the common property  $(E.A)$  of two pairs, and obtained common fixed point theorems in metric spaces. In 2005, Liu [9] used common property  $(E.A)$  to obtain the corresponding fixed point theorems. Later, in 2008, Kubiacyk and Sharma [10] introduced the property  $(E.A)$  in PM spaces and got some fixed point theorems. In 2011, Sintunavarat and

---

\*Corresponding Author: **Xiao-lan Liu:** College of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong, Sichuan 643000, China, E-mail: stellalwp@163.com

\*Corresponding Author: **Mi Zhou:** School of Science and Technology, University of Sanya, Sanya, Hainan 572000, China, E-mail: mizhou330@126.com

**Lakshmi Narayan Mishra:** Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology (VIT) University, Vellore 632 014, Tamil Nadu, India, E-mail: lakshminarayanmishra04@gmail.com

**Vishnu Narayan Mishra:** Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur, Madhya Pradesh 484 887, India, E-mail: vishnunarayanmishra@gmail.com

**Boško Damjanović:** Faculty of Agriculture, University of Belgrade, Belgrade, Serbia, E-mail: dambo@agrif.bg.ac.rs

Kumam [11] introduced  $(CLR_S)$  property and got the fixed point theorem in fuzzy metric spaces. Soon, Imdad, Pant and Chauhan introduced [12] $(CLR_{ST})$  property, and obtained some fixed point theorems in Menger spaces. In 2014, Imdad, Chauhan, Kadelburg, Vetro [13] proved  $(CLR_{ST})$  property of two pairs of non-self weakly compatible mappings under  $\varphi$ -weak contractive conditions in symmetric spaces. Singh and Jain [14] obtained a fixed point theorem of six self-mappings in Menger spaces through weak compatibility. Later, Liu [15] utilized the property (E.A) to prove common fixed point theorems in Menger spaces, which improved the result of [14]. Some applications of these kind of results can be seen in [16–20]. Inspired by the above works, this paper utilizes  $(CLR_{ST})$  property to obtain the common fixed point theorems in Menger spaces, at the same time, uniqueness of common fixed point is obtained. At last, we illustrate some examples to support our results.

To begin with, we give some basic notions with relevance to Menger spaces and distribution functions. Other definitions used here can be found within [15].

**Definition 1.1.** A real valued function  $f$  on the set of real numbers is called a distribution function if it is non-decreasing, left continuous with  $\inf_{u \in \mathbb{R}} f(u) = 0$  and  $\sup_{u \in \mathbb{R}} f(u) = 1$ .

The Heaviside function  $H$  is a distribution function defined by

$$H(u) = \begin{cases} 0, & u \leq 0, \\ 1, & u > 0. \end{cases}$$

**Definition 1.2** ([6]). Let  $X$  be a non-empty set and let  $L$  denote the set of all distribution functions defined on  $X$ , i.e.,  $L = \{F_{x,y} : x, y \in X\}$ . An ordered pair  $(X, F)$  is called a probabilistic metric space (for short, PM-space) where  $F$  is a mapping from  $X \times X$  into  $L$  if, for every pair  $(x, y) \in X$ , a distribution function  $F_{x,y}$  is assumed to satisfy the following four conditions:

- (1)  $F_{x,y}(u) = 1$  for all  $u > 0$ , if and only if  $x = y$ ;
- (2)  $F_{x,y}(u) = F_{y,x}(u)$ ;
- (3)  $F_{x,y}(0) = 0$ ;
- (4) If  $F_{x,y}(u_1) = 1$  and  $F_{y,z}(u_2) = 1$ , then  $F_{x,z}(u_1 + u_2) = 1$  for all  $x, y, z$  in  $X$  and  $u_1, u_2 \geq 0$ .

**Definition 1.3** ([14]). A  $t$ -norm is a function  $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$  which satisfies the following conditions:

- (T1)  $t(a, 1) = a$ ,  $t(0, 0) = 0$ ;
- (T2)  $t(a, b) = t(b, a)$ ;
- (T3)  $t(c, d) \geq t(a, b)$  for  $c \geq a$ ,  $d \geq b$ ;
- (T4)  $t(t(a, b), c) = t(a, t(b, c))$  for all  $a, b, c$  in  $[0, 1]$ .

**Definition 1.4** ([14]). A Menger probabilistic metric space  $(X, F, t)$  (for short, Menger-space) is an ordered triple, where  $t$  is a  $t$ -norm, and  $(X, F)$  is a probabilistic metric space which satisfies the following condition:

$$F_{x,z}(u_1 + u_2) \geq t(F_{x,y}(u_1), F_{y,z}(u_2)) \text{ for all } x, y, z \text{ in } X \text{ and } u_1, u_2 \geq 0.$$

Next, we will obtain  $(CLR_{ST})$  property of six self weakly compatible mappings under certain conditions proposed by Liu [15] in Menger spaces. Before that, we list some basic definitions with regards to property (E.A) and  $(CLR_{ST})$  property for one pair and two pairs of self mappings.

**Definition 1.5.**

(1) The pair  $(A, S)$  of self mappings of a Menger space  $(X, F, t)$  is said to satisfy the property (E.A) [15] if there exists a sequence  $\{x_n\}$  in  $X$  such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z, \text{ for some } z \in X.$$

(2) Two pairs  $(A, S)$  and  $(B, T)$  of self mappings of a Menger space  $(X, F, t)$  are said to satisfy the property (E.A) [15] if there exists two sequences  $\{x_n\}, \{y_n\}$  in  $X$  such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} By_n = \lim_{n \rightarrow \infty} Ty_n = z, \text{ for some } z \in X.$$

(3) The pair  $(A, S)$  of self mappings of a Menger space  $(X, F, t)$  is said to have the common limit range property with respect to the mapping  $S$  (denoted by  $(CLR_S)$ ) [12] if there exists a sequence  $\{x_n\}$  in  $X$  such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z, \text{ where } z \in S(X).$$

(4) Two pairs  $(A, S)$  and  $(B, T)$  of self mappings of a Menger space  $(X, F, t)$  are said to have the common limit range property with respect to mappings  $S$  and  $T$  [12] (denoted by  $(CLR_{ST})$ ) if there exists two sequences  $\{x_n\}, \{y_n\}$  in  $X$  such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} By_n = \lim_{n \rightarrow \infty} Ty_n = z, \text{ where } z \in S(X) \cap T(X).$$

**Definition 1.6** ([15]). Self mappings  $A$  and  $B$  of a Menger space  $(X, F, t)$  are said to be weakly compatible if they commute at their coincidence points, i.e., if  $Ax = Bx$  for some  $x \in X$ , then  $ABx = BAx$ .

**Lemma 1.7.** Let  $A, B, S, T, L$  and  $M$  be self mappings of a Menger space  $(X, F, t)$ , with continuous  $t$ -norm with  $t(x, x) \geq x$  for all  $x \in [0, 1]$ , satisfying the following conditions:

- (i)  $L(X) \subseteq ST(X)$  [resp.  $M(X) \subseteq AB(X)$ ];
- (ii) the pair  $(L, AB)$  satisfies the  $(CLR_{AB})$  property [resp. the pair  $(M, ST)$  satisfies the  $(CLR_{ST})$  property];
- (iii)  $ST(X)$  is a closed subset of  $X$  [resp.  $AB(X)$  is a closed subset of  $X$ ];
- (iv) there exists an upper semicontinuous function  $\phi : [0, \infty) \rightarrow [0, \infty)$  with  $\phi(0) = 0$  and  $\phi(x) < x$  for all  $x > 0$  such that

$$F_{Lp, Mq}(\phi(x)) \geq \min\{F_{ABp, Lp}(x), F_{STq, Mq}(x), F_{STq, Lp}(\beta x), F_{ABp, Mq}((1 + \beta)x), F_{ABp, STq}(x)\} \quad (1)$$

for all  $p, q \in X, \beta \geq 1$  and  $x > 0$ .

Then the pairs  $(L, AB)$  and  $(M, ST)$  share the  $(CLR_{(AB)(ST)})$  property.

*Proof.* Since the pair  $(L, AB)$  satisfies the  $(CLR_{AB})$  property, there exists a sequence  $\{x_n\}$  in  $X$  such that

$$\lim_{n \rightarrow \infty} Lx_n = \lim_{n \rightarrow \infty} ABx_n = z, \text{ where } z \in AB(X).$$

In view of (i) and (iii), for  $\{x_n\} \subset X$ , there exists a sequence  $\{y_n\} \subset X$  such that  $Lx_n = STy_n$ . It follows that

$$\lim_{n \rightarrow \infty} STy_n = \lim_{n \rightarrow \infty} Lx_n = z, \text{ where } z \in AB(X) \cap ST(X).$$

Therefore, it suffices to prove that  $\lim_{n \rightarrow \infty} My_n = z$ . In fact, by (iv), putting  $p = x_n, q = y_n$ , we can obtain that

$$\begin{aligned} F_{Lx_n, My_n}(\phi(x)) &\geq \min\{F_{ABx_n, Lx_n}(x), F_{STy_n, My_n}(x), F_{STy_n, Lx_n}(\beta x), F_{ABx_n, My_n}((1 + \beta)x), F_{ABx_n, STy_n}(x)\} \\ &= \min\{F_{ABx_n, Lx_n}(x), F_{Lx_n, My_n}(x), F_{Lx_n, Lx_n}(\beta x), F_{ABx_n, My_n}((1 + \beta)x), F_{ABx_n, Lx_n}(x)\} \\ &= \min\{F_{ABx_n, Lx_n}(x), F_{Lx_n, My_n}(x), 1, F_{ABx_n, My_n}((1 + \beta)x), F_{ABx_n, Lx_n}(x)\} \\ &\geq \min\{F_{ABx_n, Lx_n}(x), F_{Lx_n, My_n}(x), t(F_{ABx_n, Lx_n}(\beta x), F_{Lx_n, My_n}(x))\} \\ &= \min\{F_{ABx_n, Lx_n}(x), F_{Lx_n, My_n}(x), \min\{F_{ABx_n, Lx_n}(\beta x), F_{Lx_n, My_n}(x)\}\} \\ &\geq \min\{F_{ABx_n, Lx_n}(x), F_{Lx_n, My_n}(x)\}. \end{aligned}$$

If  $\min\{F_{ABx_n, Lx_n}(x), F_{Lx_n, My_n}(x)\} = F_{Lx_n, My_n}(x)$ . Since  $\phi(x) < x$  for all  $x > 0$  and  $F$  is non-decreasing, then we get  $F_{Lx_n, My_n}(\phi(x)) < F_{Lx_n, My_n}(x)$  which is a contradiction.

Therefore  $\min\{F_{ABx_n, Lx_n}(x), F_{Lx_n, My_n}(x)\} = F_{ABx_n, Lx_n}(x)$ . It follows that

$$F_{Lx_n, My_n}(\phi(x)) \geq F_{ABx_n, Lx_n}(x).$$

Letting  $n \rightarrow \infty$  in above inequality, then we have  $F_{ABx_n, Lx_n}(x) \rightarrow F_{z,z}(x) = 1$ . Thus,  $\lim_{n \rightarrow \infty} My_n = z$ . It yields that

$$\lim_{n \rightarrow \infty} Lx_n = \lim_{n \rightarrow \infty} ABx_n = \lim_{n \rightarrow \infty} STy_n = \lim_{n \rightarrow \infty} My_n = z, \text{ where } z \in AB(X) \cap ST(X).$$

i.e., the pairs  $(L, AB)$  and  $(M, ST)$  share the  $(CLR_{(AB)(ST)})$  property. □

**Remark 1.8.** It can be pointed that Lemma 1.7 generalizes Lemma 3.2 in [12], from four self-mappings to six self-mappings. Simultaneously, it is straight forward to notice that Lemma 1.7 improves Lemma 1 of [13] from symmetric spaces to Menger spaces.

## 2 Main results

Before proving our main results, we first list two lemmas which will be used in the following section.

**Lemma 2.1** ([17]). *Suppose that the function  $\phi : [0, \infty) \rightarrow [0, \infty)$  is upper semicontinuous with  $\phi(0) = 0$  and  $\phi(x) < x$  for all  $x > 0$ . Then there exists a strictly increasing continuous function  $\alpha : [0, \infty) \rightarrow [0, \infty)$  such that  $\alpha(0) = 0$  and  $\phi(x) \leq \alpha(x) < x$  for all  $x > 0$ . The function  $\alpha$  is invertible and for any  $x > 0$ ,  $\lim_{n \rightarrow \infty} \alpha^{-n} = \infty$ , where  $\alpha^{-n}$  denotes the  $n$ -th iterates of  $\alpha^{-1}$  and  $\alpha^{-1}$  denotes the inverse of  $\alpha$ .*

**Lemma 2.2** ([17]). *Suppose that  $(X, F)$  is a PM-space and  $\alpha : [0, \infty) \rightarrow [0, \infty)$  is a strictly increasing function satisfying  $\alpha(0) = 0$  and  $\alpha(x) < x$  for all  $x > 0$ . If  $x, y$  are two members in  $X$  such that*

$$F_{x,y}(\alpha(\epsilon)) \geq F_{x,y}(\epsilon),$$

for all  $\epsilon > 0$ , then  $x = y$ .

Now, we state and prove our main result.

**Theorem 2.3.** *Let  $A, B, S, T, L$  and  $M$  be self mappings of a Menger space  $(X, F, t)$ , with continuous  $t$ -norm with  $t(x, x) \geq x$  for all  $x \in [0, 1]$ . Suppose that the inequality (1) of Lemma 1.7 holds. If the pairs  $(L, AB)$  and  $(M, ST)$  share the  $(CLR_{(AB)(ST)})$  property, then  $(L, AB)$  and  $(M, ST)$  have a coincidence point each.*

Moreover, if

(i) both the pairs  $(L, AB)$  and  $(M, ST)$  are weakly compatible.

(ii)  $AB = BA, LA = AL, MS = SM$  and  $ST = TS$ .

Then  $A, B, S, T, L$  and  $M$  have a unique common fixed point.

*Proof.* Since the pairs  $(L, AB)$  and  $(M, ST)$  share the  $(CLR_{(AB)(ST)})$  property, there exist two sequences  $\{x_n\}, \{y_n\}$  in  $X$  such that

$$\lim_{n \rightarrow \infty} Lx_n = \lim_{n \rightarrow \infty} ABx_n = \lim_{n \rightarrow \infty} STy_n = \lim_{n \rightarrow \infty} My_n = z, \text{ where } z \in AB(X) \cap ST(X).$$

Since  $z \in ST(X)$ , there exists a point  $u \in X$  such that  $STu = z$ . Putting  $p = x_n$  and  $q = u$  in inequality (1), it yields that

$$F_{Lx_n, Mu}(\phi(x)) \geq \min\{F_{ABx_n, Lx_n}(x), F_{STu, Mu}(x), F_{STu, Lx_n}(\beta x), F_{ABx_n, Mu}((1 + \beta)x), F_{ABx_n, STu}(x)\}$$

Letting  $n \rightarrow \infty$ , we obtain that

$$\begin{aligned} F_{z, Mu}(\phi(x)) &\geq \min\{F_{z, z}(x), F_{z, Mu}(x), F_{z, z}(\beta x), F_{z, Mu}((1 + \beta)x), F_{z, z}(x)\} \\ &= \min\{1, F_{z, Mu}(x), 1, F_{z, Mu}((1 + \beta)x), 1\} \\ &= F_{z, Mu}(x). \end{aligned}$$

From Lemma 2.1, there exists a strictly increasing continuous function  $\alpha : [0, \infty) \rightarrow [0, \infty)$  such that  $\alpha(0) = 0$  and  $\phi(x) \leq \alpha(x) < x$  for all  $x > 0$ . Therefore,  $F_{z, Mu}(\alpha(x)) \geq F_{z, Mu}(\phi(x)) \geq F_{z, Mu}(x)$ , for all  $x > 0$ . By Lemma 2.2, we obtain that  $z = Mu$ . Hence,  $z = Mu = STu$ , which shows  $u$  is a coincidence point of the pair  $(M, ST)$ .

As  $z \in AB(X)$ , there exists a point  $v \in X$  such that  $ABv = z$ , putting  $p = v, q = y_n$  in inequality (1), we have

$$F_{Lv, My_n}(\phi(x)) \geq \min\{F_{ABv, Lv}(x), F_{STy_n, My_n}(x), F_{STy_n, Lv}(\beta x), F_{ABv, My_n}((1 + \beta)x), F_{ABv, STy_n}(x)\}.$$

Letting  $n \rightarrow \infty$ , we obtain that

$$\begin{aligned} F_{Lv, z}(\phi(x)) &\geq \min\{F_{z, Lv}(x), F_{z, z}(x), F_{z, Lv}(\beta x), F_{z, z}((1 + \beta)x), F_{z, z}(x)\} \\ &= \min\{F_{z, Lv}(x), 1, F_{z, Lv}(\beta x), 1, 1\} \end{aligned}$$

$$= F_{z,Lv}(x).$$

From Lemma 2.1, there exists a strictly increasing continuous function  $\alpha : [0, \infty) \rightarrow [0, \infty)$  such that  $\alpha(0) = 0$  and  $\phi(x) \leq \alpha(x) < x$  for all  $x > 0$ . Therefore,  $F_{Lv,z}(\alpha(x)) \geq F_{Lv,z}(\phi(x)) \geq F_{Lv,z}(x)$ , for all  $x > 0$ . By Lemma 2.2, we obtain that  $z = Lv$ . Hence,  $z = ABv = Lv$ , which shows  $v$  is a coincidence point of the pair  $(L, AB)$ .

Since the pair  $(M, ST)$  is weakly compatible, and by the previous proof,  $z = Mu = STu$ , then  $MSTu = STMu$ , it yields that  $Mz = STz$ . And since the pair  $(L, AB)$  is weakly compatible, and by the previous proof,  $z = ABv = Lv$ , then  $LABv = ABLv$ , it yields that  $Lz = ABz$ . Letting  $p = z, q = u$  in inequality (1), we obtain:

$$\begin{aligned} F_{Lz,Mu}(\phi(x)) &\geq \min\{F_{Mu,Lz}(x), F_{STu,Mu}(x), F_{STu,Lz}(\beta x), F_{ABz,Mu}((1 + \beta)x), F_{ABz,STu}(x)\} \\ &= \min\{F_{z,Lz}(x), F_{z,z}(x), F_{z,Lz}(\beta x), F_{Lz,z}((1 + \beta)x), F_{Lz,z}(x)\} \\ &= \min\{F_{z,Lz}(x), 1, F_{z,Lz}(\beta x), F_{Lz,z}((1 + \beta)x), F_{Lz,z}(x)\} \\ &= F_{z,Lz}(x). \end{aligned}$$

From Lemma 2.1, there exists a strictly increasing continuous function  $\alpha : [0, \infty) \rightarrow [0, \infty)$  such that  $\alpha(0) = 0$  and  $\phi(x) \leq \alpha(x) < x$  for all  $x > 0$ . Therefore,  $F_{Lz,Mu}(\alpha(x)) \geq F_{Lz,Mu}(\phi(x)) \geq F_{Lz,Mu}(x)$ , for all  $x > 0$ . By Lemma 2.2, we obtain that  $Lz = Mu$ . Therefore,  $Lz = Mu = z$ . Thus,  $z = Lz = ABz$ .

Sequentially, letting  $p = z, q = z$  in inequality (1), we obtain:

$$\begin{aligned} F_{Lz,Mz}(\phi(x)) &\geq \min\{F_{Mz,Lz}(x), F_{STz,Mz}(x), F_{STz,Lz}(\beta x), F_{ABz,Mz}((1 + \beta)x), F_{ABz,STz}(x)\} \\ &= \min\{F_{Mz,Lz}(x), F_{Mz,Mz}(x), F_{Mz,Lz}(\beta x), F_{Lz,Mz}((1 + \beta)x), F_{Lz,Mz}(x)\} \\ &= \min\{F_{Mz,Lz}(x), 1, F_{Mz,Lz}(\beta x), F_{Lz,Mz}((1 + \beta)x), F_{Lz,Mz}(x)\} \\ &= F_{Lz,Mz}(x). \end{aligned}$$

From Lemma 2.1, there exists a strictly increasing continuous function  $\alpha : [0, \infty) \rightarrow [0, \infty)$  such that  $\alpha(0) = 0$  and  $\phi(x) \leq \alpha(x) < x$  for all  $x > 0$ . Therefore,  $F_{Lz,Mz}(\alpha(x)) \geq F_{Lz,Mz}(\phi(x)) \geq F_{Lz,Mz}(x)$ , for all  $x > 0$ . By Lemma 2.2, we obtain  $Lz = Mz = z = STz = ABz$ . Hence,  $AB, ST, L$  and  $M$  have a common fixed point  $z$ .

Letting  $p = z, q = Sz$  in inequality (1), we obtain:

$$\begin{aligned} F_{z,Sz}(\phi(x)) &\geq \min\{F_{ABz,Lz}(x), F_{STSz,MSz}(x), F_{STSz,Lz}(\beta x), F_{ABz,MSz}((1 + \beta)x), F_{ABz,STSz}(x)\} \\ &= \min\{F_{z,z}(x), F_{Sz,Sz}(x), F_{Sz,z}(\beta x), F_{z,Sz}((1 + \beta)x), F_{z,Sz}(x)\} \\ &= \min\{1, 1, F_{Sz,z}(\beta x), F_{z,Sz}((1 + \beta)x), F_{z,Sz}(x)\} \\ &= F_{z,Sz}(x). \end{aligned}$$

From Lemma 2.1, there exists a strictly increasing continuous function  $\alpha : [0, \infty) \rightarrow [0, \infty)$  such that  $\alpha(0) = 0$  and  $\phi(x) \leq \alpha(x) < x$  for all  $x > 0$ . Therefore,  $F_{z,Sz}(\alpha(x)) \geq F_{z,Sz}(\phi(x)) \geq F_{z,Sz}(x)$ , for all  $x > 0$ . By Lemma 2.2, we obtain  $z = Sz$ . Thus,  $z = Sz = STz = TSz = Tz$ .

Letting  $p = Az, q = z$  in inequality (1), we obtain:

$$\begin{aligned} F_{Az,z}(\phi(x)) &\geq \min\{F_{ABAz,LAz}(x), F_{STz,Mz}(x), F_{STz,Lz}(\beta x), F_{ABAz,Mz}((1 + \beta)x), F_{ABAz,STz}(x)\} \\ &= \min\{F_{Az,Az}(x), F_{z,z}(x), F_{z,z}(\beta x), F_{Az,z}((1 + \beta)x), F_{Az,z}(x)\} \\ &= \min\{1, 1, 1, F_{Az,z}((1 + \beta)x), F_{Az,z}(x)\} \\ &= F_{Az,z}(x). \end{aligned}$$

From Lemma 2.1, there exists a strictly increasing continuous function  $\alpha : [0, \infty) \rightarrow [0, \infty)$  such that  $\alpha(0) = 0$  and  $\phi(x) \leq \alpha(x) < x$  for all  $x > 0$ . Therefore,  $F_{Az,z}(\alpha(x)) \geq F_{Az,z}(\phi(x)) \geq F_{Az,z}(x)$ , for all  $x > 0$ . By Lemma 2.2, we obtain  $z = Az$ . Hence,  $z = Az = ABz = BAz = Bz$ . Thus, combing with the above proof, we have  $z = Az = Bz = Lz = Mz = Sz = Tz$ .

Then,  $A, B, S, T, L$  and  $M$  have a common fixed point  $z$ .

(Uniqueness). Assume that  $t$  is another common fixed point of  $A, B, S, T, L$  and  $M$ . It follows that  $t = At = Bt = Lt = Mt = St = Tt$ . Letting  $p = z, q = t$  in inequality (1), we obtain:

$$F_{Lz,Mt}(\phi(x)) \geq \min\{F_{ABz,Lz}(x), F_{STt,Mt}(x), F_{STt,Lz}(\beta x), F_{ABz,Mt}((1 + \beta)x), F_{ABz,STt}(x)\}$$

$$\begin{aligned}
 &= \min\{F_{z,z}(x), F_{t,t}(x), F_{t,z}(\beta x), F_{z,t}((1 + \beta)x), F_{z,t}(x)\} \\
 &= \min\{1, 1, F_{t,z}(\beta x), F_{z,t}((1 + \beta)x), F_{z,t}(x)\} \\
 &= F_{z,t}(x).
 \end{aligned}$$

It yields that  $F_{z,t}(\phi(x)) \geq F_{z,t}(x)$ . From Lemma 2.1, there exists a strictly increasing continuous function  $\alpha : [0, \infty) \rightarrow [0, \infty)$  such that  $\alpha(0) = 0$  and  $\phi(x) \leq \alpha(x) < x$  for all  $x > 0$ . Therefore,  $F_{z,t}(\alpha(x)) \geq F_{z,t}(\phi(x)) \geq F_{z,t}(x)$ , for all  $x > 0$ . By Lemma 2.2, we obtain  $z = t$ . Thus,  $A, B, S, T, L$  and  $M$  have a unique common fixed point  $z$ .  $\square$

If we take  $B = T = I$  ( $I \equiv$  the identity mapping on  $X$ ), we have:

**Corollary 2.4.** *Let  $A, S, L$  and  $M$  be self mappings of a Menger space  $(X, F, t)$ , with continuous  $t$ -norm with  $t(x, x) \geq x$  for all  $x \in [0, 1]$ . Suppose that the inequality*

$$F_{Lp,Mq}(\phi(x)) \geq \min\{F_{Ap,Lp}(x), F_{Sq,Mq}(x), F_{Sq,Lp}(\beta x), F_{Ap,Mq}((1 + \beta)x), F_{Ap,Sq}(x)\}$$

holds. If the pairs  $(L, A)$  and  $(M, S)$  share the  $(CLR_{(AS)})$  property, then  $(L, A)$  and  $(M, S)$  have a coincidence point each. Moreover, if both the pairs  $(L, A)$  and  $(M, S)$  are weakly compatible,  $LA = AL, MS = SM$ , then  $A, S, L$  and  $M$  have a unique common fixed point.

**Remark 2.5.** *Theorem 2.3 generalizes Theorem 3.3 of [15]. Here, completeness of Menger space  $(X, F, t)$ , the containment of  $L(X) \subseteq ST(X), M(X) \subseteq AB(X)$  and the closure of  $AB(X)$  or  $ST(X)$  can be replaced by  $(CLR_{(AB)(ST)})$  property of the pairs  $(L, AB)$  and  $(M, ST)$ . Simultaneously,  $BL = LB, MS = SM$  can be replaced with  $AL = LA, MS = SM$ . Of course, Theorem 2.3 also improves Theorem 3.4 of [15], the containment of  $L(X) \subseteq ST(X), M(X) \subseteq AB(X)$  and the closure of  $AB(X)$  and the property (E.A.) of  $(M, ST)$  or the closure of  $ST(X)$  and the property (E.A.) of  $(L, AB)$  can be removed,  $BL = LB, MS = SM$  can be replaced with  $AL = LA, MS = SM$ . Meanwhile, Theorem 2.3 improves results of [13] from symmetric spaces to Menger spaces. In other respect, Theorem 2.3 improves Theorem 3.4 of [12], from four self mappings to six self-mappings in Menger spaces. To above all, we can deduce that the inequality (1) is different from that of [12].*

Now, we illustrate an example to show that our main result of Theorem 2.3 is valid, and at the same time, the existing literature does not hold.

**Example 2.6.** *Let  $X = [0, 3)$ , with the metric  $d$  defined by  $d(x, y) = |x - y|$  and define  $F_{x,y}(u) = H(u - d(x, y))$  for all  $x, y \in X, u > 0$  (refer to [15, Example 3.2]). It is obviously that the space  $X$  is not complete, since it is not a closed interval in real numbers  $\mathbb{R}$ . We define  $t(a, b) = \min\{a, b\}$  for all  $a, b \in [0, 1]$ . Let  $A, B, S, T, L$  and  $M$  be self mappings on  $X$  defined as*

$$\begin{aligned}
 A(x) &= \begin{cases} 2 - x, & 0 \leq x < 1, \\ 2, & 1 \leq x < 3. \end{cases} & B(x) &= \begin{cases} 2, & x = 0, \\ 1/x, & 0 < x < 1, \\ 2, & 1 \leq x < 3. \end{cases} \\
 S(x) &= \frac{1}{2}x + 1, \quad 0 \leq x < 3, & T(x) &= \frac{1}{3}(x + 4), \quad 0 \leq x < 3.
 \end{aligned}$$

And  $L(x) = M(x) = 2$ . By a simple calculation, we can check the conditions in Theorem 2.3 hold true.

(1) Consider two sequences  $\{x_n\} = \{1 + \frac{1}{n}\}$  and  $\{y_n\} = \{2 - \frac{1}{n}\}$ . Then  $Lx_n = 2, ABx_n = A(2) = 2, My_n = 2, STy_n = S(\frac{1}{3}(y_n + 4)) = \frac{1}{6}(2 - \frac{1}{n}) + \frac{5}{3} = 2 - \frac{1}{6n}$ , which consequently it yields that

$$\lim_{n \rightarrow \infty} Lx_n = \lim_{n \rightarrow \infty} ABx_n = \lim_{n \rightarrow \infty} STy_n = \lim_{n \rightarrow \infty} My_n = 2, \text{ where } 2 \in AB(X) \cap ST(X).$$

Therefore, the pairs  $(L, AB)$  and  $(M, ST)$  have the  $(CLR_{(AB)(ST)})$  property. It is obvious that  $ST(X) = [\frac{5}{3}, \frac{13}{6})$  is not closed in  $X$ .

(2) Check the inequality (1). Let  $\phi : [0, \infty] \rightarrow [0, \infty]$  defined by  $\phi(t) = kt$ ,  $k \in (0, 1)$  be an upper semicontinuous function with  $\phi(0) = 0$  and  $\phi(t) < t$  for all  $t > 0$ . For any  $p, q \in \mathbb{R}$  and  $x > 0$ , we have  $F_{Lp, Mq}(kx) = 1$  and

$$\begin{aligned} & \min\{F_{ABp, Lp}(x), F_{STq, Mq}(x), F_{STq, Lp}(\beta x), F_{ABp, Mq}((1 + \beta)x), F_{ABp, STq}(x)\} \\ &= \min\{1, F_{STq, Mq}(x), F_{STq, Lp}(\beta x), 1, F_{ABp, STq}(x)\} \\ &= \min\{F_{STq, Mq}(x), F_{STq, Lp}(\beta x), F_{ABp, STq}(x)\} \\ &= \min\{F_{STq, 2}(x), F_{STq, 2}(\beta x), F_{2, STq}(x)\} \\ &= F_{STq, 2}(x) \\ &\leq 1. \end{aligned}$$

Then  $F_{Lp, Mq}(kx) \geq \min\{F_{ABp, Lp}(x), F_{STq, Mq}(x), F_{STq, Lp}(\beta x), F_{ABp, Mq}((1 + \beta)x), F_{ABp, STq}(x)\}$  holds for  $x, y \in X$ ,  $\beta \geq 1$  and  $x > 0$ .

(3) It is obviously that  $L(x) = AB(x) = \{2\}$  for  $1 \leq x < 3$ , and  $L(AB)(x) = (AB)L(x) = \{2\}$ . Then the weakly compatibility of the pair  $(L, AB)$  is satisfied. And  $M(x) = ST(x) = \{2\}$  for  $x = 1$ , and  $M(ST)(x) = 2 = ST(2) = (ST)M(x)$  for  $x = 1$ . Then the weakly compatibility of the pair  $(M, ST)$  is also satisfied.

(4)  $AB = BA = \{2\}$ ,  $ST = TS = \frac{1}{6}x + \frac{5}{3}$ ,  $LA = LA = \{2\}$ , and  $SM = MS = \{2\}$ .

Thus, all the conditions of Theorem 2.3 are satisfied, but 2 is a unique common fixed point of  $A, B, S, T, L$  and  $M$ .

**Theorem 2.7.** Let  $A, B, S, T, L$  and  $M$  be self mappings of a Menger space  $(X, F, t)$ , with continuous  $t$ -norm with  $t(x, x) \geq x$  for all  $x \in [0, 1]$ . Suppose that the conditions (i)-(iv) of Lemma 1.7 hold. Then  $(L, AB)$  and  $(M, ST)$  have a coincidence point each.

Moreover, if

(i) both the pairs  $(L, AB)$  and  $(M, ST)$  are weakly compatible.

(ii)  $AB = BA, LA = AL, MS = SM$  and  $ST = TS$ .

Then  $A, B, S, T, L$  and  $M$  have a unique common fixed point.

*Proof.* Since the conditions (i)-(iv) of Lemma 1.7 hold, thus the pairs  $(L, AB)$  and  $(M, ST)$  have the  $(CLR_{(AB)(ST)})$  property. The rest of proof can be completed along the routine of the proof of Theorem 2.3. In order to avoid tedious presentation, we omit the rest of proof.  $\square$

It can be noted that the conclusion in Example 2.6 does not hold if we utilize Theorem 2.7. Indeed, conditions (3) of Lemma 1.7 are not satisfied, i.e., the closure of  $ST(X)$ . So we give another example, and obtain the corresponding uniqueness of common fixed point which was proposed in Theorem 2.7.

**Example 2.8.** Assume the same conditions of Example 2.6, except that

$$S(x) = T(x) = \begin{cases} \frac{8}{3}, & x = 0, \\ \frac{4}{3}, & x \in (0, 1], \\ \frac{2x+2}{3}, & x \in (1, 3). \end{cases}$$

And  $L(x) = M(x) = 2$ . First, we can check the conditions in Lemma 1.7.

(1)  $L(X) = 2, ST(X) = [\frac{4}{3}, \frac{22}{9}]$ . Thus,  $L(X) \subseteq ST(X)$ .

(2) Take  $x_n = 1 - 1/n \in X$ . Then  $\lim_{n \rightarrow \infty} AB(x_n) = \lim_{n \rightarrow \infty} AB(1 - 1/n) = \{2\}$  and  $\lim_{n \rightarrow \infty} L(x_n) = \lim_{n \rightarrow \infty} L(1 - 1/n) = \{2\}$ . Therefore,  $\lim_{n \rightarrow \infty} AB(x_n) = \lim_{n \rightarrow \infty} L(x_n)$ . It yields that the pair  $(L, AB)$  satisfies the property  $(E, A)$ .

(3)  $ST(X) = [\frac{4}{3}, \frac{22}{9}]$ . It is a closed interval in  $\mathbb{R}$ , of course, it is closed subset of  $X$ .

(4) Check the inequality (1). Let  $\phi : [0, \infty] \rightarrow [0, \infty]$  defined by  $\phi(t) = kt$ ,  $k \in (0, 1)$  be an upper semicontinuous function with  $\phi(0) = 0$  and  $\phi(t) < t$  for all  $t > 0$ . For any  $p, q \in \mathbb{R}$  and  $x > 0$ , we have  $F_{Lp, Mq}(kx) = 1$  and

$$\min\{F_{ABp, Lp}(x), F_{STq, Mq}(x), F_{STq, Lp}(\beta x), F_{ABp, Mq}((1 + \beta)x), F_{ABp, STq}(x)\}$$

$$\begin{aligned}
 &= \min\{1, F_{STq,Mq}(x), F_{STq,Lp}(\beta x), 1, F_{ABp,STq}(x)\} \\
 &= \min\{F_{STq,Mq}(x), F_{STq,Lp}(\beta x), F_{ABp,STq}(x)\} \\
 &= \min\{F_{STq,1}(x), F_{STq,1}(\beta x), F_{1,STq}(x)\} \\
 &= F_{STq,1}(x) \\
 &\leq 1.
 \end{aligned}$$

Then  $F_{Lp,Mq}(kx) \geq \min\{F_{ABp,Lp}(x), F_{STq,Mq}(x), F_{STq,Lp}(\beta x), F_{ABp,Mq}((1 + \beta)x), F_{ABp,STq}(x)\}$  holds for  $x, y \in X, \beta \geq 1$  and  $x > 0$ .

Besides, we should check weak compatibility of  $(M, ST)$ .  $M(x) = ST(x) = \{2\}$  for  $x = 2$ , and  $M(ST)(x) = M(2) = 2 = ST(2) = (ST)M(x)$  for  $x = 2$ . Then the weakly compatibility of the pair  $(M, ST)$  is also satisfied.

$$\text{At the last, } ST = TS = \begin{cases} \frac{4}{3}, & x = 0, \\ \frac{22}{9}, & x \in (0, 1], \\ \frac{4x+10}{9}, & x \in (1, 3). \end{cases} \quad \text{and } SM = MS = \{\frac{4}{3}\}.$$

Thus all the conditions of Theorem 2.7 are satisfied. From Theorem 2.7,  $A, B, S, T, L$  and  $M$  have a unique common fixed point in  $X$ . In fact, by the definition of  $A, B, S, T, L$  and  $M, 2$  is the unique common fixed point of  $A, B, S, T, L$  and  $M$  in  $X$ .

Instead of the  $(CLR_{(AB)(ST)})$  property of  $(L, AB)$  and  $(M, ST)$  in Theorem 2.3, we utilize the common property (E.A.) to obtain fixed point theorems.

**Theorem 2.9.** Let  $A, B, S, T, L$  and  $M$  be self mappings of a Menger space  $(X, F, t)$ , with continuous  $t$ -norm with  $t(x, x) \geq x$  for all  $x \in [0, 1]$ . Suppose that the inequality (1) of Lemma 1.7 and the following hypotheses hold:

- (a) the pairs  $(L, AB)$  and  $(M, ST)$  have the common property (E.A.);
- (b)  $ST(X)$  and  $AB(X)$  is closed subset of  $X$ .

Then  $(L, AB)$  and  $(M, ST)$  have a coincidence point each.

Moreover, if

- (i) both the pairs  $(L, AB)$  and  $(M, ST)$  are weakly compatible.
- (ii)  $AB = BA, LA = AL, MS = SM$  and  $ST = TS$ .

Then  $A, B, S, T, L$  and  $M$  have a unique common fixed point.

*Proof.* If the pairs  $(L, AB)$  and  $(M, ST)$  have the common property (E.A.), then there exist two sequences  $\{x_n\}$  and  $\{y_n\}$  in  $X$  such that

$$\lim_{n \rightarrow \infty} Lx_n = \lim_{n \rightarrow \infty} ABx_n = \lim_{n \rightarrow \infty} STy_n = \lim_{n \rightarrow \infty} My_n = z, \text{ for some } z \in X.$$

Since  $ST(X)$  is closed, then  $\lim_{n \rightarrow \infty} STy_n = z = STu$  for some  $u \in X$ . And  $AB(X)$  is closed, then  $\lim_{n \rightarrow \infty} ABx_n = z = ABv$  for some  $v \in X$ . The rest of the proof can runs on the lines of Theorem 2.3. □

**Corollary 2.10.** The result of Theorem 2.9 holds if condition (b') is substituted for condition (b):

(b')  $\overline{L(X)} \subseteq ST(X)$  and  $\overline{M(X)} \subseteq AB(X)$  where  $\bar{\cdot}$  denoted the closure.

**Corollary 2.11.** The result of Theorem 2.9 holds if condition (b'') is substituted for condition (b):

(b'')  $L(X)$  and  $M(X)$  is closed subset of  $X$ , and  $L(X) \subseteq ST(X)$  and  $M(X) \subseteq AB(X)$ .

**Example 2.12.** Assume the same conditions of Example 2.6 hold, except that

$$S(x) = T(x) = \begin{cases} \frac{8}{3}, & x = 0, \\ \frac{4}{3}, & x \in (0, 1], \\ \frac{6x+2}{5}, & x \in (1, 3). \end{cases}$$



$ST(X) = \{\frac{8}{3}\} \cup \{\frac{4}{3}\} \cup (\frac{2}{5}, 4)$  is not closed subset of  $X$ , but conditions  $(b')$  and  $(b'')$  of Corollary 2.10 and Corollary 2.11 are satisfied, 2 is a unique common fixed point of  $A, B, S, T, L$  and  $M$ .

**Remark 2.13.** Theorem 2.9 improves Theorem 3.4 in [15]. Here, containment of  $L(X) \subseteq ST(X)$ ,  $M(X) \subseteq AB(X)$ , and the closure of  $ST(X)$ , property  $(E.A)$  of  $(L, AB)$  are replaced by the closure of  $ST(X)$  and  $AB(X)$ , and the common property  $(E.A)$  of two pairs  $(L, AB)$  and  $(M, ST)$ . Of course,  $LB = BL, MT = TM$  are also replaced by  $LA = AL, MS = SM$ . Indeed, the common property  $(E.A)$  of two pairs  $(L, AB)$  and  $(M, ST)$  can be deduced from containment of  $L(X) \subseteq ST(X)$  and property  $(E.A)$  of  $(L, AB)$ .

In order to show that the common property  $(E.A)$  of two pairs  $(L, AB)$  and  $(M, ST)$  can be deduced from containment of  $L(X) \subseteq ST(X)$  and property  $(E.A)$  of  $(L, AB)$ , we propose the following theorem.

**Theorem 2.14.** Let  $A, B, S, T, L$  and  $M$  be self mappings of a Menger space  $(X, F, t)$ , with continuous  $t$ -norm with  $t(x, x) \geq x$  for all  $x \in [0, 1]$ . Suppose that the inequality (1) of Lemma 1.7 and the following hypotheses hold:

- (i)  $L(X) \subseteq ST(X)$ ;
- (ii)  $ST(X)$  is closed in  $X$  and  $(L, AB)$  satisfies the property  $(E.A)$ .

Then  $(L, AB)$  and  $(M, ST)$  share the common property  $(E.A)$ .

*Proof.* Since  $(L, AB)$  satisfies the property  $(E.A)$ , there exists a sequence  $\{x_n\} \subset X$  such that

$$\lim_{n \rightarrow \infty} Lx_n = \lim_{n \rightarrow \infty} ABx_n = z, \text{ for some } z \in X.$$

Since  $L(X) \subseteq ST(X)$ , for each  $x_n$ , there exists a corresponding  $y_n \in X$  such that  $Lx_n = STy_n$ . Therefore, we have

$$\lim_{n \rightarrow \infty} Lx_n = \lim_{n \rightarrow \infty} ABx_n = \lim_{n \rightarrow \infty} STy_n = z, \text{ for some } z \in X.$$

It suffices to show that  $\lim_{n \rightarrow \infty} My_n = z$ . Substituting  $p = x_n, q = y_n$  in inequality (1), we obtain

$$\begin{aligned} F_{Lx_n, My_n}(\phi(x)) &\geq \min\{F_{ABx_n, Lx_n}(x), F_{STy_n, My_n}(x), F_{STy_n, Lx_n}(\beta x), F_{ABx_n, My_n}((1 + \beta)x), F_{ABx_n, STy_n}(x)\} \\ &= \min\{F_{ABx_n, Lx_n}(x), F_{Lx_n, My_n}(x), F_{Lx_n, Lx_n}(\beta x), F_{ABx_n, My_n}((1 + \beta)x), F_{ABx_n, Lx_n}(x)\} \\ &= \min\{F_{ABx_n, Lx_n}(x), F_{Lx_n, My_n}(x), 1, F_{ABx_n, My_n}((1 + \beta)x), F_{ABx_n, Lx_n}(x)\} \\ &\geq \min\{F_{ABx_n, Lx_n}(x), F_{Lx_n, My_n}(x), t(F_{ABx_n, Lx_n}(\beta x), F_{Lx_n, My_n}(x))\} \\ &= \min\{F_{ABx_n, Lx_n}(x), F_{Lx_n, My_n}(x), \min\{(F_{ABx_n, Lx_n}(\beta x), F_{Lx_n, My_n}(x))\}\} \\ &\geq \min\{F_{ABx_n, Lx_n}(x), F_{Lx_n, My_n}(x)\}. \\ &= F_{ABx_n, Lx_n}(x). \end{aligned}$$

Letting  $n \rightarrow \infty$ , we obtain that  $F_{ABx_n, Lx_n}(x) \rightarrow F_{z,z}(x) = 1$ . So,

$$\lim_{n \rightarrow \infty} Lx_n = \lim_{n \rightarrow \infty} ABx_n = \lim_{n \rightarrow \infty} STy_n = \lim_{n \rightarrow \infty} My_n = z, \text{ for some } z \in X.$$

Thus,  $(L, AB)$  and  $(M, ST)$  share the common property  $(E.A)$ . □

**Remark 2.15.** Theorem 2.14 shows that our common property  $(E.A)$  of two pairs  $(L, AB)$  and  $(M, ST)$  is weaker than containment of  $L(X) \subseteq ST(X)$  and property  $(E.A)$  of  $(L, AB)$ . It is namely that Theorem 2.9 is indeed a generalization of Theorem 3.4 in [15].

Next, we extend common fixed point theorem of six self-mappings to six finite families of self mappings in Menger spaces.

**Theorem 2.16.** Let  $\{A_i\}_{i=1}^m, \{B_r\}_{r=1}^n, \{S_k\}_{k=1}^e, \{T_h\}_{h=1}^f, \{L_j\}_{j=1}^c$  and  $\{M_v\}_{v=1}^d$  be six finite families of self mappings of a Menger space  $(X, F, t)$ , with continuous  $t$ -norm with  $t(x, x) \geq x$  for all  $x \in [0, 1]$  where

$A = A_1A_2 \cdots A_m$ ,  $B = B_1B_2 \cdots B_n$ ,  $S = S_1S_2 \cdots S_e$ ,  $T = T_1T_2 \cdots T_f$ ,  $L = L_1L_2 \cdots L_c$  and  $M = M_1M_2 \cdots M_d$ . Suppose that the inequality (1) of Lemma 1.7 holds. If the pairs  $(L, AB)$  and  $(M, ST)$  share the  $(CLR_{(AB)(ST)})$  property, then  $(L, AB)$  and  $(M, ST)$  have a coincidence point each.

Moreover, if

(i) both the pairs  $(L, AB)$  and  $(M, ST)$  are weakly compatible.

(ii)  $AB = BA$ ,  $LA = AL$ ,  $MS = SM$  and  $ST = TS$ .

Then  $A, B, S, T, L$  and  $M$  have a unique common fixed point.

*Proof.* The proof can be completed on the lines of Theorem 4.2 in Imdad et al. [12].  $\square$

When  $A_1 = A_2 = \cdots = A_m = A$ ,  $B_1 = B_2 = \cdots = B_n = B$ ,  $S_1 = S_2 = \cdots = S_e = S$ ,  $T_1 = T_2 = \cdots = T_f = T$ ,  $L_1 = L_2 = \cdots = L_c = L$  and  $M_1 = M_2 = \cdots = M_d = M$ , then we have the following corollary:

**Corollary 2.17.** Let  $A, B, S, T, L$  and  $M$  of self mappings of a Menger space  $(X, F, t)$ , with continuous  $t$ -norm with  $t(x, x) \geq x$  for all  $x \in [0, 1]$ . Suppose that

(i) the pairs  $(L^c, A^mB^n)$  and  $(M^d, S^eT^f)$  share the  $(CLR_{(A^mB^n)(S^eT^f)})$  property,

(ii) there exists an upper semicontinuous function  $\phi : [0, \infty) \rightarrow [0, \infty)$  with  $\phi(0) = 0$  and  $\phi(x) < x$  for all  $x > 0$  such that

$$F_{L^c p, M^d q}(\phi(x)) \geq \min\{F_{A^m B^n p, L^c p}(x), F_{S^e T^f q, M^d q}(x), F_{S^e T^f q, L^c p}(\beta x), F_{A^m B^n p, M^d q}((1 + \beta)x), F_{A^m B^n p, S^e T^f q}(x)\},$$

for all  $p, q \in X$ ,  $\beta \geq 1$  and  $x > 0$ . Then  $(L^c, A^mB^n)$  and  $(M^d, S^eT^f)$  have a coincidence point each.

Moreover, if  $AB = BA$ ,  $LA = AL$ ,  $MS = SM$  and  $ST = TS$ . Then  $A, B, S, T, L$  and  $M$  have a unique common fixed point.

**Remark 2.18.** Theorem 2.16 can be taken as generalization of Theorem 2.3. When  $m = 1$ ,  $n = 1$ ,  $p = 1$ ,  $q = 1$ ,  $c = 1$ ,  $d = 1$ , Theorem 2.16 reduces to Theorem 2.3. It is worth noting that here  $AB = BA$ ,  $LA = AL$ ,  $MS = SM$  and  $ST = TS$  are weaker than the pairwise community of  $\{A_i\}_{i=1}^m$ ,  $\{B_r\}_{r=1}^n$ ,  $\{S_k\}_{k=1}^p$ ,  $\{T_h\}_{h=1}^q$ ,  $\{L_j\}_{j=1}^c$  and  $\{M_v\}_{v=1}^d$ . This can also be found from the process of proof in Theorem 4.2 in [12]. In fact, Theorem 2.16 improves results of Imdad et al. [13], Liu [15], and Imdad et al. [12].

### 3 Application to metric spaces

In this section, by means of results in the above section, we propose corresponding common fixed point theorem in metric spaces. In fact, every metric space  $(X, d)$  can be taken as a particular Menger space by  $F : X \times X \rightarrow \mathbb{R}$  defined by  $F_{x,y}(t) = H(t - d(x, y))$  for all  $x, y \in X$  in [12].

**Theorem 3.1.** Let  $A, B, S, T, L$  and  $M$  be self mappings of a metric space  $(X, d)$ . Suppose that

(i) the pairs  $(L, AB)$  and  $(M, ST)$  share the  $(CLR_{(AB)(ST)})$  property,

(ii) there exists an upper semicontinuous function  $\phi : [0, \infty) \rightarrow [0, \infty)$  with  $\phi(0) = 0$  and  $\phi(x) < x$  for all  $x > 0$  such that

$$d(Lp, Mq) \leq \phi(\max\{d(ABp, Lp), d(STq, Mq), \frac{1}{\beta}d(STq, Lp), \frac{1}{1 + \beta}d(ABp, Mq), d(ABp, STq)\}) \quad (2)$$

for all  $p, q \in X$ ,  $\beta \geq 1$  and  $x > 0$ , then  $(L, AB)$  and  $(M, ST)$  have a coincidence point each.

Moreover, if

(i) both the pairs  $(L, AB)$  and  $(M, ST)$  are weakly compatible.

(ii)  $AB = BA$ ,  $LA = AL$ ,  $MS = SM$  and  $ST = TS$ .

Then  $A, B, S, T, L$  and  $M$  have a unique common fixed point.

*Proof.* Define  $F_{x,y}(t) = H(t - d(x, y))$ ,  $t(a, b) = \min\{a, b\}$ , for all  $a, b \in [0, 1]$ . Then this metric space can be taken as a Menger space. It is worth noting that Theorem 3.1 enjoys the assumption of Theorem 2.3,

including inequality (2) reduces to inequality (1) in Theorem 2.3. For all  $p, q \in X$  and  $x > 0$ ,  $F_{Lp, Mq}(\phi(x))=1$  if  $\phi(x) > d(Lp, Mq)$ , inequality (1) in Theorem 2.3 is obviously true. Otherwise, if  $\phi(x) \leq d(Lp, Mq)$ , then

$$x \leq \max\{d(ABp, Lp), d(STq, Mq), \frac{1}{\beta}d(STq, Lp), \frac{1}{1+\beta}d(ABp, Mq), d(ABp, STq)\}$$

which implies that inequality (1) in Theorem 2.3 is satisfied. Therefore, in each respect, condition of Theorem 2.3 is satisfied. And the conclusion of Theorem 3.1 can be obtained.  $\square$

Take as a particular case, set  $\phi(x) = kx$ , for  $k \in (0, 1)$ . We derive the following corollary.

**Corollary 3.2.** *Let  $A, B, S, T, L$  and  $M$  be self mappings of a metric space  $(X, d)$ . Suppose that (i) the pairs  $(L, AB)$  and  $(M, ST)$  share the  $(CLR_{(AB)(ST)})$  property, (ii) there exists  $k \in (0, 1)$  such that*

$$d(Lp, Mq) \leq k \max\{d(ABp, Lp), d(STq, Mq), \frac{1}{\beta}d(STq, Lp), \frac{1}{1+\beta}d(ABp, Mq), d(ABp, STq)\} \quad (3)$$

for all  $p, q \in X$ ,  $\beta \geq 1$  and  $x > 0$ , then  $(L, AB)$  and  $(M, ST)$  have a coincidence point each.

Moreover, if

(i) both the pairs  $(L, AB)$  and  $(M, ST)$  are weakly compatible.

(ii)  $AB = BA, LA = AL, MS = SM$  and  $ST = TS$ .

Then  $A, B, S, T, L$  and  $M$  have a unique common fixed point.

**Remark 3.3.** *Theorem 3.1 improves the results of [12, 13], [15]. In this paper, there are corresponding common fixed point theorems for six self-mappings whereas for four self-mappings in [12]. It is important that our condition be weaker than that in [12]. On one hand, since our function  $\phi$  is upper semicontinuous, it is more general than that in [12]. On the other hand, letting  $\phi(x) = kx$ , Theorem 3.1 reduces to Corollary 3.2. At the same time, taking  $B = T = I$ , inequality (3) can be turned as follows:*

$$d(Lp, Mq) \leq k \max\{d(Ap, Lp), d(Sq, Mq), \frac{1}{\beta}d(Sq, Lp), \frac{1}{1+\beta}d(Ap, Mq), d(Ap, Sq)\}. \quad (4)$$

Inequality (4) is more weaker than inequality (5.1) of Theorem 5.1 in [12]. At the same time, we can find some applications in dynamic programming similar to [21, 22].

### Competing interests

The authors declare that they have no competing interests.

### Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

**Acknowledgement:** Xiao-lan Liu was partially supported by National Natural Science Foundation of China (Grant No.61573010), Science Research Fund of Science and Technology Department of Sichuan Province (No.2017JY0125) and Scientific Research Project of Sichuan University of Science and Engineering (No.2017RCL54). Mi Zhou was partially supported by Hainan Provincial Natural Science Foundation of China (Grant No.118MS081) and Science and Technology Cooperation Project of Sanya City (No.2018YD13).

## References

[1] Menger K., Statistical metrics, *Proc. Natl. Acad. Sci. USA*, 1942, 28, 535-537.

- [2] Schweizer B., Sklar A., Statistical metric spaces, *Pacific J. Math.*, 1960, 10, 313-334.
- [3] Sehgal V.M., Bharucha-Reid A.T., Fixed points of contraction mappings in PM-spaces, *Math. System Theory*, 1972, 6, 97-102.
- [4] Sessa S., On a weak commutativity condition of mappings in fixed point considerations, *Publ. Inst. Math. (Beograd) (NS)*, 1982, 32, 149-153.
- [5] Jungck G., Compatible mappings and common fixed points, *Int. J. Math. Sci.*, 1986, 9(4), 771-779.
- [6] Mishra S.N., Common fixed points of compatible mappings in PM-spaces, *Math. Japon.*, 1991, 36, 283-289.
- [7] Jungck G., Rhoades B.E., Fixed points for set valued functions without continuity, *Indian J. Pure Appl. Math.*, 1998, 29, 227-238.
- [8] Aamri M., Moutawakil D. El, Some new common fixed point theorems under strict contractive conditions, *J. Math. Anal. Appl.*, 2002, 270, 181-188.
- [9] Liu Y., Wu J., Li Z., Common fixed points of single-valued and multivalued maps, *Int. J. Math. Sci.*, 2005, 19, 3045-3055.
- [10] Kubiacyk L., Sharma S., Some common fixed point theorems in Menger space under strict contractive conditions, *Southeast Asian Bull. Math.*, 2008, 32, 117-124.
- [11] Sintunavarat W., Kumam P., Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces, *J. Appl. Math.*, 2011, Article ID 637958, 14 page.
- [12] Imdad M., Pant B.D., Chauhan S., Fixed point theorems in Menger spaces using the  $CLR_{ST}$  property and applications, *J. Nonlinear Anal. Optim.*, 2012, 3(2), 225-237.
- [13] Imdad M., Chauhan S., Kadelburg Z., Vetro C., Fixed point theorems for non-self mappings in symmetric spaces under  $\varphi$ -weak contractive conditions and an application to functional equations in dynamic programming, *Applied Mathematics and Computation*, 2014, 227, 469-479.
- [14] Singh B., Jain S., A fixed point theorem in Menger space through weak compatibility, *J. Math. Anal. Appl.*, 2005, 301, 439-448.
- [15] Liu X.L., Common fixed point theorem for six self mappings in Menger space, *J. Math. Anal. Appl.*, 2013, 404, 351-361.
- [16] Karapinar E., Patel D.K., Imdad M., Gopal D., Some nonunique common fixed point theorems in symmetric spaces through  $CLR_{ST}$  property, *Int. J. Math. Math. Sci.*, 2013, Article ID 753965, 8 pages.
- [17] Chang T.H., Common fixed point theorems in Menger spaces, *Bull. Inst. Math. Sinica.*, 1994, 22, 17-29.
- [18] Deepmala and Pathak H.K., Study on existence of solutions for some nonlinear functional-integral equations with applications, *Mathematical Communications*, 2013, 18, 97-107.
- [19] Deepmala and Pathak H.K., A study on some problems on existence of solutions for nonlinear functional-integral equations, *Acta Mathematica Scientia*, 2013, 33 B(5), 1305-1313.
- [20] Deepmala, Existence theorems for solvability of a functional equation arising in dynamic programming, *International Journal of Mathematics and Mathematical Sciences*, 2014, Article ID 706585, 9 pages.
- [21] Pathak H.K. and Deepmala, Existence and uniqueness of solutions of functional equations arising in dynamic programming, *Applied Mathematics and Computation*, 2012, 218 (13), 7221-7230.
- [22] Pathak H.K. and Deepmala, Common fixed point theorems for PD-operator pairs under Relaxed conditions with applications, *Journal of Computational and Applied Mathematics*, 2013, 239 103-113.