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COMMON FIXED POINT THEOREMS FOR MAPPINGS SATISFYING COMMON PROPERTY (E.A.) IN SYMMETRIC SPACES

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Abstract

In this paper, common fixed point theorems for mappings satisfying a generalized contractive condition are obtained in symmetric spaces by using the notion of common property (E.A.). In the process, a host of previously known results are improved and generalized. We also derive results on common fixed point in probabilistic symmetric spaces.

1 Introduction and preliminaries

The practice of coining weaker forms of commutativity to ensure the existence of common fixed point for self mappings on metric spaces is still on. The weak conditions of commutativity of a pair of selfmappings was initiated by Sessa [18] with the introduction of the notion of weakly commuting pair. Later on, Jungck [13] enlarged the class of weakly commuting mappings by introducing the notion of compatible mappings which was further widened by Jungck [14] with the notion of weakly compatible mappings. This concept of weak compatibility is most optimal and widely used concept among all the weak commutativity concepts thus far. The existing literature contains numerous weak conditions of commutativity whose systematic survey (up to 2001) is available in Murthy [16].

In recent years, Hicks and Rhoades [9] established some common fixed point theorems in symmetric spaces (see also [19]) using the fact that full force of metric conditions are not required in the proofs of certain metrical fixed point theorems. Recently, Ali and Imdad [3] proved some common fixed point theorems for mappings satisfying common property (E.A.) by replacing the usual involved contractive condition with a suitable implicit function and also highlighted it's unifying power with the help of numerous examples. On the other hand, Branciari [4] initiated a study of

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contractive conditions of integral type, giving an integral version of Banach contraction principle (extendable to more general contractive conditions) whereas Aliouche [2] established a common fixed point theorem for self mappings in a symmetric space under a contractive condition of integral type. Recently, Di Bari and Vetro [7] established some common fixed point theorems for mappings satisfying generalized contractive condition which include integral type contractive conditions. In 2008, Cho et al. [6] proved interesting fixed point theorems for nonexpansive type mappings which rectify and generalize some results of Imdad et al. [11] and also carry out a systematic study of crucial conditions such as $(W_3), (W_4), (HE)$ and $(1C)$ (to be defined shortly) which can be fruitful to the researchers of this domain.

In this paper, we prove some common fixed point theorems for mappings satisfying generalized contractive conditions in symmetric spaces. While proving our results, we utilize the idea of common property (E.A.) keeping in view the fact that it buys the required containment of range of one mapping into the other.

The following definitions and results will be needed in the sequel.

Definition 1. *A symmetric d (introduced by K. Menger in 1928) on a non-empty set X is a function $d : X \times X \to [0, \infty)$ which satisfies (for all x,y in X)*

(i) $d(x, y) = 0$ *if and only if* $x = y$,

(ii) $d(x, y) = d(y, x)$.

Let d be a symmetric on a set X, $\varepsilon > 0$ and $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$. A topology $\tau(d)$ on X is given by the sets U (along with empty set) in which for each $x \in U$, $B(x, \varepsilon) \subset U$ for some $\varepsilon > 0$. A set $S \subset X$ is a neighborhood of $x \in X$ if and only if there is a U containing x such that $x \in U \subset S$. A symmetric d is said to be a semi-metric if for each $x \in X$ and for each $\varepsilon > 0$, $B(x, \varepsilon)$ is a neighborhood of x in the topology $\tau(d)$. Thus a symmetric (resp. a semi-metric) space X is a topological space whose topology $\tau(d)$ on X is induced by a symmetric (resp. a semi-metric) d. Notice that $\lim_{n,+\infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology $\tau(d)$.

Since a symmetric space is not essentially Hausdorff, therefore in order to prove fixed point theorems some additional axioms are required. The following axioms, which are available in Galvin and Shore [8], Wilson [20], Aliouche [2] and Imdad and Soliman [12], are relevant to this presentation.

Definition 2. *(W_3 [20]): Given $\{x_n\}$, x and y in X, $d(x_n, x) \to 0$ and $d(x_n, y) \to 0$ imply $x = y$.*

Definition 3. *(W_4 [20]): Given $\{x_n\}$, $\{y_n\}$ and $x \in X$, $d(x_n, x) \to 0$ and $d(x_n, y_n) \to 0$ imply $d(y_n, x) \to 0$.*

Definition 4. *(HE [2]): Given $\{x_n\}$, $\{y_n\}$ and $x \in X$, $d(x_n, x) \to 0$ and $d(y_n, x) \to 0$ imply $d(x_n, y_n) \to 0$.*

Definition 5. *(1C [8]): A symmetric d is said $1-$continuous if $\lim_{n,+\infty} d(x_n, x) = 0$ implies $\lim_{n,+\infty} d(x_n, y) = d(x, y)$ for every $x, y \in X$.*

Clearly, (W4) implies (W3) but other possible implications amongst $(W_3), (W_4)$, (HE) are not generally true. Also notice that (1C) implies (W_3).

As usual, a sequence $\{x_n\}$ in a symmetric space (X, d) is said to be d-Cauchy sequence if it satisfies the standard metric condition. It is interesting to note that in a symmetric space, Cauchy convergence criterion is not a necessary condition for the convergence of a sequence but this criterion becomes a necessary condition if symmetric d is suitably restricted (see Wilson [20]). In 1972, Burke [5] furnished an illustrative example to show that a convergent sequence in a semi-metric space need not admit a Cauchy subsequence. Burke was able to formulate an equivalent condition under which every convergent sequence in a semi-metric space admits a Cauchy subsequence. There are several concepts of completeness in semi-metric spaces, e.g. $S-$completeness, d-Cauchy completeness, strong and weak completeness whose details are available in Wilson [20], but we omit the details and give only the following definition.

Definition 6. *A symmetric space (X, d) is $S-$complete if for every $d-$Cauchy sequence $\{x_n\}$, there exists some $x \in X$ such that $\lim_{n,+\infty} d(x_n, x) = 0$.*

Lastly, we list the remaining relevant definitions to our presentation which can be found in [3, 12] and references mentioned therein.

Definition 7. *We recall that a pair of self mappings (f, g) defined on a symmetric (or semi-metric) space (X, d) is said to be*

(i) compatible if $\lim_{n,+\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n,+\infty} fx_n = \lim_{n,+\infty} gx_n = t$ for some $t \in X$,

(ii) non-compatible if there exists some sequence $\{x_n\}$ such that $\lim_{n,+\infty} fx_n = \lim_{n,+\infty} gx_n = t$ for some $t \in X$ but $\lim_{n,+\infty} d(fgx_n, gfx_n)$ is either non-zero or non-existent,

(iii) tangential (or satisfy the property (E.A.)) if there exists some sequence $\{x_n\}$ such that $\lim_{n,+\infty} fx_n = \lim_{n,+\infty} gx_n = t$ for some $t \in X$,

(iv) weakly commuting (or partially commuting or coincidentally commuting) if the pair commutes on the set of coincidence points,

(v) occasionally weakly compatible (see [1, 15]) if there is at least one coincidence point x of (f, g) in X at which (f, g) commutes.

Clearly compatible as well as non-compatible mappings satisfy the property (E.A.).

Definition 8. *Two pairs of self mappings (A, B) and (S, T) defined on a symmetric (or semi-metric) space (X, d) are said to satisfy the common property (E.A.) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n,+\infty} Ax_n = \lim_{n,+\infty} Bx_n = \lim_{n,+\infty} Sy_n = \lim_{n,+\infty} Ty_n = t$, for some $t \in X$.*

For more on (E.A.) and common property (E.A.), we refer to [3, 12].

2 Results in symmetric spaces

In this section we prove some common fixed point theorems for mappings satisfying the common property (E.A.).

Let (X, d) be a symmetric (or semi-metric) space and A, B, S and T be self mappings of X. For all $x, y \in X$, we denote

$$m(x, y; A, B, S, T) := \max\{d(Sx, By), d(Ty, Ax), d(Ty, By), d(Sx, Ax), d(Sx, Ty)\}.$$

Let $G, \psi : [0, \infty) \to [0, \infty)$ and we consider the following properties:

(i) G is nondecreasing, continuous and $G(0) = 0 < G(t)$ for every $t > 0$,

(ii) ψ is nondecreasing, right continuous and $\psi(t) < t$ for every $t > 0$.

In the following, we denote $\mathcal{G} := \{G : [0, \infty) \to [0, \infty) : G$ satisfies $(i)\}$ and $\Psi := \{\psi : [0, \infty) \to [0, \infty) : \psi$ satisfies $(ii)\}$.

Theorem 1. *Let (X, d) be a symmetric space satisfying $(1C)$ and (HE). Let S, T, A_k, for $k = 1, 2, \ldots,$ be self mappings of X. Assume that there exist $G \in \mathcal{G}$ and $\psi \in \Psi$ such that*

$$G(d(A_1x, A_ky)) \leq \psi(G(m(x, y; A_1, A_k, S, T))), \tag{1}$$

for all $x, y \in X$ and $k > 1$.

Suppose that the pairs (A_1, S) and (A_k, T) for $k > 1$, share the common property (E.A.), $S(X)$ and $T(X)$ are closed subset of X. Then the pairs (A_1, S) and (A_k, T) for $k > 1$ have a coincidence point. Moreover, S, T and all the A_k have a unique common fixed point provided both the pairs (A_1, S) and (A_k, T) for $k > 1$ are weakly compatible.

Proof. Since the pairs (A_1, S) and (A_k, T), for $k > 1$, share the common property (E.A.), there exist two sequences $\{x_n\}, \{y_n\}$ in X such that

$$\lim_{n,+\infty} d(A_1x_n, t) = \lim_{n,+\infty} d(Sx_n, t) = \lim_{n,+\infty} d(A_ky_n, t) = \lim_{n,+\infty} d(Ty_n, t) = 0,$$

for $k > 1$ and some $t \in X$. By (HE), we have

$$\lim_{n,+\infty} d(A_1x_n, Sx_n) = \lim_{n,+\infty} d(A_ky_n, Ty_n) = 0.$$

Since $S(X)$ is a closed subset of X, $\lim_{n,+\infty} Sx_n = t \in S(X)$. Therefore, there exists a point $u \in X$ such that $Su = t$. Subsequently, we have $\lim_{n,+\infty} d(A_1x_n, Su) = \lim_{n,+\infty} d(Sx_n, Su) = \lim_{n,+\infty} d(A_ky_n, Su) = \lim_{n,+\infty} d(Ty_n, Su) = 0$.

Now, we assert that $A_1u = Su$. If not, then using (1), we have

$$\begin{aligned} &G(d(A_1u, A_ky_n)) \\ &\quad \leq \psi(G(\max\{d(Su, A_ky_n), d(Ty_n, A_1u), d(Ty_n, A_ky_n), d(Su, A_1u), d(Su, Ty_n)\})). \end{aligned}$$

Making $n \to +\infty$ and using (1C) and (HE), we get

$$G(d(A_1u, Su)) \leq \psi(G(\max\{0, d(Su, A_1u), 0, d(Su, A_1u), 0\})) < G(d(A_1u, Su)),$$

a contradiction. Hence $A_1u = Su$. Therefore, u is a coincidence point of the pair (A_1, S).

Since $T(X)$ is a closed subset of X, $\lim_{n,+\infty} Ty_n = t \in T(X)$. Therefore, there exists a point $w \in X$ such that $Tw = t$. Now, we assert that $A_k w = Tw$. If not, then using (1), we have

$$\begin{aligned} &G(d(A_1x_n, A_kw)) \\ &\leq \psi(G(\max\{d(Sx_n, A_kw), d(Tw, A_1x_n), d(Tw, A_kw), d(Sx_n, A_1x_n), d(Sx_n, Tw)\})). \end{aligned}$$

Letting $n \to +\infty$ and using (1C) and (HE), we get

$$G(d(Tw, A_kw)) \leq \psi(G(d(Tw, A_kw))) < G(d(Tw, A_kw)),$$

a contradiction. Hence $A_kw = Tw$, which shows that w is a coincidence point of the pair (A_k, T). Since the pair (A_1, S) is weakly compatible and $A_1u = Su$, hence $A_1t = A_1Su = SA_1u = St$.

Now, we assert that t is a common fixed point of the pair (A_1, S). Suppose $A_1t \neq t$, then using (1), we have

$$\begin{aligned} &G(d(A_1t, t)) = G(d(A_1t, Tw)) = G(d(A_1t, A_kw)) \\ &\leq \psi(G(\max\{d(St, A_kw), d(Tw, A_1t), d(Tw, A_kw), d(St, A_1t), d(St, Tw)\})) \\ &< G(d(A_1t, t)), \end{aligned}$$

a contradiction. As the pair (A_k, T) is also weakly compatible and $A_kw = Tw$, therefore $A_kt = A_kTw = TA_kw = Tt$. Suppose that $A_kt \neq t$, then using (1), we again arrive at a contradiction to our assumption. Therefore, $A_kt = t$, which shows that t is a common fixed point of the pair (A_k, T) and henceforth t is a common fixed point of both the pairs (A_1, S) and (A_k, T). Uniqueness of t is an easy consequence of (1). This completes the proof. □

Example 1. *Consider $X = (-1, 1)$ equipped with the symmetric defined by $d(x, y) = (x - y)^2$ for all $x, y \in X$. Define self mappings A_k, S and T on X as*

$$A_1x = \begin{cases} \frac{3}{5} & \text{if } -1 < x < -1/2 \\ \frac{x}{4} & \text{if } -1/2 \leq x \leq 1/2 \\ \frac{3}{5} & \text{if } 1/2 < x < 1, \end{cases} \quad A_kx = \begin{cases} \frac{3}{5} & \text{if } -1 < x < -1/2 \\ -\frac{x}{4k} & \text{if } -1/2 \leq x \leq 1/2 \quad (k > 1) \\ \frac{3}{5} & \text{if } 1/2 < x < 1, \end{cases}$$

$$Sx = \begin{cases} \frac{3}{4} & \text{if } -1 < x < -1/2 \\ \frac{x}{2} & \text{if } -1/2 \leq x \leq 1/2 \\ -\frac{3}{4} & \text{if } 1/2 < x < 1, \end{cases} \quad Tx = \begin{cases} -\frac{3}{4} & \text{if } -1 < x < -1/2 \\ -\frac{x}{2} & \text{if } -1/2 \leq x \leq 1/2 \\ \frac{3}{4} & \text{if } 1/2 < x < 1. \end{cases}$$

Consider sequences $\{x_n\} = \{\frac{1}{n+1}\}$ and $\{y_n\} = \{-\frac{1}{n+1}\}$ in X. Clearly,

$$\lim_{n,+\infty} A_1x_n = \lim_{n,+\infty} Sx_n = \lim_{n,+\infty} A_ky_n = \lim_{n,+\infty} Ty_n = 0$$

which shows that (A_1, S) and (A_k, T) for $k > 1$ share the common property (E.A.).

Set G to be identity mapping and $\psi(s) = cs$ with $c \in (\frac{12}{15}, 1)$. By a routine calculation, one can verify that the condition (1) holds. Also, $A_1(X) = \{\frac{3}{5}\} \cup [-\frac{1}{8}, \frac{1}{8}] \not\subset S(X) = \{\frac{-3}{4}, \frac{3}{4}\} \cup [-\frac{1}{4}, \frac{1}{4}]$ and $A_k(X) = \{\frac{3}{5}\} \cup [-\frac{1}{8k}, \frac{1}{8k}] \not\subset T(X) = \{\frac{-3}{4}, \frac{3}{4}\} \cup [-\frac{1}{4}, \frac{1}{4}]$. Therefore, all the conditions of Theorem 1 are satisfied and 0 is the unique common fixed point of the pairs (A_1, S) and (A_k, T). Here it is worth noting that none of the theorems, e.g. Di Bari and Vetro [7], Pathak et al. [17], Zhu et al. [21], can be used in the context of this example as Theorem 1 never requires any condition on the containment of ranges of the involved mappings. Further, all the mappings involved in this example are discontinuous.

The following example shows that the axioms (1C) and (HE) cannot be dropped in Theorem 1. The idea of this example appears in Cho et al. [6].

Example 2. *Consider $X = [0, \infty)$ equipped with the symmetric defined by $d(0, 0) = 0$ and*

$$d(x, y) = \begin{cases} |x - y| & \text{if } x \neq 0 \text{ and } y \neq 0 \\ \frac{1}{x} & \text{if } x \neq 0. \end{cases}$$

Define self mappings S, T and A_k, for $k \geq 1$, on X as

$$Sx = Tx = x, \quad A_k x = \begin{cases} \frac{1}{3}x & \text{if } x > 0 \\ \frac{1}{3} & \text{if } x = 0. \end{cases}$$

Thus, (X, d) is a symmetric space where d does not satisfy (1C) and (HE) for $\{x_n\} = \{n\}$, $\{y_n\} = \{n+1\}$. Now, define G to be the identity mapping, $\psi(s) = cs$ and consider the sequence $\{x_n\} = \{n\}$. By a routine calculation, one can verify that the condition (1) holds with $c \in (\frac{1}{2}, 1)$. Notice that the pairs (A_1, S) and (A_k, T) for $k > 1$, share the common property (E.A.), $S(X)$ and $T(X)$ are closed subsets of X. But the pairs (A_1, S) and (A_k, T) (for $k > 1$) have no coincidence points and henceforth no common fixed point.

By choosing $A_1 = A$ and $A_k = B$ in the above Theorem 1, we get the following corollary for two pairs of mappings which is an improvement over the corresponding result of Di Bari and Vetro [7].

Corollary 1. *Let (X, d) be a symmetric space satisfying $(1C)$ and (HE). Let A, B, S and T be self mappings of X. Assume that there exist $G \in \mathcal{G}$ and $\psi \in \Psi$ such that*

$$G(d(Ax, By)) \leq \psi(G(m(x, y; A, B, S, T))), \tag{2}$$

for all $x, y \in X$. Suppose that the pairs (A, S) and (B, T) share the common property (E.A.), $S(X)$ and $T(X)$ are closed subset of X. Then the pairs (A, S) and (B, T) have a coincidence point. Moreover, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Next, we consider the function $G : [0, \infty) \to [0, \infty)$, defined by $G(s) = \int_0^s \varphi(t)dt$ (where $\varphi : [0, \infty) \to [0, \infty)$ is a function Lebesgue integrable such that $\int_0^\varepsilon \varphi(t)dt > 0$ whenever $\varepsilon > 0$. The function $G \in \mathcal{G}$ and from Corollary 1 we deduce the following corollary.

Corollary 2. *Let (X,d) be a symmetric space satisfying $(1C)$ and (HE). Let A, B, S and T be self mappings of X. Assume that there exists $\psi \in \Psi$ such that*

$$\int_0^{d(Ax,By)} \varphi(t)dt \leq \psi(\int_0^{m(x,y;A,B,S,T)} \varphi(t)dt). \tag{3}$$

Suppose that the pairs (A,S) and (B,T) share the common property $(E.A.)$, $S(X)$ and $T(X)$ are closed subset of X. Then the pairs (A,S) and (B,T) have a coincidence point. Moreover, A, B, S and T have a unique common fixed point provided both the pairs (A,S) and (B,T) are weakly compatible.

Since a pair of mappings without any point of coincidence can also be realized as a weakly compatible pair (as requirements of the definition are vacuously satisfied), therefore we get the following result. Precisely, it may be pointed out that the axioms (1C) and (HE) are not required if we consider occasionally weakly compatible mappings.

Theorem 2. *Let (X,d) be a symmetric space and let S, T, A_k, for $k = 1, 2, \ldots,$ be self mappings of X. Assume that there exist $G \in \mathcal{G}$ and $\psi \in \Psi$ such that*

$$G(d(A_1x, A_ky)) \leq \psi(G(m(x, y; A_1, A_k, S, T))), \tag{4}$$

for all $x, y \in X$. Then S, T and all the A_k have a unique common fixed point provided both the pairs (A_1, S) and (A_k, T) for $k > 1$ are occasionally weakly compatible.

Proof. Since the pairs (A_1, S) and (A_k, T) are each occasionally weakly compatible, there exist points $x, y_k \in X$ such that $A_1x = Sx$ and $A_ky_k = Ty_k$ for $k > 1$. We claim that $A_1x = A_ky_k$ for all $k > 1$. If it is not, then by (4) we have

$$\begin{aligned} G(d(A_1x, A_ky_k)) \leq &\psi(G(m(x, y_k; A_1, A_k, S, T))) \\ = &\psi(G(d(A_1x, A_ky_k))) < G(d(A_1x, A_ky_k)), \end{aligned}$$

a contradiction. Therefore, $A_1x = A_ky_k$ for all $k > 1$. Moreover, if there is another point z such that $A_1z = Sz$, then in view of earlier deduction $A_1z = A_ky_k$ for all $k > 1$ which in turn yields that $A_1z = A_1x = Sx = Sz$. Hence $w = A_1x = Sx$ is the unique point of coincidence of A_1 and S. By Lemma 1 of [15], w is the unique common fixed point of A_1 and S. By symmetry, $r_k = A_ky_k = Ty_k$ is the unique common fixed point of A_k and T for $k > 1$. Since $w = r_k$ for all $k > 1$, we obtain that w is the unique common fixed point of S, T and all the A_k. □

For all $x, y \in X$ and $0 < \alpha < 2$, we denote

$$\begin{aligned} &m_1(x, y; A, B, S, T) := \\ &\max\{d(Sx, Ty), \alpha\frac{d(Ax, Sx) + d(By, Ty)}{2}, \alpha\frac{d(Ax, Ty) + d(By, Sx)}{2}\}. \end{aligned}$$

Now, we are ready to state and prove the following result.

Corollary 3. *Let (X, d) be a symmetric space satisfying $(1C)$ and (HE). Let S, T, A_k, for $k = 1, 2, \ldots$, be self mappings of X. Assume that there exists $G \in \mathcal{G}$ and $\psi \in \Psi$ such that*

$$G(d(A_1x, A_ky)) \leq \psi(G(m_1(x, y; A_1, A_k, S, T))), \tag{5}$$

for all $x, y \in X$, $0 < \alpha < 2$ and $k > 1$.

Suppose that the pairs (A_1, S) and (A_k, T) for $k > 1$, share the common property (E.A.), $S(X)$ and $T(X)$ are closed subset of X. Then the pairs (A_1, S) and (A_k, T) for $k > 1$ have a coincidence point. Moreover, S, T and all the A_k have a unique common fixed point provided both the pairs (A_1, S) and (A_k, T) for $k > 1$ are weakly compatible.

Proof. Since $m_1(x, y; A_1, A_k, S, T) \leq m(x, y; A_1, A_k, S, T)$, the proof of this corollary follows from Theorem 1. □

Now, we get the following corollary which improves the corresponding results of Cho et al. [6] and also rectifies the relevant results of Imdad et al.[11].

Corollary 4. *Let (X, d) be a symmetric space satisfying $(1C)$ and (HE). Let A, B, S and T be self mappings of X. Assume that there exists $F : [0, \infty) \to [0, \infty)$ that is increasing, continuous and $F(0) = 0 < F(t)$ for every $t > 0$, such that*

$$F(d(Ax, By)) < F(m_1(x, y; A, B, S, T)), \tag{6}$$

for all $x, y \in X$.

Suppose that the pairs (A, S) and (B, T), share the common property (E.A.), $S(X)$ and $T(X)$ are closed subset of X. Then the pairs (A, S) and (B, T) for $k > 1$ have a coincidence point. Moreover, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof. Since the pairs (A_1, S) and (A_k, T), for $k > 1$, share the common property (E.A.), then there exist two sequences $\{x_n\}, \{y_n\}$ in X such that

$$\lim_{n,+\infty} d(A_1x_n, t) = \lim_{n,+\infty} d(Sx_n, t) = \lim_{n,+\infty} d(A_ky_n, t) = \lim_{n,+\infty} d(Ty_n, t) = 0,$$

for $k > 1$ and some $t \in X$. By (HE), we have

$$\lim_{n,+\infty} d(A_1x_n, Sx_n) = \lim_{n,+\infty} d(A_ky_n, Ty_n) = 0.$$

Since $S(X)$ is a closed subset of X, $\lim_{n,+\infty} Sx_n = t \in S(X)$. Therefore, there exists a point $u \in X$ such that $Su = t$. Subsequently, we have

$$\lim_{n,+\infty} d(A_1x_n, Su) = \lim_{n,+\infty} d(Sx_n, Su) = \lim_{n,+\infty} d(A_ky_n, Su) = \lim_{n,+\infty} d(Ty_n, Su) = 0.$$

Now, we assert that $A_1u = Su$. If not, then using (6), we have

$$\begin{aligned}&F(d(A_1u, A_ky_n))\\&< F(\max\{d(Su, Ty_n), \alpha\frac{d(A_1u, Su) + d(A_ky_n, Ty_n)}{2}, \alpha\frac{d(A_1u, Ty_n) + d(A_ky_n, Su)}{2}\}).\end{aligned}$$

Making $n \to +\infty$ and using (1C) and (HE), we get

$$F(d(A_1u, Su)) \leq F(\max\{0, \alpha\frac{d(A_1u, Su)}{2}, \alpha\frac{d(A_1u, Su)}{2}\}) < F(d(A_1u, Su)),$$

a contradiction. Hence $A_1u = Su$. Therefore, u is a coincidence point of the pair (A_1, S). The rest of the proof of this theorem can be completed on the lines of the proof of Theorem 1, hence details are omitted. □

3 Results via an implicit relation

Let Φ be the family of lower semi-continuous functions (l.s.c.) $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(ϕ_1) $\phi(t, 0, t, 0, 0, t) > 0$, for all $t > 0$,

(ϕ_2) $\phi(t, 0, 0, t, t, 0) > 0$, for all $t > 0$,

(ϕ_3) $\phi(t, t, 0, 0, t, t) > 0$, for all $t > 0$.

Example 3. *Define* $\phi(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ *as*

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \psi(\max\{t_2, t_3, t_4, t_5, t_6\}),$$

where $\psi(t) < t$ *for all* $t \in \mathbb{R}_+$. *Then,* ϕ *satisfies the conditions* $(\phi_1), (\phi_2), (\phi_3)$ *and so* $\phi \in \Phi$.

Further, the following examples of $\phi \in \Phi$ are indeed contained in Ali and Imdad [3].

Example 4. *Define* $\phi(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ *as*

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k(\max\{t_2, t_3, t_4, t_5, t_6\}),$$

where $k \in [0, 1)$.

Example 5. *Define* $\phi(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ *as*

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k(\max\{t_2, t_3, t_3t_5, t_4t_6\}),$$

where $k \in [0, 1)$.

Example 6. *Define* $\phi(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ *as*

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k(\max\{t_2^2, t_3t_4, t_5t_6, t_3t_5, t_4t_6\})^{\frac{1}{2}},$$

where $k \in [0, 1)$.

Example 7. *Define* $\phi(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ *as*

$$\begin{aligned}\phi(t_1, t_2, t_3, t_4, t_5, t_6) = & t_1 - \alpha[\beta \max\{t_2, t_3, t_4, t_5, t_6\} \\ & + (1 - \beta)(\max\{t_2^2, t_3t_4, t_5t_6, t_3t_6, t_4t_5\})^{\frac{1}{2}}],\end{aligned}$$

where $\alpha \in [0, 1)$ *and* $\beta \geq 0$.

Example 8. *Define* $\phi(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ *as*

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - \alpha \max\{t_2^2, t_3^2, t_4^2\} - \beta \max\{t_3 t_5, t_4 t_6\} - \gamma t_5 t_6,$$

where $\alpha, \beta, \gamma \geq 0$ *and* $\alpha + \gamma < 1$.

Example 9. *Define* $\phi(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ *as*

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = (1 + \alpha t_2)t_1 - \alpha \max\{t_3 t_4, t_5 t_6\} - \beta \max\{t_2, t_3, t_4, t_5, t_6\},$$

where $\alpha \geq 0$ *and* $\beta \in [0, 1)$.

Example 10. *Define* $\phi(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ *as*

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \alpha t_2 - \beta \max\{t_3, t_4\} - \gamma \max\{t_3 + t_4, t_5 + t_6\},$$

where $\alpha, \beta, \gamma \geq 0$ *and* $\alpha + \beta + 2\gamma < 1$.

Example 11. *Define* $\phi(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ *as*

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \theta(\max\{t_2, t_3, t_4, t_5, t_6\}),$$

where $\theta(t) : \mathbb{R}_+ \to \mathbb{R}$ *is an upper semi-continuous function such that* $\theta(0) = 0$ *and* $\theta(t) < t$ *for all* $t > 0$.

Example 12. *Define* $\phi(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ *as*

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \theta(t_2, t_3, t_4, t_5, t_6),$$

where $\theta(t) : \mathbb{R}_+^5 \to \mathbb{R}$ *is an upper semi-continuous function such that, for each* $t > 0$, $\max\{\theta(0, t, 0, 0, t), \theta(0, 0, t, t, 0), \theta(t, 0, 0, t, t)\} < t$.

Example 13. *Define* $\phi(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ *as*

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - \theta(t_2^2, t_3 t_4, t_5 t_6, t_3 t_6, t_4 t_5),$$

where $\theta(t) : \mathbb{R}_+^5 \to \mathbb{R}$ *is an upper semi-continuous function such that, for each* $t > 0$, $\max\{\theta(0, 0, 0, t, 0), \theta(0, 0, 0, 0, t), \theta(t, 0, t, 0, 0)\} < t$.

Example 14. *Define* $\phi(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ *as*

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = \begin{cases} t_1 - \alpha t_2 - \beta \frac{t_3^2 + t_4^2}{t_3 + t_4} - \gamma(t_5 + t_6) & \textit{if } t_3 + t_4 \neq 0 \\ t_1 & \textit{if } t_3 + t_4 = 0, \end{cases}$$

where $\alpha, \beta, \gamma \geq 0$ *and* $\beta + \gamma < 1$.

Example 15. *Define* $\phi(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ *as*

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = \begin{cases} t_1^p - k t_2^p - \frac{t_3 t_4^p + t_5 t_6^p}{t_3 + t_4} & \textit{if } t_3 + t_4 \neq 0 \\ t_1 & \textit{if } t_3 + t_4 = 0, \end{cases}$$

where $p \geq 1$ *and* $0 \leq k < \infty$.

Example 16. *Define* $\phi(t_1,t_2,t_3,t_4,t_5,t_6):\mathbb{R}_+^6\to\mathbb{R}$ *as*

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6)=\begin{cases} t_1-\alpha t_2-\beta\frac{t_5^2+t_6^2}{t_5+t_6}-\gamma(t_3+t_4) & \text{if } t_5+t_6\neq 0\\ t_1 & \text{if } t_5+t_6=0,\end{cases}$$

where $\alpha,\beta,\gamma\geq 0$ *and* $\beta+\gamma<1$.

Example 17. *Define* $\phi(t_1,t_2,t_3,t_4,t_5,t_6):\mathbb{R}_+^6\to\mathbb{R}$ *as*

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6)=\begin{cases} t_1-kt_2-\frac{t_3t_4+t_5t_6}{t_5+t_6} & \text{if } t_5+t_6\neq 0\\ t_1 & \text{if } t_5+t_6=0,\end{cases}$$

where $0\leq k<\infty$.

Example 18. *Define* $\phi(t_1,t_2,t_3,t_4,t_5,t_6):\mathbb{R}_+^6\to\mathbb{R}$ *as*

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6)=\begin{cases} t_1-kt_2-\frac{t_3t_4+t_5t_6}{t_3+t_4}-\frac{t_3t_5+t_4t_6}{t_5+t_6} & \text{if } t_3+t_4\neq 0 \text{ and } t_5+t_6\neq 0\\ t_1 & \text{if } t_3+t_4=0 \text{ or } t_5+t_6=0,\end{cases}$$

where $0\leq k<\infty$.

Example 19. *Define* $\phi(t_1,t_2,t_3,t_4,t_5,t_6):\mathbb{R}_+^6\to\mathbb{R}$ *as*

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6)=t_1-\frac{t_3t_4+t_5t_6}{1+t_2}.$$

Example 20. *Define* $\phi(t_1,t_2,t_3,t_4,t_5,t_6):\mathbb{R}_+^6\to\mathbb{R}$ *as*

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6)=t_1-\alpha t_2-\beta\frac{t_3+t_4}{1+t_5t_6},$$

where $\alpha,\beta\in[0,1)$.

Example 21. *Define* $\phi(t_1,t_2,t_3,t_4,t_5,t_6):\mathbb{R}_+^6\to\mathbb{R}$ *as*

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6)=t_1^2-\alpha t_2^2-\beta\frac{t_5t_6}{1+t_3^2+t_4^2},$$

where $\alpha,\beta\geq 0$ *and* $\alpha+\beta<1$.

Example 22. *Define* $\phi(t_1,t_2,t_3,t_4,t_5,t_6):\mathbb{R}_+^6\to\mathbb{R}$ *as*

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6)=t_1^3-\frac{t_3^2t_4^2+t_5^2t_6^2}{1+t_2}.$$

Example 23. *Define* $\phi(t_1,t_2,t_3,t_4,t_5,t_6):\mathbb{R}_+^6\to\mathbb{R}$ *as*

$$\phi(t_1,t_2,t_3,t_4,t_5,t_6)=t_1^3-\alpha t_1^2t_2-\beta t_1t_3t_4-\gamma t_5^2t_6-\eta t_5t_6^2,$$

where $\alpha,\beta,\gamma,\eta\geq 0$ *and* $\alpha+\gamma+\eta<1$.

Since verification of requirements $(\phi_1), (\phi_2)$ and (ϕ_3) for Examples 4–23 are easy, we omit the details.

Theorem 3. *Let (X, d) be a symmetric space satisfying $(1C)$ and (HE). Let S, T, A_k, for $k = 1, 2, \ldots$, be self mappings of X. Assume that there exists $\phi \in \Phi$ such that*

$$\phi(d(A_1x, A_ky), d(Sx, Ty), d(Sx, A_1x), d(Ty, A_ky), d(Sx, A_ky), d(Ty, A_1x)) \leq 0, \qquad (7)$$

for all $x, y \in X$. Suppose that the pairs (A_1, S) and (A_k, T) for $k > 1$ share the common property (E.A.), $S(X)$ and $T(X)$ are closed subsets of X. Then the pairs (A_1, S) and (A_k, T) have a coincidence point. Moreover S, T and all the A_k have a unique common fixed point provided both the pairs (A_1, S) and (A_k, T) for $k > 1$ are weakly compatible.

Proof. Since the pairs (A_1, S) and (A_k, T) share the common property (EA), there exist two sequences $\{x_n\}$, $\{y_n\}$ in X such that for some $z \in X$

$$\lim_{n,+\infty} d(A_1x_n, z) = \lim_{n,+\infty} d(Sx_n, z) = \lim_{n,+\infty} d(A_ky_n, z) = \lim_{n,+\infty} d(Ty_n, z) = 0.$$

By (HE), we have

$$\lim_{n,+\infty} d(A_1x_n, Sx_n) = \lim_{n,+\infty} d(A_ky_n, Ty_n) = 0.$$

Since S(X) is a closed subset of X, $\lim_{n,+\infty} Sx_n = z \in S(X)$. Therefore, there exists a point $p \in X$ such that $Sp = z$. Subsequently, we have

$$\lim_{n,+\infty} d(A_1x_n, Sp) = \lim_{n,+\infty} d(Sx_n, Sp) = \lim_{n,+\infty} d(A_ky_n, Sp) = \lim_{n,+\infty} d(Ty_n, Sp) = 0.$$

Now we assert that $A_1p = Sp$. If not, then using (7), we have

$$\phi(d(A_1p, A_ky_n), d(Sp, Ty_n), d(Sp, A_1p), d(Ty_n, A_ky_n), d(Sp, A_ky_n), d(Ty_n, A_1p)) \leq 0.$$

Letting $n \to \infty$ and using the l.s.c. of ϕ, (1C) and (HE) we get

$$\phi(d(A_1p, Sp), 0, d(Sp, A_1p), 0, 0, d(Sp, A_1p)) \leq 0,$$

a contradiction to (ϕ_1). Hence $A_1p = Sp$. Therefore, p is a coincidence point of the pair (A_1, S).

As $T(X)$ is a closed subset of X, $\lim_{n,+\infty} Ty_n = z \in T(X)$. Therefore, there exists a point $q \in X$ such that $Tq = z$.

Now, we assert that $A_kq = Tq$. If not, then using (7), we have

$$\phi(d(A_1x_n, A_kq), d(Sx_n, Tq), d(Sx_n, A_1x_n), d(Tq, A_kq), d(Sx_n, A_kq), d(Tq, A_1x_n)) \leq 0.$$

Letting $n \to \infty$ and using the l.s.c. of ϕ, (1C) and (HE) we get

$$\phi(d(Tq, A_kq), 0, 0, d(Tq, A_kq), d(Tq, A_kq), 0) \leq 0,$$

a contradiction to (ϕ_2). Hence $A_kq = Tq$, which shows that q is a coincidence point of the pair (A_k, T). Since the pair (A_1, S) is weakly compatible and $A_1p = Sp$, hence $A_1z = A_1Sp = SA_1p = Sz$. Now, we assert that z is a common fixed point of the pair (A_1, S). Suppose $A_1z \neq z$, then using (7) and (ϕ_3), we get a contradiction to our assumption. As the pair (A_k, T) is weakly compatible and $A_kq = Tq$, therefore $A_kz = A_kTq = TA_kq = Tz$. Suppose that $A_kz \neq z$, then using again (7), we arrive at a contradiction to our assumption. Therefore $A_kz = z$ which shows that z is a common fixed point of the pair (A_k, T). Hence z is a common fixed point of both the pairs (A_1, S) and (A_k, T). Uniqueness of z is an easy consequence of (7). This completes the proof. □

Example 24. *If in the setting of Theorem 1,we define $\phi(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as*

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k(\max\{t_2, t_3, t_4, t_5, t_6\}),$$

then all the conditions of Theorem 1 are satisfied with $k \in (\frac{15}{16}, 1)$ enabling us to demonstrate Theorem 3 with the aid of Example 1.

As corollaries, we give the following results which improve the results of Kumar et al. [10], Ali and Imdad [3], Pathak et al. [17] and Zhu et al. [21].

Corollary 5. *Let (X, d) be a symmetric space satisfying $(1C)$ and (HE). Let A, B, S and T be self mappings of X. Assume that there exists $\phi \in \Phi$ such that*

$$\phi(d(Ax, By), d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)) \leq 0, \qquad (8)$$

for all $x, y \in X$. Suppose that the pairs (A, S) and (B, T) share the common property (E.A.), $S(X)$ and $T(X)$ are closed subsets of X. Then the pairs (A, S) and (B, T) have a coincidence point. Moreover A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

We denote $D := d(A_1x, Sx) + d(A_ky, Ty)$ and $D_1 := d(Sx, A_ky) + d(Ty, A_1x)$, for all $x, y \in X$.

Corollary 6. *The conclusion of Theorem 3 will remain true if the inequality (7) of Theorem 3 is replaced by one of the following contractive conditions. For all $x, y \in X$:*

(i) $d(A_1x, A_ky) \leq w \max\{d(Sx, Ty), d(A_1x, Sx), d(A_ky, Ty), d(Sx, A_ky), d(Ty, A_1x)\}$, *where $w \in [0, 1)$;*

(ii) $d(A_1x, A_ky) \leq w \max\{d(Sx, Ty), d(A_1x, Sx), d(A_1x, Sx)d(Sx, A_ky), d(A_ky, Ty)d(Ty, A_1x)\}$, *where $w \in [0, 1)$;*

(iii) $d(A_1x, A_ky) \leq w(\max\{d^2(Sx, Ty), d(A_1x, Sx)d(A_ky, Ty), d(Sx, A_ky)d(Ty, A_1x), d(A_1x, Sx)d(Sx, A_ky), d(A_ky, Ty)d(Ty, A_1x)\})^{\frac{1}{2}}$, *where $w \in [0, 1)$;*

(iv) $d(A_1x, A_ky) \leq \alpha[\beta \max\{d(Sx, Ty), d(A_1x, Sx), d(A_ky, Ty), d(Sx, A_ky), d(Ty, A_1x)\} + (1-\beta)(\max\{d^2(Sx, Ty), d(A_1x, Sx)d(A_ky, Ty), d(Sx, A_ky)d(Ty, A_1x), d(A_1x, Sx)d(Ty, A_1x), d(A_ky, Ty)d(Sx, A_ky)\})^{\frac{1}{2}}]$, *where $\alpha \in [0, 1)$ and $\beta \geq 0$;*

(v) $d^2(A_1x, A_ky) \leq \alpha \max\{d^2(Sx, Ty), d^2(A_1x, Sx), d^2(A_ky, Ty)\} + \beta \max\{d(A_1x, Sx)d(Sx, A_ky), d(A_ky, Ty)d(Ty, A_1x)\} + \gamma d(Sx, A_ky)d(Ty, A_1x)$, *where* $\alpha, \beta, \gamma \geq 0$ *and* $\alpha + \gamma < 1$;

(vi) $(1+\alpha d(Sx, Ty))d(A_1x, A_ky) \leq \alpha \max\{d(A_1x, Sx)d(A_ky, Ty), d(Sx, A_ky)d(Ty, A_1x)\} + \beta \max\{d(Sx, Ty), d(A_1x, Sx), d(A_ky, Ty), d(Sx, A_ky), d(Ty, A_1x)\}$, *where* $\alpha \geq 0$ *and* $\beta \in [0, 1)$;

(vii) $d(A_1x, A_ky) \leq \alpha d(Sx, Ty) + \beta \max\{d(A_1x, Sx), d(A_ky, Ty)\} + \gamma \max\{d(A_1x, Sx) + d(A_ky, Ty), d(Sx, A_ky) + d(Ty, A_1x)\}$, *where* $\alpha, \beta, \gamma \geq 0$ *and* $\alpha + \beta + 2\gamma < 1$;

(viii) $d(A_1x, A_ky) \leq \theta(\max\{d(Sx, Ty), d(A_1x, Sx), d(A_ky, Ty), d(Sx, A_ky), d(Ty, A_1x)\})$ *where* $\theta(t) : \mathbb{R}_+ \to \mathbb{R}$ *is an upper semi-continuous function such that* $\theta(0) = 0$ *and* $\theta(t) < t$ *for each* $t > 0$;

(ix) $d(A_1x, A_ky) \leq \theta(d(Sx, Ty), d(A_1x, Sx), d(A_ky, Ty), d(Sx, A_ky), d(Ty, A_1x))$ *where* $\theta(t) : \mathbb{R}^5_+ \to \mathbb{R}$ *is an upper semi-continuous function such that, for each* $t > 0$, $\max\{\theta(0, t, 0, 0, t), \theta(0, 0, t, t, 0), \theta(t, 0, 0, t, t)\} < t$;

(x) $d^2(A_1x, A_ky) \leq \theta(d^2(Sx, Ty), d(A_1x, Sx)d(A_ky, Ty), d(Sx, A_ky)d(Ty, A_1x), d(A_1x, Sx)d(Ty, A_1x), d(A_ky, Ty)d(Sx, A_ky))$ *where* $\theta(t) : \mathbb{R}^5_+ \to \mathbb{R}$ *is an upper semi-continuous function such that, for each* $t > 0$, $\max\{\theta(0, 0, 0, t, 0), \theta(0, 0, 0, 0, t), \theta(t, 0, t, 0, 0)\} < t$;

(xi)

$$d(A_1x, A_ky) \leq \begin{cases} \alpha d(Sx, Ty) + \beta \frac{d^2(A_1x, Sx) + d^2(A_ky, Ty)}{d(A_1x, Sx) + d(A_ky, Ty)} \\ +\gamma(d(Sx, A_ky) + d(Ty, A_1x)) & \text{if } D \neq 0 \\ 0 & \text{if } D = 0, \end{cases}$$

where $\alpha, \beta, \gamma \geq 0$ *and* $\beta + \gamma < 1$;

(xii)

$$d^p(A_1x, A_ky) \leq \begin{cases} wd^p(Sx, Ty) \\ +\frac{d(A_1x, Sx)d^p(A_ky, Ty) + d(Sx, A_ky)d^p(Ty, A_1x)}{d(A_1x, Sx) + d(A_ky, Ty)} & \text{if } D \neq 0 \\ 0 & \text{if } D = 0, \end{cases}$$

where $p \geq 1$ *and* $0 \leq w < \infty$;

(xiii)

$$d(A_1x, A_ky) \leq \begin{cases} \alpha d(Sx, Ty) + \beta \frac{d^2(Sx, A_ky) + d^2(Ty, A_1x)}{d(Sx, A_ky) + d(Ty, A_1x)} \\ +\gamma(d(A_1x, Sx) + d(A_ky, Ty)) & \text{if } D_1 \neq 0 \\ 0 & \text{if } D_1 = 0, \end{cases}$$

where $\alpha, \beta, \gamma \geq 0$ *and* $\beta + \gamma < 1$;

(xiv)

$$d(A_1x, A_ky) \leq \begin{cases} wd(Sx,Ty) \\ +\frac{d(A_1x,Sx)d(A_ky,Ty)+d(Sx,A_ky)d(Ty,A_1x)}{d(Sx,A_ky)+d(Ty,A_1x)} & \textit{if } D_1 \neq 0 \\ 0 & \textit{if } D_1 = 0, \end{cases}$$

where $0 \leq w < \infty$*;*

(xv)

$$d(A_1x, A_ky) \leq \begin{cases} wd(Sx,Ty) \\ +\frac{d(A_1x,Sx)d(A_ky,Ty)+d(Sx,A_ky)d(Ty,A_1x)}{d(A_1x,Sx)+d(A_ky,Ty)} \\ +\frac{d(A_1x,Sx)d(Sx,A_ky)+d(A_ky,Ty)d(Ty,A_1x)}{d(Sx,A_ky+d(Ty,A_1x)} & \textit{if } D \neq 0 \textit{ and } D_1 \neq 0 \\ 0 & \textit{if } D = 0 \textit{ or } D_1 = 0, \end{cases}$$

where $0 \leq w < \infty$*;*

(xvi) $d(A_1x, A_ky) \leq \frac{d(A_1x,Sx)d(A_ky,Ty)+d(Sx,A_ky)d(Ty,A_1x)}{1+d(Sx,Ty)}$;

(xvii) $d(A_1x, A_ky) \leq \alpha d(Sx,Ty) + \beta\frac{d(A_1x,Sx)+d(A_ky,Ty)}{1+d(Sx,A_ky)d(Ty,A_1x)}$, *where* $\alpha, \beta \in [0,1)$*;*

(xviii) $d^2(A_1x, A_ky) \leq \alpha d^2(Sx,Ty) + \beta\frac{d(Sx,A_ky)d(Ty,A_1x)}{1+d^2(A_1x,Sx)+d^2(A_ky,Ty)}$, *where* $0 \leq \alpha + \beta < 1$*;*

(xix) $d^3(A_1x, A_ky) \leq \frac{d^2(A_1x,Sx)d^2(A_ky,Ty)+d(^2Sx,A_ky)d^2(Ty,A_1x)}{1+d(Sx,Ty)}$;

(xx) $d(^3A_1x, A_ky) \leq \alpha d^2(A_1x,A_ky)d(Sx,Ty) + \beta d(A_1x,A_ky)d(A_1x,Sx)d(A_ky,Ty)$ $+\gamma d^2(Sx,A_ky)d(Ty,A_1x) + \eta d(Sx,A_ky)d^2(Ty,A_1x)$, *where* $\alpha, \beta, \gamma, \eta \geq 0$ *and* $\alpha + \gamma + \eta < 1$.

Proof. The proof of the corollaries corresponding to contractive conditions (i)-(xx) follows from Theorem 3 and Examples 4-23. □

Corollary 7. *Let* (X, d) *be a symmetric space satisfying* $(1C)$ *and* (HE)*. Let* A, B, S *and* T *be self mappings of* X*. Assume that there exist a Lebesgue integrable function* $\phi : \mathbb{R} \to [0,\infty)$ *and a function* $\psi : \mathbb{R}^6_+ \to \mathbb{R}$ *such that, for all* $x, y \in X$*,*

$$\int_0^{\psi(d(Ax,By),d(Sx,Ty),d(Ax,Sy),d(By,Ty),d(Sx,By),d(Ax,Ty))} \varphi(s)ds \leq 0, \tag{9}$$

$$\int_0^{\psi(t,0,t,0,0,t)} \varphi(s)ds > 0 \textit{ for all } t > 0, \tag{10}$$

$$\int_0^{\psi(t,0,0,t,t,0)} \varphi(s)ds > 0 \textit{ for all } t > 0, \tag{11}$$

$$\int_0^{\psi(t,t,0,0,t,t)} \varphi(s)ds > 0 \textit{ for all } t > 0. \tag{12}$$

Suppose that the pairs (A, S) *and* (B, T) *share the common property* $(E.A.)$*,* $S(X)$ *and* $T(X)$ *are closed subsets of* X*. Then the pairs* (A, S) *and* (B, T) *have a coincidence point. Moreover* A, B, S *and* T *have a unique common fixed point provided both the pairs* (A, S) *and* (B, T) *are weakly compatible.*

Proof. The function $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ defined by

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = \int_0^{\psi(t_1,t_2,t_3,t_4,t_5,t_6)} \varphi(s)ds$$

belongs to Φ for conditions (10)-(12) and so condition (9) is a special case of condition (7). Thus, the result follows immediately from Theorem 3. □

To conclude this section, we state a result for occasionally weakly compatible self mappings via implicit relations.

Theorem 4. *Let (X, d) be a symmetric space and let S, T, A_k, for $k = 1, 2, \ldots,$ be self mappings of X. Assume that there exists $\phi \in \Phi$ such that*

$$\phi(d(A_1x, A_ky), d(Sx, Ty), d(Sx, A_1x), d(Ty, A_ky), d(Sx, A_ky), d(Ty, A_1x)) \leq 0, \quad (13)$$

for all $x, y \in X$. Then S, T and all the A_k have a unique common fixed point provided both the pairs (A_1, S) and (A_k, T) for $k > 1$ are occasionally weakly compatible.

4 Results in probabilistic symmetric spaces

A real valued function f on the set of real numbers is called a distribution function if it is nondecreasing, left continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$.

We denote by Δ the set of all distribution functions defined on the set of real numbers and by $\Delta^+ := \{f \in \Delta, f(0) = 0\}$.

We shall use the Heaviside distribution function defined by

$$H(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > 0. \end{cases}$$

In the sequel, we need the following definitions and results which are given in [9].

Definition 9. *A probabilistic symmetric on a non-empty set X is a mapping F from $X \times X$ into Δ^+ satisfying the following conditions*:

(i) $F_{x,y}(t) = H(t)$ if and only if $x = y$,

(ii) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t \in \mathbb{R}$.

The pair (X, F) is a probabilistic symmetric space.

Let F be a probabilistic symmetric on a set X and $\varepsilon > 0$, we write $B(x, \varepsilon) := \{y \in X : F_{x,y}(\varepsilon) > 1 - \varepsilon\}$. A T_1 topology $t(F)$ on X is obtained as follows: $U \in t(F)$ if for each $x \in U$, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$. Now $B(x, \varepsilon)$ may not be a $t(F)$ neighborhood of x. If it is so, then $t(F)$ is said to be topological.

Definition 10. *A probabilistic symmetric space* (X, F) *is complete if for every Cauchy sequence* $\{x_n\}$ *convergent in* X*, i.e. every sequence such that, for all* $t > 0$, $\lim_{n,m,+\infty} F_{x_n,x_m}(t) = 1$*, there exists some* $x \in X$ *with* $\lim_{n,+\infty} F_{x_n,x}(t) = 1$ *for all* $t > 0$.

Hicks and Rhoades [9] proved that each probabilistic symmetric space (X, F) admits a compatible symmetric d such that the probabilistic symmetric F is related to the symmetric d. To be precise:

Theorem 5. *[9] Let* (X, F) *be a probabilistic symmetric space. Define* $d : X \times X \to [0, \infty)$ *as*

$$d(x, y) = \begin{cases} 0 & \text{if } y \in U_x(t, t) \text{ for all } t > 0 \\ \sup\{t : y \notin U_x(t, t), 0 < t < 1\} & \text{otherwise.} \end{cases}$$

Then

(i) $d(x, y) < t$ *if and only if* $F_{x,y}(t) > 1 - t$,

(ii) d *is a compatible symmetric for* $t(F)$,

(iii) (X, F) *is complete if and only if* (X, d) *is* S*-complete,*

(iv) If $t(F)$ *is topological,* d *is a semi-metric.*

The condition (HE) for compatible symmetric d is equivalent to the following condition:

(PHE) For all $t > 0$, $F_{x_n,x}(t) \to 1$ and $F_{y_n,x}(t) \to 1$ imply $F_{x_n,y_n}(t) \to 1$.

We also consider the following condition:

(P1C) For all $t > 0$, $F_{x_n,x}(t) \to 1$ implies $F_{x_n,y}(t) \to F_{x,y}(t)$ for all $y \in X$.

Proposition 1. *Let* (X, F) *be a probabilistic symmetric space and* d *the compatible symmetric for* $t(F)$*. If* (X, F) *satisfies the condition* $(P1C)$*, then* (X, d) *satisfies the condition* $(1C)$.

Proof. If $d(x_n, y) \nrightarrow d(x, y)$, then there exist $\sigma > 0$ and a subsequence of $\{x_n\}$, say $\{x_{n_k}\}$, such that $|d(x_{n_k}, y) - d(x, y)| > \sigma$ for all k. If $d(x_{n_k}, y) < d(x, y) - \sigma = t - \sigma$ holds for infinite values of k, by Theorem 5 (i), we have

$$F_{x_{n_k},y}(t) \geq F_{x_{n_k},y}(t - \sigma) > 1 - t + \sigma.$$

So, being $F_{x,y}(t) \leq 1 - t$, it follows

$$F_{x_{n_k},y}(t) - F_{x,y}(t) > 1 - t + \sigma - 1 + t = \sigma,$$

for infinite values of k, a contradiction. Now, if $d(x_{n_k}, y) > \tau = d(x, y) + \sigma$ for infinite values of k, by Theorem 5 (i), we have

$$F_{x,y}(\tau) \geq F_{x,y}(\tau - \sigma/2) > 1 - \tau + \sigma/2.$$

So, being $F_{x_{n_k},y}(\tau) \leq 1 - \tau$, it follows

$$F_{x,y}(\tau) - F_{x_n,y}(\tau) \geq 1 - \tau + \sigma/2 - 1 + \tau = \sigma/2$$

for infinite values of k, a contradiction. We conclude that (P1C) for F implies (1C) for d. □

Let $S, T, A_k,\ for\ k = 1, 2, \ldots$, be self mappings of X and let

$$K(x, y, t) := \min\{F_{Sx,Ty}(t), F_{Sx,A_1x}(t), F_{Ty,A_ky}(t), F_{Sx,A_ky}(t), F_{Ty,A_1x}(t)\}$$

for all $x, y \in X$ and $t > 0$.

Now we state and prove the following result.

Theorem 6. *Let (X, F) be a probabilistic symmetric space satisfying $(P1C)$ and (PHE). Let $S, T, A_k,\ for\ k = 1, 2, \ldots,$ be self mappings of X. Assume that there exists $\psi \in \Psi$ with*

$$F_{A_1x,A_ky}(\psi(t)) > 1 - \psi(t), \tag{14}$$

for all $x, y \in X$ and $t > 0$ such that $K(x, y, t) > 1 - t$. Suppose that the pairs (A_1, S) and (A_k, T) for $k > 1$ share the common property (E.A.), $S(X)$ and $T(X)$ are closed subsets of X. Then the pairs (A_1, S) and (A_k, T) for $k > 1$ have a coincidence point. Moreover, S, T and all the A_k have a unique common fixed point in X provided both the pairs (A_1, S) and (A_k, T) are weakly compatible.

Proof. We show that Theorem 6 reduces to Theorem 1. Let d be the compatible symmetric and also compute $m(x, y; A_1, A_k, S, T)$. Notice that d satisfies properties (HE) and (1C). So, given $\varepsilon > 0$, we set $t = \varepsilon + m(x, y; A_1, A_k, S, T)$, then $m(x, y; A_1, A_k, S, T) < t$ if and only if all elements $d(u, v)$ in $m(x, y; A_1, A_k, S, T)$ are minor than t which in turn yields (by Theorem 5 (i)) that all the elements $F_{u,v}$ in $K(x, y, t)$ are major than $1 - t$ so that by condition (14) it follows that $F_{A_1x,A_ky}(\psi(t)) > 1 - \psi(t)$ for all $x, y \in X$ and $t > 0$. Then, by Theorem 5 (i), we have

$$d(A_1x, A_ky) < \psi(t) = \psi(\varepsilon + m(x, y; A_1, A_k, S, T)).$$

Now, making $\varepsilon \to 0^+$ (as ψ is a right continuous function), we have

$$d(A_1x, A_ky) \leq \psi(m(x, y; A_1, A_k, S, T)),$$

for all $x, y \in X$. This is a special case of condition (1), whenever G is the identity mapping on $[0, +\infty)$. So the result follows immediately from Theorem 1. □

Finally, we give a result for occasionally weakly compatible mappings.

Theorem 7. *Let (X, F) be a probabilistic symmetric space satisfying $(P1C)$ and (PHE). Let S, T, A_k, for $k = 1, 2, \dots,$ be self mappings of X. Assume that there exists $\psi \in \Psi$ with*

$$F_{A_1x,A_ky}(\psi(t)) > 1 - \psi(t), \tag{15}$$

for all $x, y \in X$ and $t > 0$ such that $K(x, y, t) > 1 - t$. Then S, T and all the A_k have a unique common fixed point in X provided both the pairs (A_1, S) and (A_k, T) are occasionally weakly compatible.

Proof. Using the arguments of the proof of Theorem 6, it is easy to show that Theorem 7 reduces to Theorem 2. □

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