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Research Article

Common Fixed Point Theorems for Weakly Compatible Pairs on Cone Metric Spaces

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We prove several fixed point theorems on cone metric spaces in which the cone does not need to be normal. These theorems generalize the recent results of Huang and Zhang (2007), Abbas and Jungck (2008), and Vetro (2007). Furthermore as corollaries, we obtain recent results of Rezapour and Hamlborani (2008).

1. Introduction and Preliminaries

Recently, Abbas and Jungck [1], have studied common fixed point results for noncommuting mappings without continuity in cone metric space with normal cone. In this paper, our results are related to the results of Abbas and Jungck, but our assumptions are more general, and also we generalize some results of [1–3], and [4] by omitting the assumption of normality in the results.

Let us mention that nonconvex analysis, especially ordered normed spaces, normal cones, and topical functions ([2, 4–9]) have some applications in optimization theory. In these cases, an order is introduced by using vector space cones. Huang and Zhang [2] used this approach, and they have replaced the real numbers by ordering Banach space and defining cone metric space. Consistent with Huang and Zhang [2], the following definitions and results will be needed in the sequel.

Let E be a real Banach space. A subset P of E is called a cone if and only if:

(i) P is closed, nonempty, and $P \neq \{0\}$;

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$;

(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$ (interior of P). A cone $P \subset E$ is called normal if there are a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\|y\|. \tag{1.1}$$

The least positive number satisfying the above inequality is called the normal constant of P. It is clear that $K \geq 1$. From [4] we know that there exists ordered Banach space E with cone P which is not normal but with $\operatorname{int} P \neq \varnothing$.

Definition 1.1 (see [2]). Let X be a nonempty set. Suppose that the mapping $d : X \times X \mapsto E$ satisfies

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space. The concept of a cone metric space is more general than of a metric space.

Definition 1.2 (see [2]). Let (X, d) be a cone metric space. We say that $\{x_n\}$ is

(e) Cauchy sequence if for every c in E with $0 \ll c$, there is an N such that for all $n, m > N, d(x_n, x_m) \ll c$;

(f) convergent sequence if for every c in E with $0 \ll c$, there is an N such that for all $n > N, d(x_n, x) \ll c$ for some fixed x in X.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. The sequence $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. It is a Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Remark 1.3. (see [10]) Let E be an ordered Banach (normed) space. Then c is an interior point of P, if and only if $[-c, c]$ is a neighborhood of 0.

Corollary 1.4 (see, e.g., [11] without proof). *(1) If $a \leq b$ and $b \ll c$, then $a \ll c$.*

Indeed, $c - a = (c - b) + (b - a) \geq c - b$ implies $[-(c - a), c - a] \supseteq [-(c - b), c - b]$.

(2) If $a \ll b$ and $b \ll c$, then $a \ll c$.

Indeed, $c - a = (c - b) + (b - a) > c - b$ implies $[-(c - a), c - a] \supset [-(c - b), c - b]$.

(3) If $0 \leq u \ll c$ for each $c \in \operatorname{int}P$, then $u = 0$.

Remark 1.5. If $c \in \operatorname{int} P$, $0 \leq a_n$ and $a_n \to 0$, then there exists n_0 such that for all $n > n_0$ we have $a_n \ll c$.

Proof. Let $0 \ll c$ be given. Choose a symmetric neighborhood V such that $c + V \subset P$. Since $a_n \to 0$, there is n_0 such that $a_n \in V = -V$ for $n > n_0$. This means that $c \pm a_n \in c + V \subset P$ for $n > n_0$, that is, $a_n \ll c$. □

From this it follows that: the sequence $\{x_n\}$ converges to $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$, and $\{x_n\}$ is a Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$. In the situation with non-normal cone, we have only half of the lemmas 1 and 4 from [2]. Also, the fact that $d(x_n, y_n) \to d(x, y)$ if $x_n \to x$ and $y_n \to y$ is not applicable.

Remark 1.6. Let $0 \ll c$. If $0 \le d(x_n, x) \le b_n$ and $b_n \to 0$, then eventually $d(x_n, x) \ll c$, where x_n, x are sequence and given point in X.

Proof. It follows from Remark 1.5, Corollary 1.4(1), and Definition 1.2(f). □

Remark 1.7. If $0 \le a_n \le b_n$ and $a_n \to a, b_n \to b$, then $a \le b$, for each cone P.

Remark 1.8. If E is a real Banach space with cone P and if $a \le \lambda a$ where $a \in P$ and $0 < \lambda < 1$, then $a = 0$.

Proof. The condition $a \le \lambda a$ means that $\lambda a - a \in P$, that is, $-(1-\lambda)a \in P$. Since $a \in P$ and $1 - \lambda > 0$, then also $(1-\lambda)a \in P$. Thus we have $(1-\lambda)a \in P \cap (-P) = \{0\}$ and $a = 0$. □

Remark 1.9. Let (X, d) be a cone metric space. Let us remark that the family $\{N(x, e) : x \in X, 0 \ll e\}$, where $N(x, e) = \{y \in X : d(y, x) \ll e\}$, is a subbasis for topology on X. We denote this cone topology by τ_c, and note that τ_c is a Hausdorff topology (see, e.g., [11] without proof).

For the proof of the last statement, we suppose that for each c, $0 \ll c$ we have $N(x, c) \cap N(y, c) \neq \varnothing$. Thus, there exists $z \in X$ such that $d(z, x) \ll c$ and $d(z, y) \ll c$. Hence, $d(x, y) \le d(x, z) + d(z, y) \ll c/2 + c/2 = c$. Clearly, for each n, we have $c/n \in \operatorname{int} P$, so $c/n - d(x, y) \in \operatorname{int} P \subset P$. Now, $0 - d(x, y) \in P$, that is, $d(x, y) \in -P \cap P$, and we have $d(x, y) = 0$.

We find it convenient to introduce the following definition.

Definition 1.10. Let (X, d) be a cone metric space and P a cone with nonempty interior. Suppose that the mappings $f, g : X \mapsto X$ are such that the range of g contains the range of f, and $f(X)$ or $g(X)$ is a complete subspace of X. In this case we will say that the pair (f, g) is Abbas and Jungck's pair, or shortly AJ's pair.

Definition 1.11 (see [1]). Let f and g be self-maps of a set X (i.e., $f, g : X \to X$) . If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. Self-maps f and g are said to be weakly compatible if they commute at their coincidence point, that is, if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

Proposition 1.12 (see [1]). *Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

2. Main Results

In this section we will prove some fixed point theorems of contractive mappings for cone metric space. We generalize some results of [1–4] by omitting the assumption of normality in the results.

Theorem 2.1. *Suppose that (f, g) is AJ's pair, and that for some constant $\lambda \in (0, 1)$ and for every $x, y \in X$, there exists*

$$u \equiv u(x, y) \in \left\{ d(gx, gy), d(fx, gx), d(fy, gy), \frac{d(fx, gy) + d(fy, gx)}{2} \right\}, \tag{2.1}$$

such that

$$d(fx, fy) \leq \lambda \cdot u. \tag{2.2}$$

Then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Let $x_0 \in X$, and let $x_1 \in X$ be such that $gx_1 = fx_0 = y_0$. Having defined $x_n \in X$, let $x_{n+1} \in X$ be such that $gx_{n+1} = fx_n = y_n$.

We first show that

$$d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n), \quad \text{for } n \geq 1. \tag{2.3}$$

We have that

$$d(y_n, y_{n+1}) = d(fx_n, fx_{n+1}) \leq \lambda \cdot u, \tag{2.4}$$

where

$$\begin{aligned} u &\in \left\{ d(gx_n, gx_{n+1}), d(fx_n, gx_n), d(fx_{n+1}, gx_{n+1}), \frac{d(fx_n, gx_{n+1}) + d(fx_{n+1}, gx_n)}{2} \right\} \\ &= \left\{ d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{d(y_{n-1}, y_{n+1})}{2} \right\}. \end{aligned} \tag{2.5}$$

Now we have to consider the following three cases.

If $u = d(y_{n-1}, y_n)$ then clearly (2.3) holds. If $u = d(y_n, y_{n+1})$ then according to Remark 1.8 $d(gx_n, gx_{n+1}) = 0$, and (2.3) is immediate. Finally, suppose that $u = (1/2)d(y_{n-1}, y_{n+1})$. Now,

$$d(y_n, y_{n+1}) \leq \lambda \frac{d(y_{n-1}, y_{n+1})}{2} \leq \frac{\lambda}{2} d(y_{n-1}, y_n) + \frac{1}{2}(y_n, y_{n+1}). \tag{2.6}$$

Hence, $d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n)$, and we proved (2.3).

Now, we have

$$d(y_n, y_{n+1}) \leq \lambda^n d(y_0, y_1). \tag{2.7}$$

We will show that $\{y_n\}$ is a Cauchy sequence. For $n > m$, we have

$$d(y_n, y_m) \leq d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + \cdots + d(y_{m+1}, y_m), \tag{2.8}$$

and we obtain

$$\begin{aligned} d(y_n, y_m) &\leq (\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda^m) d(y_1, y_0) \\ &\leq \frac{\lambda^m}{1-\lambda} d(y_1, y_0) \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \end{aligned} \tag{2.9}$$

From Remark 1.5 it follows that for $0 \ll c$ and large $m : \lambda^m (1-\lambda)^{-1} d(y_1, y_0) \ll c$; thus, according to Corollary 1.4(1), $d(y_n, y_m) \ll c$. Hence, by Definition 1.2(e), $\{y_n\}$ is a Cauchy sequence. Since $f(X) \subseteq g(X)$ and $f(X)$ or $g(X)$ is complete, there exists a $q \in g(X)$ such that $gx_n \to q \in g(X)$ as $n \to \infty$. Consequently, we can find $p \in X$ such that $gp = q$.

Let us show that $fp = q$. For this we have

$$d(fp, q) \leq d(fp, fx_n) + d(fx_n, q) \leq \lambda \cdot u_n + d(fx_n, q), \tag{2.10}$$

where

$$u_n \in \left\{ d(gx_n, gp), d(fx_n, gx_n), d(fp, gp), \frac{d(fx_n, gp) + d(fp, gp)}{2} \right\}. \tag{2.11}$$

Let $0 \ll c$. Clearly at least one of the following four cases holds for infinitely many n.

(Case 1^0)

$$d(fp, q) \leq \lambda \cdot d(gx_n, gp) + d(fx_n, q) \ll \lambda \cdot \frac{c}{2\lambda} + \frac{c}{2} = c. \tag{2.12}$$

(Case 2^0)

$$\begin{aligned} d(fp, q) &\leq \lambda \cdot d(fx_n, gx_n) + d(fx_n, q) \\ &\leq \lambda \cdot d(fx_n, q) + \lambda \cdot d(q, gx_n) + d(fx_n, q) \\ &= (\lambda + 1) \cdot d(fx_n, q) + \lambda \cdot d(q, gx_n) \\ &\ll (\lambda + 1) \cdot \frac{c}{2(\lambda + 1)} + \lambda \cdot \frac{c}{2\lambda} = c. \end{aligned} \tag{2.13}$$

(Case 3^0)

$$\begin{aligned} &d(fp, q) \leq \lambda \cdot d(fp, gp) + d(fx_n, q), \text{ that is,} \\ &d(fp, q) \ll \frac{1}{1-\lambda} \cdot \frac{c}{1/(1-\lambda)} = c. \end{aligned} \tag{2.14}$$

(Case 4^0)

$$
\begin{aligned}
d(fp,q) &\leq \lambda \cdot \frac{d(fx_n,gp)+d(fp,gp)}{2} + d(fx_n,q) \\
&\leq \frac{\lambda d(fx_n,gp)}{2} + \frac{1}{2}d(fp,gp) + d(fx_n,q), \text{ that is,} \\
d(fp,q) &\leq (\lambda+2)d(fx_n,q) \ll (\lambda+2)\frac{c}{(\lambda+2)} = c.
\end{aligned}
\tag{2.15}
$$

In all cases, we obtain $d(fp,q) \ll c$ for each $c \in \operatorname{int} P$. Using Corollary 1.4(3), it follows that $d(fp,q) = 0$, or $fp = q$.

Hence, we proved that f and g have a coincidence point $p \in X$ and a point of coincidence $q \in X$ such that $q = f(p) = g(p)$. If q_1 is another point of coincidence, then there is $p_1 \in X$ with $q_1 = fp_1 = gp_1$. Now,

$$d(q,q_1) = d(fp,fp_1) \leq \lambda \cdot u, \tag{2.16}$$

where

$$
\begin{aligned}
u &\in \left\{ d(gp,gp_1), d(fp,gp), d(fp_1,gp_1), \frac{d(fp,gp_1)+d(fp_1,gp)}{2} \right\} \\
&= \left\{ d(q,q_1), 0, \frac{d(q,q_1)+d(q_1,q)}{2} \right\} = \{0, d(q,q_1)\}.
\end{aligned}
\tag{2.17}
$$

Hence, $d(q,q_1) = 0$, that is, $q = q_1$.

Since $q = f(p) = g(p)$ is the unique point of coincidence of f and g, and f and g are weakly compatible, q is the unique common fixed point of f and g by Proposition 1.12 [1]. □

In the next theorem, among other things, we generalize the well-known Zamfirescu result [12, (21″)].

Theorem 2.2. *Suppose that* (f,g) *is AJ's pair, and that for some constant* $\lambda \in (0,1)$ *and for every* $x, y \in X$, *there exists*

$$u \equiv u(x.y) \in \left\{ d(gx,gy), \frac{d(fx,gx)+d(fy,gy)}{2}, \frac{d(fx,gy)+d(fy,gx)}{2} \right\}, \tag{2.18}$$

such that

$$d(fx,fy) \leq \lambda \cdot u. \tag{2.19}$$

Then f and g have a unique coincidence point in X. Moreover, if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Let $x_0 \in X$, and let $x_1 \in X$ be such that $gx_1 = fx_0 = y_0$. Having defined $x_n \in X$, let $x_{n+1} \in X$ be such that $gx_{n+1} = fx_n = y_n$.

We first show that

$$d(y_n, y_{n+1}) \le \lambda d(y_{n-1}, y_n) \quad \text{for } n \ge 1. \tag{2.20}$$

Notice that

$$d(y_n, y_{n+1}) = d(fx_n, fx_{n+1}) \le \lambda \cdot u_n, \tag{2.21}$$

where

$$\begin{aligned} u_n &\in \left\{ d(gx_n, gx_{n+1}), \frac{d(fx_n, gx_n) + d(fx_{n+1}, gx_{n+1})}{2}, \frac{d(fx_n, gx_{n+1}) + d(fx_{n+1}, gx_n)}{2} \right\} \\ &= \left\{ d(y_{n-1}, y_n), \frac{d(y_{n-1}, y_n) + d(y_n, y_{n+1})}{2}, \frac{d(y_{n-1}, y_{n+1})}{2} \right\}. \end{aligned} \tag{2.22}$$

As in Theorem 2.1, we have to consider three cases.

If $u = d(y_{n-1}, y_n)$, then clearly (2.20) holds. If $u = [d(y_{n-1}, y_n) + d(y_n, y_{n+1})]/2$, then from (2.19) with $x = x_n$ and $y = x_{n+1}$, as $\lambda \in (0,1)$, we have

$$d(y_n, y_{n+1}) \le \lambda \frac{d(y_{n-1}, y_n) + d(y_n, y_{n+1})}{2} \le \lambda \frac{d(y_{n-1}, y_n)}{2} + \frac{d(y_n, y_{n+1})}{2}. \tag{2.23}$$

Hence, $d(y_n, y_{n+1}) \le \lambda d(y_{n-1}, y_n)$, and in this case (2.20) holds. Finally, if $u = [d(y_{n-1}, y_{n+1})]/2$, then

$$\begin{aligned} d(y_n, y_{n+1}) &\le \lambda \frac{d(y_{n-1}, y_{n+1})}{2} \le \lambda \frac{d(y_{n-1}, y_n) + d(y_n, y_{n+1})}{2} \\ &\le \lambda \frac{d(y_{n-1}, y_n)}{2} + \frac{d(y_n, y_{n+1})}{2}, \end{aligned} \tag{2.24}$$

and (2.20) holds. Thus, we proved that in all three cases (2.20) holds.

Now, from the proof of Theorem 2.1, we know that $\{gx_{n+1}\} = \{fx_n\} = \{y_n\}$ is a Cauchy sequence. Hence, there exist q in $g(X)$ and $p \in X$ such that $gx_n \to q$, $n \to \infty$, and $g(p) = q$.

Now we have to show that $fp = q$. For this we have

$$d(fp, q) \le d(fp, fx_n) + d(fx_n, q) \le \lambda \cdot u_n + d(fx_n, q), \tag{2.25}$$

where

$$u_n \in \left\{ d(gx_n, gp), \frac{d(fx_n, gx_n) + d(fp, gp)}{2}, \frac{d(fx_n, gp) + d(fp, gx_n)}{2} \right\}. \tag{2.26}$$

Let $0 \ll c$. Clearly at least one of the following three cases holds for infinitely many n.

(Case 1^0)

$$d(fp, q) \leq \lambda \cdot d(gx_n, gp) + d(fx_n, q) \ll \lambda \cdot \frac{c}{2\lambda} + \frac{c}{2} = c. \tag{2.27}$$

(Case 2^0)

$$\begin{aligned} d(fp, q) &\leq \lambda \cdot \frac{d(fx_n, gx_n) + d(fp, gp)}{2} + d(fx_n, q) \\ &\leq \frac{\lambda d(fx_n, gx_n)}{2} + \frac{d(fp, gp)}{2} + d(fx_n, q), \text{ that is,} \\ d(fp, q) &\leq (\lambda + 2) d(fx_n, q) + \lambda d(gx_n, q) \ll (\lambda + 2) \frac{c}{2(\lambda + 2)} + \lambda \frac{c}{2\lambda} = c. \end{aligned} \tag{2.28}$$

(Case 3^0)

$$\begin{aligned} d(fp, q) &\leq \lambda \cdot \frac{d(fx_n, gp) + d(fp, gx_n)}{2} + d(fx_n, q) \\ &\leq \frac{\lambda d(fx_n, gp)}{2} + \frac{1}{2} d(fp, q) + \frac{\lambda}{2} d(q, gx_n) + d(fx_n, q), \text{ that is,} \\ d(fp, q) &\leq (\lambda + 2) d(fx_n, q) + \lambda d(gx_n, q) \ll (\lambda + 2) \frac{c}{2(\lambda + 2)} + \lambda \frac{c}{2\lambda} = c. \end{aligned} \tag{2.29}$$

In all cases we obtain $d(fp, q) \ll c$ for each $c \in \operatorname{int} P$. Using Corollary 1.4(3), it follows that $d(fp, q) = 0$, or $fp = q$.

Thus we showed that f and g have a coincidence point $p \in X$, that is, point of coincidence $q \in X$ such that $q = fp = gp$. If q_1 is another point of coincidence then there is $p_1 \in X$ with $q_1 = fp_1 = gp_1$. Now from (2.19), it follows that

$$d(q, q_1) = d(fp, fp_1) \leq \lambda \cdot u, \tag{2.30}$$

where

$$u \in \left\{ d(gp, gp_1), \frac{d(fp, gp) + d(fp_1, gp_1)}{2}, \frac{d(fp, gp_1) + d(fp_1, gp)}{2} \right\}$$
$$= \left\{ d(q, q), 0, \frac{d(q, q_1) + d(q_1, q)}{2} \right\} = \{0, d(q, q_1)\}. \tag{2.31}$$

Hence, $d(q, q_1) = 0$, that is, $q = q_1$. If f and g are weakly compatible, then as in the proof of Theorem 2.1, we have that q is a unique common fixed point of f and g. The assertion of the theorem follows. □

Now as corollaries, we recover and generalize the recent results of Huang and Zhang [2], Abbas and Jungck [1], and Vetro [3]. Furthermore as corollaries, we obtain recent results of Rezapour and Hamlborani [4].

Corollary 2.3. *Suppose that (f, g) is AJ's pair, and that for some constant $\lambda \in (0, 1)$ and for every $x, y \in X$,*

$$d(fx, fy) \leq \lambda \cdot d(gx, gy). \tag{2.32}$$

Then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Corollary 2.4. *Suppose that (f, g) is AJ's pair, and that for some constant $\lambda \in (0, 1)$ and for every $x, y \in X$,*

$$d(fx, fy) \leq \lambda \cdot \frac{d(fx, gx) + d(fy, gy)}{2}. \tag{2.33}$$

Then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Corollary 2.5. *Suppose that (f, g) is AJ's pair, and that for some constant $\lambda \in (0, 1)$ and for every $x, y \in X$,*

$$d(fx, fy) \leq \lambda \cdot \frac{d(fx, gy) + d(fy, gx)}{2}. \tag{2.34}$$

Then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

In the next corollary, among other things, we generalize the well-known result [12, (9′)].

Corollary 2.6. *Suppose that (f, g) is AJ's pair, and that for some constant $\lambda \in (0,1)$ and for every $x, y \in X$, there exists*

$$u = u(x, y) \in \{d(gx, gy), d(fx, gx), d(fy, gy)\}, \tag{2.35}$$

such that

$$d(fx, fy) \leq \lambda \cdot u. \tag{2.36}$$

Then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Now, we generalize the well-known Bianchini result [12, (5)].

Corollary 2.7. *Suppose that (f, g) is AJ's pair, and that for some constant $\lambda \in (0,1)$ and for every $x, y \in X$, there exists*

$$u = u(x, y) \in \{d(fx, gx), d(fy, gy)\}, \tag{2.37}$$

such that

$$d(fx, fy) \leq \lambda \cdot u. \tag{2.38}$$

Then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

When in the next theorem we set $g = I_X$, the identity map on $X, E = (-\infty, +\infty)$ and $P = [0, +\infty[$, we get the theorem of Hardy and Rogers [12, (18)].

Theorem 2.8. *Suppose that (f, g) is AJ's pair, and that there exist nonnegative constants a_i satisfying $\sum_{i=1}^{5} a_i < 1$ such that, for each $x, y \in X$,*

$$d(fx, fy) \leq a_1 d(gx, gy) + a_2 d(gx, fx) + a_3 d(gy, fy) + a_4 d(gx, fy) + a_5 d(gy, fx). \tag{2.39}$$

Then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Let us define the sequences x_n and y_n as in the proof of Theorem 2.1 We have to show that

$$d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n), \quad \text{for some } \lambda \in (0,1),\ n \geq 1. \tag{2.40}$$

From

$$\begin{aligned} d(y_{n+1}, y_n) &= d(fx_{n+1}, fx_n) \le a_1 d(gx_{n+1}, gx_n) + a_2 d(gx_{n+1}, fx_{n+1}) \\ &\quad + a_3 d(gx_n, fx_n) + a_4 d(gx_{n+1}, fx_n) + a_5 d(gx_n, fx_{n+1}) \\ &= a_1 d(y_n, y_{n-1}) + a_2 d(y_n, y_{n+1}) + a_3 d(y_{n-1}, y_n) + a_5 d(y_{n-1}, y_{n+1}), \\ d(y_n, y_{n+1}) &= d(fx_n, fx_{n+1}) \le a_1 d(gx_n, gx_{n+1}) + a_2 d(gx_n, fx_n) \\ &\quad + a_3 d(gx_{n+1}, fx_{n+1}) + a_4 d(gx_n, fx_{n+1}) + a_5 d(gx_{n+1}, fx_n) \\ &= a_1 d(y_{n-1}, y_n) + a_2 d(y_{n-1}, y_n) + a_3 d(y_n, y_{n+1}) + a_4 d(y_{n-1}, y_{n+1}), \end{aligned} \tag{2.41}$$

we obtain

$$\begin{aligned} 2d(y_n, y_{n+1}) &\le 2a_1 d(y_{n-1}, y_n) + 2a_2 d(y_{n-1}, y_n) + a_3 d(y_n, y_{n+1}) \\ &\quad + a_4 d(y_{n-1}, y_{n+1}) + (a_3 + a_4 + a_5) d(y_n, y_{n+1}). \end{aligned} \tag{2.42}$$

Thus,

$$d(y_n, y_{n+1}) \le \frac{2a_1 + a_2 + a_3 + a_4}{2 - a_3 - a_4 - a_5} \cdot d(y_{n-1}, y_n) = \lambda \cdot d(y_{n-1}, y_n), \tag{2.43}$$

where $\lambda = (2a_1 + a_2 + a_3 + a_4)(2 - a_3 - a_4 - a_5)^{-1} \in (0, 1)$, and we proved (2.40).

Now, from the proof of Theorem 2.1, we know that $\{gx_{n+1}\} = \{fx_n\} = \{y_n\}$ is a Cauchy sequence. Hence, there exist q in $g(X)$ and $p \in X$ such that $gx_n \to q$, $n \to \infty$, and $gp = q$.

We have to show that $fp = q$. For this we have

$$\begin{aligned} d(fp, q) &\le d(fp, fx_n) + d(fx_n, q) \le a_1 d(gx_n, gp) + a_2 d(gx_n, fx_n) \\ &\quad + a_3 d(gp, fp) + a_4 d(gx_n, fp) + a_5 d(gp, fx_n) + d(fx_n, q) \\ &\le a_1 d(gx_n, q) + a_2 d(gx_n, q) + a_2 d(q, fx_n) + a_3 d(q, fp) \\ &\quad + a_4 d(gx_n, q) + a_4 d(q, fp) + (a_5 + 1) d(q, fx_n), \text{that is,} \\ d(fp, q) &\le \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} d(gx_n, q) + \frac{1 + a_2 + a_5}{1 - a_3 - a_4} d(fx_n, q) \\ &\ll \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} \frac{c}{2((a_1 + a_2 + a_4)/(1 - a_3 - a_4))} \\ &\quad + \frac{1 + a_2 + a_5}{1 - a_3 - a_4} \frac{c}{2((1 + a_2 + a_5)/(1 - a_3 - a_4))} = c. \end{aligned} \tag{2.44}$$

Then according to Corollary 1.4(3), $d(fp, q) = 0$, that is, $fp = q$.

Thus we showed that f and g have a coincidence point $p \in X$, that is, point of coincidence $q \in X$ such that $q = fp = gp$. If q_1 is another point of coincidence then there is $p_1 \in X$ with $q_1 = fp_1 = gp_1$. Now,

$$\begin{aligned} d(q,q_1) &= d(fp,fp_1) \\ &\leq a_1 d(gp,gp_1) + a_2 d(gp,fp) + a_3 d(gp_1,fp_1) \\ &\quad + a_4 d(gp,fp_1) + a_5 d(gp_1,fp) \\ &= (a_1 + a_4 + a_5) d(q,q_1). \end{aligned} \tag{2.45}$$

According to Remark 1.8, and because $0 \leq a_1 + a_4 + a_5 \leq \sum_{i=1}^{5} a_i < 1$, we get $d(q,q_1) = 0$, that is, $q = q_1$. If f and g are weakly compatible, then as in the proof of Theorem 2.1, we have that q is a unique common fixed point of f and g. The assertion of the theorem follows.

It is clear that, for the special choice of a_i in Theorem 2.8, all the results from Corollaries 2.3, 2.4, and 2.5, could be obtained. □

Finally, we add an example with Banach type contraction on non-normal cone metric space (see Corollary 2.3).

Example 2.9. Let $X = \mathbb{R}, E = C^1_{\mathbb{R}}[0,1]$, and $P = \{\varphi \in E : \varphi \geq 0\}$. Define $d : X \times X \to E$ by $d(x,y) = |x-y|\varphi$ where $\varphi : [0,1] \to \mathbb{R}$ such that $\varphi(t) = e^t$. It is easy to see that d is a cone metric on X. Consider the mappings $f,g : X \to X$ in the following manner:

$$fx = \begin{cases} \dfrac{1}{1+\alpha}x + \beta, & x \neq 0, \\ 0, & x = 0, \end{cases} \qquad gx = \begin{cases} x + (1+\alpha)\beta, & x \neq 0, \\ 0, & x = 0, \end{cases} \tag{2.46}$$

where $\alpha > 1, \beta \in \mathbb{R}$. One can see that

$$d(fx,fy) \leq k d(gx,gy), \tag{2.47}$$

for all $x,y \in X$, where $k = 1/\alpha \in (0,1)$. The mappings f and g commute at $x = 0$, the only coincidence point. So f and g are weakly compatible. All the conditions of the Corollary 2.3 hold, then f and g have a common fixed point.

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References

[1] M. Abbas and G. Jungck, "Common fixed point results for noncommuting mappings without continuity in cone metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 1, pp. 416–420, 2008.

[2] L.-G. Huang and X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1468–1476, 2007.

[3] P. Vetro, "Common fixed points in cone metric spaces," *Rendiconti del Circolo Matematico di Palermo*, vol. 56, no. 3, pp. 464–468, 2007.

[4] Sh. Rezapour and R. Hamlbarani, "Some notes on the paper: "Cone metric spaces and fixed point theorems of contractive mappings"," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 2, pp. 719–724, 2008.

[5] C. D. Aliprantis and R. Tourky, *Cones and Duality*, vol. 84 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, USA, 2007.

[6] H. Mohebi, "Topical functions and their properties in a class of ordered Banach spaces," in *Continuous Optimization*, vol. 99 of *Applied Optimization*, pp. 343–361, Springer, New York, NY, USA, 2005.

[7] P. Raja and S. M. Vaezpour, "Some extensions of Banach's contraction principle in complete cone metric spaces," *Fixed Point Theory and Applications*, Article ID 768294, 11 pages, 2008.

[8] D. Ilić and V. Rakočević, "Common fixed points for maps on cone metric space," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 876–882, 2008.

[9] D. Ilić and V. Rakočević, "Quasi-contraction on a cone metric spacestar, open," *Applied Mathematics Letters*, vol. 22, no. 5, pp. 728–731, 2009.

[10] Y.-C. Wong and K.-F. Ng, *Partially Ordered Topological Vector Spaces*, Oxford Mathematical Monograph, Clarendon Press, Oxford, UK, 1973.

[11] Sh. Rezapour, "A review on topological properties of cone metric spaces," in *Analysis, Topology and Applications (ATA '08)*, Vrnjacka Banja, Serbia, May-June 2008.

[12] B. E. Rhoades, "A comparison of various definitions of contractive mappings," *Transactions of the American Mathematical Society*, vol. 226, pp. 257–290, 1977.