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Common fixed points for single-valued and multi-valued maps satisfying a generalized contraction in G-metric spaces

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Abstract

In this article, we establish some common fixed point theorems for a hybrid pair {*g*, *T*} of single valued and multi-valued maps satisfying a generalized contractive condition defined on *G*-metric spaces. Our results unify, generalize and complement various known comparable results from the current literature.

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1. Introduction and preliminaries

Nadler [1] initiated the study of fixed points for multi-valued contraction mappings and generalized the well known Banach fixed point theorem. Then after, many authors studied many fixed point results for multi-valued contraction mappings see [2-13].

Mustafa and Sims [14] introduced the *G*-metric spaces as a generalization of the notion of metric spaces. Mustafa et al. [15-19] obtained some fixed point theorems for mappings satisfying different contractive conditions. Abbas and Rhoades [20] initiated the study of common fixed point in *G*-metric spaces. While Saadati et al. [21] studied some fixed point theorems in generalized partially ordered *G*-metric spaces. Gajić and Crvenković [22,23] proved some fixed point results for mappings with contractive iterate at a point in *G*-metric spaces. For other studies in *G*-metric spaces, we refer the reader to [24-38]. Consistent with Mustafa and Sims [14], the following definitions and results will be needed in the sequel.

Definition 1.1. (See [14]). *Let X be a non-empty set,* $G : X \times X \times X \to \mathbb{R}^+$ *be a function satisfying the following properties*

(G1) $G(x, y, z) = 0$ *if* $x = y = z$,

(G2) $0 < G(x, x, y)$ *for all* $x, y \in X$ *with* $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *with* $y \neq z$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (*symmetry in all three variables*),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$ (*rectangle inequality*).

Then the function G is called a generalized metric, or, more specially, a G-metric on X, and the pair (X, G) *is called a G-metric space.*

Definition 1.2. (See [14]). *Let* (X, G) *be a G-metric space, and let* (x_n) *be a sequence of points of X, therefore, we say that* (x_n) *is G-convergent to* $x \in X$ *if* $\lim_{n,m\to+\infty} G(x, x_n, x_m) = 0$, *that is, for any* $\varepsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that* $G(x, x_n, x_m) < \varepsilon$, *for all* $n, m \geq N$. *We call x the limit of the sequence and write* $x_n \to x$ *or* $\lim_{n\to+\infty} x_n = x$.

Proposition 1.1. (See [14]). *Let* (X, G) *be a G-metric space. The following statements are equivalent:*

(1) (x_n) *is G-convergent to x,*

(2) $G(x_n, x_n, x) \to 0$ *as* $n \to +\infty$,

(3) $G(x_n, x, x) \to 0$ *as* $n \to +\infty$,

(4) $G(x_n, x_m, x) \to 0$ *as* $n, m \to +\infty$.

Definition 1.3. (See [14]). *Let* (X, G) *be a G-metric space. A sequence* (x_n) *is called a G-Cauchy sequence if for any* $\varepsilon > 0$, *there is* $N \in \mathbb{N}$ *such that* $G(x_n, x_m, x_l) < \varepsilon$ *for all* $m, n, l \geq N$, *that is,* $G(x_n, x_m, x_l) \to 0$ *as* $n, m, l \to +\infty$.

Proposition 1.2. (See [14]). *Let* (X, G) *be a G-metric space. Then the following statements are equivalent:*

(1) *the sequence* (x_n) *is G-Cauchy,*

(2) *for any* $\varepsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that* $G(x_n, x_m, x_m) < \varepsilon$, *for all* $m, n \geq N$.

Definition 1.4. (See [14]). *A G-metric space* (X, G) *is called G-complete if every G-Cauchy sequence is G-convergent in* (X,G).

Every G-metric on X defines a metric d_G on X given by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \text{ for all} x, y \in X. \tag{1}$$

Recently, Kaewcharoen and Kaewkhao [34] introduced the following concepts. Let X be a G-metric space. We shall denote $CB(X)$ the family of all nonempty closed bounded subsets of X. Let $H(.,.,.)$ be the Hausdorff G-distance on $CB(X)$, i.e.,

$$H_G(A, B, C) = \max\left\{\sup_{x\in A} G(x, B, C), \sup_{x\in B} G(x, C, A), \sup_{x\in C} G(x, A, B)\right\},$$

where

$$\begin{aligned} G(x, B, C) &= d_G(x, B) + d_G(B, C) + d_G(x, C), \\ d_G(x, B) &= \inf\left\{d_G(x, y), y \in B\right\}, \\ d_G(A, B) &= \inf\{d_G(a, b), a \in A, b \in B\}. \end{aligned}$$

Recall that $G(x, y, C) = \inf\{G(x, y, z), z \in C\}$. A mapping $T : X \to 2^X$ is called a multi-valued mapping. A point $x \in X$ *is* called a fixed point of T if $x \in Tx$.

Definition 1.5. *Let X be a given non empty set. Assume that* $g : X \to X$ *and* $T : X \to 2^X$.

If $w = gx \in Tx$ *for some* $x \in X$, *then x is called a coincidence point of g and T and w is a point of coincidence of g and T.*

Mappings g and T are called weakly compatible if $gx \in Tx$ *for some* $x \in X$ *implies* $gT(x) \subseteq Tg(x)$.

Proposition 1.3. (*see* [34]). *Let X be a given non empty set. Assume that* $g : X \to X$ *and* $T : X \to 2^X$ *are weakly compatible mappings. If g and T have a unique point of coincidence* $w = gx \in Tx$, *then w is the unique common fixed point of g and T.*

In this article, we establish some common fixed point theorems for a hybrid pair $\{g, T\}$ of single valued and multi-valued maps satisfying a generalized contractive condition defined on G-metric spaces. Also, an example is presented.

2. Main results

We start this section with the following lemma, which is the variant of the one given in Nadler [1] or Assad and Kirk [4]. Its proof is a simple consequence of the definition of the Hausdorff G-distance $H_G(A, B, B)$.

Lemma 2.1. *If $A, B \in CB(X)$ and $a \in A$, then for each $\varepsilon > 0$, there exists $b \in B$ such that $G(a,b,b) \leq H_G(A, B, B) + \varepsilon$.*

The main result of the article is the following.

Theorem 2.1. *Let (X, G) be a G-metric space. Set $g : X \to X$ and $T : X \to CB(X)$. Assume that there exists a function $\alpha : [0,+\infty) \to [0,1)$ satisfying $\limsup_{r\to t^+} \alpha(r) < 1$ for every $t \geq 0$ such that*

$$H_G(Tx, Ty, Tz) \leq \alpha\left(G\left(gx, gy, gz\right)\right) G\left(gx, gy, gz\right), \tag{2}$$

for all $x, y, z \in X$. If for any $x \in X$, $Tx \subseteq g(X)$ and $g(X)$ is a G-complete subspace of X, then g and T have a point of coincidence in X. Furthermore, if we assume that $gp \in Tp$ and $gq \in Tq$ implies $G(gq, gp, gp) \leq H_G(Tq, Tp, Tp)$, then

(i) g and T have a unique point of coincidence.

(ii) If in addition g and T are weakly compatible, then g and T have a unique common fixed point.

Proof. Let x_0 be arbitrary in X. Since $Tx_0 \subseteq g(X)$, choose $x_1 \in X$ such that $gx_1 \in Tx_0$. If $gx_1 = gx_0$, we finished. Assume that $gx_0 \neq gx_1$, so $G(gx_0, gx_1, gx_1) > 0$. We can choose a positive integer n_1 such that

$$\alpha^{n_1}\left(G\left(gx_0, gx_1, gx_1\right)\right) \leq \left[1 - \alpha\left(G\left(gx_0, gx_1, gx_1\right)\right)\right] G\left(gx_0, gx_1, gx_1\right).$$

By Lemma 2.1 and the fact that $Tx_1 \subseteq g(X)$, there exists $gx_2 \in Tx_1$ such that

$$G\left(gx_1, gx_2, gx_2\right) \leq H_G\left(Tx_0, Tx_1, Tx_1\right) + \alpha^{n_1}\left(G\left(gx_0, gx_1, gx_1\right)\right).$$

Using the two above inequalities and (2), it follows that

$$\begin{aligned} G\left(gx_1, gx_2, gx_2\right) &\leq H_G\left(Tx_0, Tx_1, Tx_1\right) + \alpha^{n_1}\left(G\left(gx_0, gx_1, gx_1\right)\right) \\ &\leq \alpha\left(G\left(gx_0, gx_1, gx_1\right)\right) G\left(gx_0, gx_1, gx_1\right) + \left[1 - \alpha\left(G\left(gx_0, gx_1, gx_1\right)\right)\right] G\left(gx_0, gx_1, gx_1\right) \\ &= G\left(gx_0, gx_1, gx_1\right). \end{aligned}$$

If $gx_1 = gx_2$, we finished. Assume that $gx_1 \neq gx_2$. Now we choose a positive integer $n_2 > n_1$ such that

$$\alpha^{n_2}\left(G\left(gx_1, gx_2, gx_2\right)\right) \leq \left[1 - \alpha\left(G\left(gx_1, gx_2, gx_2\right)\right)\right] G\left(gx_2, gx_2, gx_2\right).$$

Since $Tx_2 \in CB(X)$ and the fact that $Tx_2 \subseteq g(X)$, we may select $gx_3 \in Tx_2$ such that from Lemma 2.1

$$G\left(gx_2, gx_3, gx_3\right) \leq H_G\left(Tx_1, Tx_2, Tx_2\right) + \alpha^{n_2}\left(G\left(gx_1, gx_2, gx_2\right)\right),$$

and then, similarly to the previous case, we have

$$\begin{aligned} G(gx_2, gx_3, gx_3) &\le H_G(Tx_1, Tx_2, Tx_2) + \alpha^{n_2}(G(gx_1, gx_2, gx_2)) \\ &\le \alpha(G(gx_1, gx_2, gx_2))\, G(gx_1, gx_2, gx_2) + [1 - \alpha(G(gx_1, gx_2, gx_2))]\, G(gx_1, gx_2, gx_2) \\ &= G(gx_1, gx_2, gx_2). \end{aligned}$$

By repeating this process, for each $k \in \mathbb{N}^*$, we may choose a positive integer n_k such that

$$\alpha^{n_k}(G(gx_{k-1}, gx_k, gx_k)) \le [1 - \alpha(G(gx_{k-1}, gx_k, gx_k))]\, G(gx_{k-1}, gx_k, gx_k).$$

Again, we may select $gx_{k+1} \in Tx_k$ such that

$$G(gx_k, gx_{k+1}, gx_{k+1}) \le H_G(Tx_{k-1}, Tx_k, Tx_k) + \alpha^{n_k}(G(gx_{k-1}, gx_k, gx_k)). \tag{3}$$

The last two inequalities together imply that

$$G(gx_k, gx_{k+1}, gx_{k+1}) \le G(gx_{k-1}, gx_k, gx_k),$$

which shows that the sequence of nonnegative numbers $\{d_k\}$, given by $d_k = G(gx_{k-1}, gx_k, gx_k)$, $k = 1, 2, \ldots$, is non-increasing. This means that there exists $d \ge 0$ such that

$$\lim_{k \to +\infty} d_k = d.$$

Let now prove that the $\{gx_k\}$ is a G-Cauchy sequence.

Using the fact that, by hypothesis for $t = d$, $\limsup_{r \to d^+} \alpha(t) < 1$, it results that there exists a rank k_0 such that for $k \ge k_0$, we have $\alpha(d_k) < h$, where

$$\limsup_{t \to d^+} \alpha(t) < h < 1.$$

Now, by (3) we deduce that the sequence $\{d_k\}$ satisfies the following recurrence inequality

$$d_{k+1} \le H_G(Tx_{k-1}, Tx_k, Tx_k) + \alpha^{n_k}(d_k) \le \alpha(d_k) d_k + \alpha^{n_k}(d_k), \quad k \ge 1. \tag{4}$$

By induction, from (4), we get

$$d_{k+1} \le \prod_{i=1}^{k} \alpha(d_i) d_1 + \sum_{m=1}^{k-1} \prod_{i=m+1}^{k} \alpha(d_i) \alpha^{n_m}(d_m) + \alpha^{n_k}(d_k), \quad k \ge 1,$$

which, by using the fact that $\alpha < 1$, can be simplified to

$$d_{k+1} \le \prod_{i=1}^{k} \alpha(d_i) d_1 + \sum_{m=1}^{k-1} \prod_{i=\max\{k_0, m+1\}}^{k} \alpha(d_i) \alpha^{n_m}(d_m) + \alpha^{n_k}(d_k), \quad k \ge 1,$$

Referring to the proof of Theorem 2.1 in [11] or Lemma 3.2 in [12], we may obtain

$$\prod_{i=1}^{k} \alpha(d_i) d_1 + \sum_{m=1}^{k-1} \prod_{i=\max\{k_0, m+1\}}^{k} \alpha(d_i) \alpha^{n_m}(d_m) + \alpha^{n_k}(d_k) \le ch^k,$$

where c is a positive constant. We deduce that

$$d_{k+1} = G(gx_k, gx_{k+1}, gx_{k+1}) \le ch^k.$$

Now for $k \geq k_0$ and m is a positive arbitrary integer, we have using the property (G4)

$$\begin{aligned} G(gx_k, gx_{k+m}, gx_{k+m}) &\leq G(gx_k, gx_{k+1}, gx_{k+1}) + G(gx_{k+1}, gx_{k+2}, gx_{k+2}) \\ &\quad + \cdots + G(gx_{k+m-2}, gx_{k+m-1}, gx_{k+m-1}) + G(gx_{k+m-1}, gx_{k+m}, gx_{k+m}) \\ &\leq c\left[h^k + h^{k+1} + \cdots + h^{k+m-1}\right] \\ &\leq c\frac{h^k}{1-h} \to 0 \text{ as } k \to +\infty, \end{aligned}$$

since $0 < h < 1$. This shows that the sequence $\{gx_n\}$ is G-Cauchy in the complete subspace $g(X)$. Thus, there exists $q \in g(X)$ such that, from Proposition 1.1

$$\lim_{n\to+\infty} G(gx_n, gx_n, q) = \lim_{n\to+\infty} G(gx_n, q, q) = 0. \tag{5}$$

Since $q \in g(X)$, then there exists $p \in X$ such that $q = gp$. From (5), we have

$$\lim_{n\to+\infty} G(gx_n, gx_n, gp) = \lim_{n\to+\infty} G(gx_n, gp, gp) = 0. \tag{6}$$

We claim that $gp \in Tp$. Indeed, from (2), we have

$$\begin{aligned} G(gx_{n+1}, Tp, Tp) &\leq H_G(Tx_n, Tp, Tp) \\ &\leq \alpha(G(gx_n, gp, gp))G(gx_n, gp, gp). \end{aligned} \tag{7}$$

Letting $n \to +\infty$ in (7) and using (6), we get

$$G(gp, Tp, Tp) = \lim_{n\to+\infty} G(gx_{n+1}, Tp, Tp) = 0,$$

that is, $gp \in Tp$. That is T and g have a point of coincidence. Now, assume that if $gp \in Tp$ and $gq \in Tq$, then $G(gq, gp, gp) \leq H_G(Tq, Tp, Tp)$. We will prove the uniqueness of a point of coincidence of g and T. Suppose that $gp \in Tp$ and $gq \in Tq$. By (2) and this assumption, we have

$$\begin{aligned} G(gq, gp, gp) &\leq H_G(Tq, Tp, Tp) \\ &\leq \alpha(G(gq, gp, gp))G(gq, gp, gp), \end{aligned} \tag{8}$$

and since $\alpha(G(gq, gp, gp)) < G(gq, gp, gp)$, so necessarily from (8), we have $G(gq, gp, gp) = 0$, *i.e.*, $gp = gq$. In view of

$$H_G(Tq, Tp, Tp) \leq \alpha(G(gq, gp, gp))G(gq, gp, gp) = 0,$$

we get $Tq = Tp$. Thus, T and g have a unique point of coincidence. Suppose that g and T are weakly compatible. By applying Proposition 1.3, we obtain that g and T have a unique common fixed point.

Corollary 2.1. *Let* (X,G) *be a complete G-metric space. Assume that* $T : X \to CB(X)$ *satisfies the following condition*

$$H_G(Tx, Ty, Tz) \leq \alpha(G(x, y, z))G(x, y, z), \tag{9}$$

for all $x, y, z \in X$, *where* $\alpha : [0,+\infty) \to [0,1)$ *satisfies* $\limsup_{r\to t^+} \alpha(r) < 1$ *for every* $t \geq 0$. *Then T has a fixed point in X. Furthermore, if we assume that* $p \in Tp$ *and* $q \in Tq$ *implies* $G(q, p, p) \leq H_G(Tq, Tp, Tp)$, *then T has a unique fixed point.*

Proof. It follows by taking g the identity on X in Theorem 2.1.

Corollary 2.2. *Let* (X, G) *be a G-metric space. Assume that* $g : X \to X$ *and* $T : X \to CB(X)$ *satisfy the following condition*

$$H_G(Tx, Ty, Tz) \leq kG(gx, gy, gz), \tag{10}$$

for all x, y, z ∈ X, where k ∈ [0,1). *If for any x ∈ X, Tx ⊆ g(X) and g(X) is a G-complete subspace of X, then g and T have a point of coincidence in X. Furthermore, if we assume that gp ∈ Tp and gq ∈ Tq implies* $G(gq, gp, gp) \leq H_G(Tq, Tp, Tp)$, *then*

(*i*) *g and T have a unique point of coincidence.*

(*ii*) *If in addition g and T are weakly compatible, then g and T have a unique common fixed point.*

Proof. It follows by taking $\alpha(t) = k$, $k \in [0,1)$, in Theorem 2.1.

In the case of single-valued mappings, that is, if $T : X \to X$, (i.e., $Tx = \{Tx\}$ for any $x \in X$), it is obviously that

$$H_G(Tx, Ty, Tz) = G(Tx, Ty, Tz), \quad \forall x, y, z \in X.$$

Furthermore, if $gp \in Tp$ (i.e., $gp = Tp$) and $gq \in Tq$ (i.e., $gq = Tq$), then clearly,

$$G(gq, gp, gp) = G(Tq, Tp, Tp) = H_G(Tq, Tp, Tp),$$

that is, the assumption given in Theorem 2.1 is verified.

Also, the single-valued mappings $T, g : X \to X$ are said weakly compatible if $Tgx = gTx$ whenever $Tx = gx$ for some $x \in X$.

Now, we may state the following corollaries from Theorem 2.1 and the precedent corollaries:

Corollary 2.3. *Let* (*X, G*) *be a complete G-metric space. Assume that* $T : X \to X$ *satisfies the following condition*

$$G(Tx, Ty, Tz) \leq \alpha(G(x, y, z))G(x, y, z) \tag{11}$$

for all x, y, z ∈ X, where $\alpha : [0, +\infty) \to [0, 1)$ *satisfies* $\limsup_{r \to t^+} \alpha(r) < 1$ *for every* $t \geq 0$. *Then, T has a unique fixed point.*

Corollary 2.4. *Let* (*X, G*) *be a G-metric space. Assume that* $g : X \to X$ *and* $T : X \to X$ *satisfy the following condition*

$$G(Tx, Ty, Tz) \leq \alpha(G(gx, gy, gz))G(gx, gy, gz) \tag{12}$$

for all x, y, z ∈ X, where $\alpha : [0, +\infty) \to [0, 1)$ *satisfies* $\limsup_{r \to t^+} \alpha(r) < 1$ *for every* $t \geq 0$. *If T(X) ⊆ g(X) and g(X) is a G-complete subspace of X, then*

(*i*) *g and T have a unique point of coincidence.*

(*ii*) *Furthermore, if g and T are weakly compatible, then g and T have a unique common fixed point.*

Now, we introduce an example to support the useability of our results.

Example 2.1. *Let* $X = [0, 1]$. *Define* $T : X \to CB(X)$ *by* $Tx = \left[0, \frac{1}{16}x\right]$ *and define* $g : X \to X$ *by* $gx = \sqrt{x}$. *Define a G-metric on X by* $G(x, y, z) = \max\{|x-y|, |x-z|, |y-z|\}$. *Also, define* $\alpha : [0, +\infty) \to [0, 1)$ *by* $\alpha(t) = \frac{1}{2}$ *Then:*

(1) *Tx ⊆ g(X) for all x ∈ X.*

(2) *g(X) is a G-complete subspace of X.*

(3) *g and T are weakly compatible.*

(4) $H_G(Tx, Ty, Tz) \leq \alpha(G(gx, gy, gz))G(gx, gy, gz)$ *for all x, y, z ∈ X.*

Proof. The proofs of (1), (2), and (3) are clear. By (1), we have

$$d_G(x,y) = G(x,y,y) + G(y,x,x) = 2|x-y| \text{ for all } x,y \in X.$$

To prove (4), let x, y, $z \in X$. If $x = y = z = 0$, then

$$H_G(Tx,Ty,Tz) = 0 \le \alpha(G(gx,gy,gz))\,G(gx,gy,gz).$$

Thus, we may assume that x, y, and z are not all zero. With out loss of generality, we assume that $x \le y \le z$. Then

$$H_G(Tx,Ty,Tz) = H_G\left(\left[0,\frac{1}{16}x\right],\left[0,\frac{1}{16}y\right],\left[0,\frac{1}{16}z\right]\right)$$
$$= \max\left\{\sup_{0\le a\le \frac{1}{16}x} G\left(a,\left[0,\frac{1}{16}y\right],\left[0,\frac{1}{16}z\right]\right),\ \sup_{0\le b\le \frac{1}{16}y} G\left(b,\left[0,\frac{1}{16}z\right],\left[0,\frac{1}{16}x\right]\right),\right.$$
$$\left.\sup_{0\le c\le \frac{1}{16}z} G\left(c,\left[0,\frac{1}{16}x\right],\left[0,\frac{1}{16}y\right]\right)\right\}.$$

Since $x \le y \le z$, *so* $\left[0,\frac{1}{16}x\right] \subseteq \left[0,\frac{1}{16}y\right] \subseteq \left[0,\frac{1}{16}z\right]$ This implies that

$$d_G([0,\frac{1}{16}x],[0,\frac{1}{16}y]) = d_G([0,\frac{1}{16}y],[0,\frac{1}{16}z]) = d_G([0,\frac{1}{16}x],[0,\frac{1}{16}z]) = 0.$$

For each $0 \le a \le \frac{1}{16}x$, we have

$$G\left(a,\left[0,\frac{1}{16}y\right],\left[0,\frac{1}{16}z\right]\right) = d_G\left(a,\left[0,\frac{1}{16}y\right]\right) + d_G\left(\left[0,\frac{1}{16}y\right],\left[0,\frac{1}{16}z\right]\right) + d_G\left(a,\left[0,\frac{1}{16}z\right]\right) = 0.$$

Also, for each $0 \le b \le \frac{1}{16}y$, we have

$$G\left(b,\left[0,\frac{1}{16}z\right],\left[0,\frac{1}{16}x\right]\right) = d_G\left(b,\left[0,\frac{1}{16}z\right]\right) + d_G\left(\left[0,\frac{1}{16}z\right],\left[0,\frac{1}{16}x\right]\right) + d_G\left(b,\left[0,\frac{1}{16}x\right]\right)$$
$$= \begin{cases} 0 \text{ if } b \le \frac{x}{16} \\ 2b - \frac{x}{8} \text{ if } b \ge \frac{x}{16}. \end{cases}$$

This yields that

$$\sup_{0\le b\le \frac{1}{16}y} G\left(b,\left[0,\frac{1}{16}z\right],\left[0,\frac{1}{16}x\right]\right) = \frac{y}{8} - \frac{x}{8}.$$

Moreover, for each $0 \leq c \leq \frac{1}{16}z$, we have

$$G\left(c, \left[0, \frac{1}{16}x\right], \left[0, \frac{1}{16}y\right]\right) = d_G\left(c, \left[0, \frac{1}{16}x\right]\right) + d_G\left(\left[0, \frac{1}{16}x\right], \left[0, \frac{1}{16}y\right]\right) + d_G\left(c, \left[0, \frac{1}{16}y\right]\right)$$
$$= \begin{cases} 0 \text{ if } c \leq \frac{x}{16} \\ 2c - \frac{x}{8} \text{ if } \frac{x}{16} \leq c \leq \frac{y}{16} \\ 4c - \frac{x}{8} - \frac{y}{8} \text{ if } c \geq \frac{y}{16}. \end{cases}$$

This yields that

$$\sup_{0 \leq c \leq \frac{1}{16}z} G\left(c, \left[\frac{1}{16}c\right], \left[0, \frac{1}{16}y\right]\right) = \frac{z}{4} - \frac{x}{8} - \frac{y}{8}.$$

We deduce that

$$\begin{aligned} H_G(Tx, Ty, Tz) &= \frac{z}{4} - \frac{x}{8} - \frac{y}{8} \\ &\leq \frac{1}{4}(z - x) \\ &= \frac{1}{2}\left(\frac{1}{2}(z - x)\right) \\ &\leq \frac{1}{2}\left(\frac{z - x}{\sqrt{x} + \sqrt{z}}\right) \\ &= \frac{1}{2}(\sqrt{z} - \sqrt{x}) \end{aligned}$$

On the other hand, it is obvious that all other hypotheses of Theorem 2.1 are satisfied and so g and T have a unique common fixed point, which is $u = 0$.

Remark 1• *Theorem 2.1 improves Kaewcharoen and Kaewkhao* [[34], Theorem 3.3] (*in case* $b = c = d = 0$).

- *Corollary 2.3 generalizes Mustafa* [[15], Theorem 5.1.7] *and Shatanawi* [[35], Corollary 3.4].

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Authors' contributions
The authors have contributed in obtaining the new results presented in this article. All authors read and approved the final manuscript.

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