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COMMON FIXED POINTS FOR WEAKLY COMPATIBLE MAPS IN SYMMETRIC SPACES WITH APPLICATION TO PROBABILISTIC SPACES*

Mohamed Aamri, Abdelhafid Bassou and Driss EL Moutawakil†

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Abstract

In this paper, we prove a common fixed point theorem in symmetric spaces for weakly compatible maps without appeal to continuity, which generalizes the result of Hicks and Rhoades [1]. At the end, we give an application of our main Theorem to probabilistic spaces.

1 Introduction

In 1968, Jungck [2] introduced the concept of compatibility, which is more general than that of weak commutativity introduced by Sessa [5], as follows.

DEFINITION 1.1 ([2]). Let T and S be two selfmappings of a metric space (X, d). S and T are said to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever (x_n) is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

In 1998, Jungck and Rhoades [3] introduced the following concept of weak compatibility.

DEFINITION 1.2 ([3]). Two selfmappings T and S of a metric space X are said to be weakly compatible if they commute at three coincidence points, i.e., if $Tu = Su$ for some $u \in X$, then $TSu = STu$.

In 1999, Hicks and Rhoades [1] proved a common fixed point theorem for commuting and continuous maps in symmetric spaces.

THEOREM 1 ([1]). Let d be a bounded symmetric (semi-metric) for X that satisfies $(W.3)$ below. Suppose (X, d) is S-complete (d-Cauchy complete) and $f : X \longrightarrow X$ is d-continuous (t(d)-continuous). Then f has a fixed point if and only if there exists $\alpha \in (0, 1)$ and a d-continuous (t(d)-continuous) function $g : X \longrightarrow X$ which commutes with f and satisfies

$$g(X) \subseteq f(X) \text{ and } d(gx, gy) \leq \alpha d(fx, fy)), \text{ for all } x, y \in X. \quad (1)$$

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†Department of Mathematics and Informatics, Faculty of Sciences Ben M'Sik, Casablanca, Morocco.

Indeed, f and g have a unique common fixed point if (1) holds.

Further, they pointed out that if one adds condition $(W.4)$ below, then one can replace commuting condition in this Theorem with compatibility.

Our purpose in this paper is to prove that these assumptions (continuity and compatibility) are still too strong. Indeed, we claim that this theorem can be improved in two ways: we do not use any continuity requirement neither for f nor for g and one can replace the compatibility condition with weak compatibility.

We begin by recalling some basic concepts of the theory of symmetric spaces needed in the sequel. A symmetric function on a set X is a nonnegative real valued function d on $X \times X$ such that (1) $d(x, y) = 0$ if and only if $x = y$, and (2) $d(x, y) = d(y, x)$.

Let d be a symmetric on a set X and for $r > 0$ and any $x \in X$, let $B(x, r) = \{y \in X : d(x, y) < r\}$. A topology $t(d)$ on X is given by $U \in t(d)$ if and only if for each $x \in U$, $B(x, r) \subset U$ for some $r > 0$. A symmetric d is a semi-metric if for each $x \in X$ and each $r > 0$, $B(x, r)$ is a neighborhood of x in the topology $t(d)$. Note that $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $x_n \longrightarrow x$ in the topology $t(d)$.

In order to unify the notation, we need the following two axioms $(W.3)$ and $(W.4)$ given by Wilson [5] in a symmetric space (X, d):

$(W.3)$ Given $\{x_n\}, x$ and y in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply $x = y$.

$(W.4)$ Given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ imply that $\lim_{n\to\infty} d(y_n, x) = 0$.

A sequence in X is said to be a d-Cauchy sequence if it satisfies the usual metric condition. There are several concepts of completeness in this setting (see [1]):

(i) X is S-complete if for every d-Cauchy sequence (x_n), there exists x in X with $\lim_{n\to\infty} d(x, x_n) = 0$.

(ii) X is d-Cauchy complete if for every d-Cauchy sequence $\{x_n\}$, there exists x in X with $x_n \longrightarrow x$ in the topology $t(d)$.

REMARK 1.1. Let (X, d) be a symmetric space and let $\{x_n\}$ be a d-Cauchy sequence. If X is S-complete, then there exists $x \in X$ such that $\lim_{n\to\infty} d(x, x_n) = 0$. Therefore S-completeness implies d-Cauchy completeness.

2 Main results

In what follows, $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a nondecreasing function satisfying, for all $t \in (0, +\infty)$, $\lim_{n\to\infty} \psi^n(t) = 0$. It is easy to see that under these conditions, the function ψ satisfies also $\psi(t) < t$ for all $t > 0$.

THEOREM 2.1. Let (X, d) be a d-bounded symmetric space that satisfies $(W.3)$. Let A and B be two weakly compatible selfmappings of X such that:

(i) $d(Ax, Ay) \le \psi(d(Bx, By)), \quad \forall x, y \in X,$

(ii) $AX \subseteq BX$.

If the range of A or B is a S-complete subspace of X, then A and B have a unique fixed point.

PROOF. Let $x_0 \in X$. Choose $x_1 \in X$ such that $Ax_0 = Bx_1$. Choose $x_2 \in X$ such that $Ax_1 = Bx_2$. Continuing in this fashion, choose $x_n \in X$ such that $Ax_{n-1} = Bx_n$. We claim that $(Ax_n), n = 1, 2, ...,$ is a d-Cauchy sequence. Indeed, we have:

$$\begin{aligned} d(Ax_n, Ax_{n+m}) &\leq \psi(d(Bx_n, Bx_{n+m})) = \psi(d(Ax_{n-1}, Ax_{n+m-1})) \\ &\leq \psi^2(d(Bx_{n-1}, Bx_{n+m-1})) = \psi^2(d(Ax_{n-2}, Ax_{n+m-2})) \\ &. \\ &. \\ &. \\ &\leq \psi^n(d(Ax_0, Ax_m)) \leq \psi^n(\delta_d(X)) \end{aligned}$$

where $\delta_d(X) = \sup\{d(x, y)/x, y \in X\}$. Hence (Ax_n) is a d-Cauchy sequence. Suppose that BX is S-complete, then $\lim_{n\to\infty} d(Bu, Ax_n) = 0$ for some $u \in X$, and therefore $\lim_{n\to\infty} d(Bu, Bx_n) = 0$. We show that $Au = Bu$. Indeed:

$$d(Au, Ax_n) \leq \psi(d(Bu, Bx_n))$$

therefore $\lim_{n\to\infty} d(Au, Ax_n) = \lim_{n\to\infty} d(Bu, Ax_n) = 0$ and $(W.3)$ implies that $Au = Bu$. The assumption that A and B are weakly compatible implies $ABu = BAu$. Suppose that $d(Bu, BBu) \neq 0$. From (i), it follows

$$d(Bu, BBu) = d(Au, ABu) \leq \psi(d(Bu, BBu)) < d(Bu, BBu)$$

which is a contradiction. Thus $d(Bu, BBu) = 0$ and therefore $BBu = Bu$. Also $ABu = BAu = BBu = Bu$ which implies that Bu is a common fixed point of A and B. Now, if the range of A is a S-complete subspace of X, then $\lim_{n\to\infty} d(Ax, Ax_n) = 0$ for some $x \in X$. Since $AX \subseteq BX$, there exists $u \in X$ such that $Ax = Bu$ and the proof that Bu is a common fixed point of A and B is the same as that given when BX is S-complete. Finally to prove uniqueness, suppose that there exists $u, v \in X$ such that $Au = Bu = u$ and $Av = Bv = v$. If $d(u, v) \neq 0$, then

$$d(u, v) = d(Au, Av) \leq \psi(d(Bu, Bv)) = \psi(d(u, v)) < d(u, v)$$

which is a contradiction. Consequently $d(u, v) = 0$ and therefore $u = v$. The proof is complete.

When $\psi(t) = \alpha t$, $\alpha \in [0, 1)$, Theorem 2.1 gives a generalization of Theorem 1 in [1] in the following way:

COROLLARY 2.1. Let (X, d) be a d-bounded symmetric space that satisfies $(W.3)$. Let A and B be two weakly compatible selfmappings of X such that:

(i) $d(Ax, Ay) \leq \alpha d(Bx, By)$, $\alpha \in [0, 1)$, $\forall x, y \in X$,

(ii) $AX \subseteq BX$.

If the range of A or B is a S-complete subspace of X, then A and B have a unique fixed point.

3 Application

In this section, our goal is to give an application of our main Theorem to probabilistic spaces. We start with some definitions and recent results regarding these spaces. Throughout this section, a distribution function f is a nondecreasing, left continuous real-valued function f defined on the set of real numbers, with $\inf f = 0$ and $\sup f = 1$.

DEFINITION 3.1. Let X be a set and $\Im$ a function defined on $X \times X$ such that $\Im(x, y) = F_{x,y}$ is a distribution function. Consider the following conditions:

I. $F_{x,y}(0) = 0$ for all $x, y \in X$.

II. $F_{x,y} = H$ if and only if $x = y$, where H denotes the distribution function defined by $H(x) = 0$ if $x \leq 0$ and $H(x) = 1$ if $x > 0$.

III. $F_{x,y} = F_{y,x}$.

IV. If $F_{x,y}(\epsilon) = 1$ and $F_{y,z}(\delta) = 1$ then $F_{x,z}(\epsilon + \delta) = 1$.

If $\Im$ satisfies I and II, then it is called a PPM-structure on X and the pair $(X, \Im)$ is called a PPM space, while $\Im$ satisfying III is said to be symmetric. A symmetric PPM-structure $\Im$ satisfying IV is a probabilistic metric structure and the pair $(X, \Im)$ is a probabilistic metric space.

Let $(X, \Im)$ be a symmetric PPM-space. For $\epsilon, \lambda > 0$ and x in X, let $N_x(\epsilon, \lambda) = \{y \in X : F_{x,y}(\epsilon) > 1 - \lambda\}$. A T_1 topology $t(\Im)$ on X is defined as follows:

$$t(\Im) = \{U \subseteq X | \text{ for each } x \in U, \text{ there exists } \epsilon > 0, \text{ such that } N_x(\epsilon, \epsilon) \subseteq U\}.$$

Recall that a sequence $\{x_n\}$ is called a fundamental sequence if $\lim_{n\to\infty} F_{x_n,x_m}(t) = 1$ for all $t > 0$. The space $(X, \Im)$ is called F-complete if for every fundamental sequence $\{x_n\}$ there exists x in X such that $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$. Note that condition $(W.3)$, defined earlier, is equivalent to the following condition:

$$P(3) \lim_{n\to\infty} F_{x_n,x}(t) = 1 \text{ and } \lim_{n\to\infty} F_{x_n,y}(t) = 1 \text{ imply } x = y.$$

In [1], Hicks and Rhoades proved that each symmetric PPM-space admits a compatible symmetric function as follows:

THEOREM 2 ([1]). Let $(X, \Im)$ be a symmetric PPM-space. Let $p : X \times X \longrightarrow I\!R^+$ be a function defined as follows:

$$d(x,y) = \begin{cases} 0 & \text{if } y \in N_x(t,t) \text{ for all } t > 0. \\ \sup\{t : y \notin N_x(t,t), \ 0 < t < 1\} & \text{otherwise.} \end{cases}$$

Then

(1) $d(x, y) < t$ if and only if $F_{x,y}(t) > 1 - t$.

(2) d is a compatible symmetric for $t(\Im)$.

(3) $(X, \Im)$ is F-complete if and only if (X, d) is S-complete.

REMARK 3.1. In the sequel, we consider a nondecreasing, right continuous function $\psi : I\!R^+ \longrightarrow I\!R^+$ such that $\lim_{n\to\infty}\psi^n(t) = 0$ for $t \in (0, +\infty)$.

Under the above properties, ψ satisfies $\psi(t) < t$ for all $t > 0$ and therefore $\psi(0) = 0$.

THEOREM 3.1. Let $(X, \Im)$ be a symmetric PPM space satisfying $P(3)$ and d a compatible symmetric function for $t(\Im)$. Let A and B be two weakly compatible selfmappings of X such that:

(1) $F_{Bx,By}(t) > 1 - t$ implies $F_{Ax,Ay}(\psi(t)) > 1 - \psi(t)$, for all $t > 0$, $\forall x, y \in X$,

(2) $AX \subset BX$.

If the range of A or B is a F-complete subspace of X, then A and B have a unique common fixed point.

PROOF. In view of Theorem 3.1, (X, d) is d-bounded and BX is a S-complete subspace of X. Also $d(x, y) < t$ if and only if $F_{x,y}(t) > 1 - t$. Let $\epsilon > 0$ be given, and set $t = d(Bx, By) + \epsilon$. Then $d(Bx, By) < t$ gives $F_{Bx,By}(t) > 1 - t$ and therefore $F_{Ax,Ay}(\psi(t)) > 1 - \psi(t)$ which implies that $d(Ax, Ay) < \psi(t) = \psi(d(Bx, By) + t)$. On letting ϵ to 0, we have $d(Ax, Ay) \leq \psi(d(Bx, By))$. Now apply Theorem 2.1.

For $\psi(t) = kt,\ k \in [0, 1)$, Theorem 3.1 is reduced to the following new result:

COROLLARY 3.1. Let $(X, \Im)$ be a symmetric PPM space satisfying $P(3)$ and d a compatible symmetric function for $t(\Im)$. Let A and B be two weakly compatible selfmappings of X such that:

(1) $F_{Bx,By}(t) > 1 - t$ implies $F_{Ax,Ay}(kt) > 1 - kt$, $k \in [0, 1[$, for all $t > 0$, $\forall x, y \in X$,

(2) $AX \subset BX$.

If the range of A or B is a F-complete subspace of X, then A and B have a unique common fixed point.

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