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Common fixed points of almost generalized $(\psi, \varphi)_s$ -contractive mappings in ordered b -metric spaces

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Abstract

In this paper, we introduce the notion of almost generalized $(\psi, \varphi)_s$ -contractive mappings and we establish some fixed and common fixed point results for this class of mappings in ordered complete b -metric spaces. Our results generalize several well-known comparable results in the literature. Finally, two examples support our results.

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1 Introduction

A fundamental principle in computer science is iteration. Iterative techniques are used to find roots of equations and solutions of linear and nonlinear systems of equations and differential equations. So, the attractiveness of the fixed point iteration is understandable to a large number of mathematicians.

The Banach contraction principle [1] is a very popular tool for solving problems in non-linear analysis. Some authors generalized this interesting theorem in different ways (see, e.g., [2–16]).

Berinde in [17, 18] initiated the concept of almost contractions and obtained many interesting fixed point theorems for a Ćirić strong almost contraction.

Now, let us recall the following definition.

Definition 1 [17] A single-valued mapping $f : X \rightarrow X$ is called a Ćirić strong almost contraction if there exist a constant $\alpha \in [0, 1)$ and some $L \geq 0$ such that

$$d(fx, fy) \leq \alpha M(x, y) + Ld(y, fx)$$

for all $x, y \in X$, where

$$M(x, y) = \max \left\{ d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2} \right\}.$$

Babu in [19] introduced the class of mappings which satisfy condition (B).

Definition 2 [19] Let (X, d) be a metric space. A mapping $f : X \rightarrow X$ is said to satisfy condition (B) if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(fx, fy) \leq \delta d(x, y) + L \min\{d(x, fx), d(x, fy), d(y, fx)\}$$

for all $x, y \in X$.

Moreover, Babu in [19] proved the existence of a fixed point for such mappings on complete metric spaces.

Ćirić *et al.* in [20] introduced the concept of almost generalized contractive condition and they proved some existing results.

Definition 3 [20] Let (X, \preceq) be a partially ordered set. Two mappings $f, g : X \rightarrow X$ are said to be strictly weakly increasing if $fx \prec gfx$ and $gx \prec fgx$, for all $x \in X$.

Definition 4 [20] Let f and g be two self mappings on a metric space (X, d) . Then they are said to satisfy almost generalized contractive condition, if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(fx, fy) \leq \delta \max\left\{d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(y, fx)}{2}\right\} + L \min\{d(x, fx), d(x, gy), d(y, fx)\} \tag{1.1}$$

for all $x, y \in X$.

Ćirić *et al.* in [20] proved the following theorems.

Theorem 1 Let (X, \preceq) be a partially ordered set and suppose that there exists a metric d on X such that the metric space (X, d) is complete. Let $f : X \rightarrow X$ be a strictly increasing continuous mapping with respect to \preceq . Suppose that there exist a constant $\delta \in [0, 1)$ and some $L \geq 0$ such that

$$d(fx, fy) \leq \delta M(x, y) + L \min\{d(x, fx), d(x, fy), d(y, fx)\}$$

for all comparable elements $x, y \in X$, where

$$M(x, y) = \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}\right\}.$$

If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point in X .

Theorem 2 Let (X, \preceq) be a partially ordered set and suppose that there exists a metric d on X such that the metric space (X, d) is complete. Let $f, g : X \rightarrow X$ be two strictly weakly increasing mappings which satisfy (1.1) with respect to \preceq , for all comparable elements $x, y \in X$. If either f or g is continuous, then f and g have a common fixed point in X .

Khan *et al.* [21] introduced the concept of an altering distance function as follows.

Definition 5 [21] The function $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ is called an altering distance function, if the following properties hold:

1. φ is continuous and non-decreasing.
2. $\varphi(t) = 0$ if and only if $t = 0$.

So far, many authors have studied fixed point theorems which are based on altering distance functions (see, e.g., [2, 21–30]).

The concept of a b -metric space was introduced by Czerwik in [31]. After that, several interesting results about the existence of a fixed point for single-valued and multi-valued operators in b -metric spaces have been obtained (see [2, 32–42]). Pacurar [40] proved some results on sequences of almost contractions and fixed points in b -metric spaces. Recently, Hussain and Shah [37] obtained some results on KKM mappings in cone b -metric spaces.

Consistent with [31] and [42], the following definitions and results will be needed in the sequel.

Definition 6 [31] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \rightarrow R^+$ is a b -metric iff for all $x, y, z \in X$, the following conditions hold:

- (b₁) $d(x, y) = 0$ iff $x = y$,
- (b₂) $d(x, y) = d(y, x)$,
- (b₃) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b -metric space.

It should be noted that, the class of b -metric spaces is effectively larger than the class of metric spaces, since a b -metric is a metric, when $s = 1$.

Here, we present an example to show that in general, a b -metric need not necessarily be a metric (see also [42, p.264]):

Example 1 Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. We show that ρ is a b -metric with $s = 2^{p-1}$.

Obviously, conditions (b₁) and (b₂) of Definition 6 are satisfied.

If $1 < p < \infty$, then the convexity of the function $f(x) = x^p$ ($x > 0$) implies

$$\left(\frac{a+b}{2}\right)^p \leq \frac{1}{2}(a^p + b^p),$$

and hence, $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ holds.

Thus, for each $x, y, z \in X$,

$$\begin{aligned} \rho(x, y) &= (d(x, y))^p \leq (d(x, z) + d(z, y))^p \leq 2^{p-1}((d(x, z))^p + (d(z, y))^p) \\ &= 2^{p-1}(\rho(x, z) + \rho(z, y)). \end{aligned}$$

So, condition (b₃) of Definition 6 is also satisfied and ρ is a b -metric.

Definition 7 [35] Let (X, d) be a b -metric space. Then a sequence $\{x_n\}$ in X is called:

- (a) *b*-convergent if and only if there exists $x \in X$ such that $d(x_n, x) \rightarrow 0$, as $n \rightarrow +\infty$. In this case, we write $\lim_{n \rightarrow \infty} x_n = x$.
- (b) *b*-Cauchy if and only if $d(x_n, x_m) \rightarrow 0$ as $n, m \rightarrow +\infty$.

Proposition 1 (See Remark 2.1 in [35]) *In a b-metric space (X, d) the following assertions hold:*

- (p₁) *A b-convergent sequence has a unique limit.*
- (p₂) *Each b-convergent sequence is b-Cauchy.*
- (p₃) *In general, a b-metric is not continuous.*

Definition 8 [35] *The b-metric space (X, d) is b-complete if every b-Cauchy sequence in X b-converges.*

It should be noted that, in general a *b*-metric function $d(x, y)$ for $s > 1$ is not jointly continuous in all its variables. The following example is an example of a *b*-metric which is not continuous.

Example 2 (see Example 3 in [36]) *Let $X = \mathbb{N} \cup \{\infty\}$ and let $D : X \times X \rightarrow \mathbb{R}$ be defined by*

$$D(m, n) = \begin{cases} 0 & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}| & \text{if } m, n \text{ are even or } mn = \infty, \\ 5 & \text{if } m \text{ and } n \text{ are odd and } m \neq n, \\ 2 & \text{otherwise.} \end{cases}$$

Then it is easy to see that for all $m, n, p \in X$, we have

$$D(m, p) \leq 3(D(m, n) + D(n, p)).$$

Thus, (X, D) is a *b*-metric space with $s = 3$. Let $x_n = 2n$ for each $n \in \mathbb{N}$. Then

$$D(2n, \infty) = \frac{1}{2n} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

that is, $x_n \rightarrow \infty$, but $D(x_{2n}, 1) = 2 \not\rightarrow D(\infty, 1)$ as $n \rightarrow \infty$.

Aghajani *et al.* [2] proved the following simple lemma about the *b*-convergent sequences.

Lemma 1 *Let (X, d) be a b-metric space with $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ b-converge to x, y , respectively. Then, we have*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n \rightarrow \infty} d(x_n, y_n) \leq \limsup_{n \rightarrow \infty} d(x_n, y_n) \leq s^2 d(x, y).$$

In particular, if $x = y$, then, $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$ we have

$$\frac{1}{s} d(x, z) \leq \liminf_{n \rightarrow \infty} d(x_n, z) \leq \limsup_{n \rightarrow \infty} d(x_n, z) \leq s d(x, z).$$

In this paper, we introduce the notion of an almost generalized $(\psi, \varphi)_s$ -contractive mapping and we establish some results in complete ordered b -metric spaces, where ψ and φ are altering distance functions. Our results generalize Theorems 1 and 2 and all results in [28] and several comparable results in the literature.

2 Main results

In this section, we define the notion of almost generalized $(\psi, \varphi)_s$ -contractive mapping and prove our new results. In particular, we generalize Theorems 2.1, 2.2 and 2.3 of Ćirić *et al.* in [20].

Let (X, \preceq, d) be an ordered b -metric space and let $f : X \rightarrow X$ be a mapping. Set

$$M_s(x, y) = \max \left\{ d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2s} \right\}$$

and

$$N(x, y) = \min \{ d(x, fx), d(y, fy) \}.$$

Definition 9 Let (X, d) be a b -metric space. We say that a mapping $f : X \rightarrow X$ is an almost generalized $(\psi, \varphi)_s$ -contractive mapping if there exist $L \geq 0$ and two altering distance functions ψ and φ such that

$$\psi(sd(fx, fy)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)) + L\psi(N(x, y)) \tag{2.1}$$

for all $x, y \in X$.

Now, let us prove our first result.

Theorem 3 Let (X, \preceq) be a partially ordered set and suppose that there exists a b -metric d on X such that (X, d) is a b -complete b -metric space. Let $f : X \rightarrow X$ be a non-decreasing continuous mapping with respect to \preceq . Suppose that f satisfies condition (2.1), for all comparable elements $x, y \in X$. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof Let $x_0 \in X$. Then, we define a sequence (x_n) in X such that $x_{n+1} = fx_n$, for all $n \geq 0$. Since $x_0 \preceq fx_0 = x_1$ and f is non-decreasing, we have $x_1 = fx_0 \preceq x_2 = fx_1$. Again, as $x_1 \preceq x_2$ and f is non-decreasing, we have $x_2 = fx_1 \preceq x_3 = fx_2$. By induction, we have

$$x_0 \preceq x_1 \preceq \dots \preceq x_n \preceq x_{n+1} \preceq \dots$$

If $x_n = x_{n+1}$, for some $n \in \mathbb{N}$, then $x_n = fx_n$ and hence x_n is a fixed point of f . So, we may assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. By (2.1), we have

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &\leq \psi(sd(x_n, x_{n+1})) \\ &= \psi(sd(fx_{n-1}, fx_n)) \\ &\leq \psi(M_s(x_{n-1}, x_n)) - \varphi(M_s(x_{n-1}, x_n)) + L\psi(N(x_{n-1}, x_n)), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
 M_s(x_{n-1}, x_n) &= \max \left\{ d(x_{n-1}, x_n), d(x_{n-1}, fx_{n-1}), d(x_n, fx_n), \frac{d(x_{n-1}, fx_n) + d(x_n, fx_{n-1})}{2s} \right\} \\
 &= \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2s} \right\} \\
 &\leq \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{sd(x_{n-1}, x_n) + sd(x_n, x_{n+1})}{2s} \right\} \\
 &= \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} \right\} \\
 &= \max \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \}
 \end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
 N(x_{n-1}, x_n) &= \min \{ d(x_{n-1}, fx_{n-1}), d(x_n, fx_n) \} \\
 &= \min \{ d(x_{n-1}, x_n), 0 \} = 0.
 \end{aligned} \tag{2.4}$$

From (2.2)-(2.4) and the properties of ψ and φ , we get

$$\begin{aligned}
 \psi(d(x_n, x_{n+1})) &\leq \psi(\max \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \}) - \varphi(\max \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \}) \\
 &< \psi(\max \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \}).
 \end{aligned} \tag{2.5}$$

If

$$\max \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \} = d(x_n, x_{n+1}),$$

then by (2.5) we have

$$\begin{aligned}
 \psi(d(x_n, x_{n+1})) &\leq \psi(d(x_n, x_{n+1})) - \varphi(d(x_n, x_{n+1})) \\
 &< \psi(d(x_n, x_{n+1})),
 \end{aligned}$$

which gives a contradiction. Thus,

$$\max \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \} = d(x_{n-1}, x_n).$$

Therefore (2.5) becomes

$$\psi(d(x_n, x_{n+1})) \leq \psi(d(x_n, x_{n-1})) - \varphi(d(x_{n-1}, x_n)) < \psi(d(x_n, x_{n-1})). \tag{2.6}$$

Since ψ is a non-decreasing mapping, $\{d(x_n, x_{n+1}) : n \in \mathbb{N} \cup \{0\}\}$ is a non-increasing sequence of positive numbers. So, there exists $r \geq 0$ such that

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = r.$$

Letting $n \rightarrow \infty$ in (2.6), we get

$$\psi(r) \leq \psi(r) - \varphi(r) \leq \psi(r).$$

Therefore, $\varphi(r) = 0$, and hence $r = 0$. Thus, we have

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \tag{2.7}$$

Next, we show that $\{x_n\}$ is a b -Cauchy sequence in X . Suppose the contrary, that is, $\{x_n\}$ is not a b -Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i, \quad d(x_{m_i}, x_{n_i}) \geq \varepsilon. \tag{2.8}$$

This means that

$$d(x_{m_i}, x_{n_i-1}) < \varepsilon. \tag{2.9}$$

From (2.8), (2.9) and using the triangular inequality, we get

$$\begin{aligned} \varepsilon &\leq d(x_{m_i}, x_{n_i}) \\ &\leq sd(x_{m_i}, x_{m_i-1}) + sd(x_{m_i-1}, x_{n_i}) \\ &\leq sd(x_{m_i}, x_{m_i-1}) + s^2 d(x_{m_i-1}, x_{n_i-1}) + s^2 d(x_{n_i-1}, x_{n_i}). \end{aligned}$$

Using (2.7) and taking the upper limit as $i \rightarrow \infty$, we get

$$\frac{\varepsilon}{s^2} \leq \limsup_{i \rightarrow \infty} d(x_{m_i-1}, x_{n_i-1}).$$

On the other hand, we have

$$d(x_{m_i-1}, x_{n_i-1}) \leq sd(x_{m_i-1}, x_{m_i}) + sd(x_{m_i}, x_{n_i-1}).$$

Using (2.7), (2.9) and taking the upper limit as $i \rightarrow \infty$, we get

$$\limsup_{i \rightarrow \infty} d(x_{m_i-1}, x_{n_i-1}) \leq \varepsilon s.$$

So, we have

$$\frac{\varepsilon}{s^2} \leq \limsup_{i \rightarrow \infty} d(x_{m_i-1}, x_{n_i-1}) \leq \varepsilon s. \tag{2.10}$$

Again, using the triangular inequality, we have

$$\begin{aligned} d(x_{m_i-1}, x_{n_i}) &\leq sd(x_{m_i-1}, x_{n_i-1}) + sd(x_{n_i-1}, x_{n_i}), \\ \varepsilon &\leq d(x_{m_i}, x_{n_i}) \leq sd(x_{m_i}, x_{m_i-1}) + sd(x_{m_i-1}, x_{n_i}) \end{aligned}$$

and

$$\varepsilon \leq d(x_{m_i}, x_{n_i}) \leq sd(x_{m_i}, x_{n_i-1}) + sd(x_{n_i-1}, x_{n_i}).$$

Taking the upper limit as $i \rightarrow \infty$ in the first and second inequalities above, and using (2.7) and (2.10) we get

$$\frac{\varepsilon}{s} \leq \limsup_{i \rightarrow \infty} d(x_{m_i-1}, x_{n_i}) \leq \varepsilon s^2. \tag{2.11}$$

Similarly, taking the upper limit as $i \rightarrow \infty$ in the third inequality above, and using (2.7) and (2.9), we get

$$\frac{\varepsilon}{s} \leq \limsup_{i \rightarrow \infty} d(x_{m_i}, x_{n_i-1}) \leq \varepsilon. \tag{2.12}$$

From (2.1), we have

$$\begin{aligned} &\psi(sd(x_{m_i}, x_{n_i})) \\ &= \psi(sd(fx_{m_i-1}, fx_{n_i-1})) \\ &\leq \psi(M_s(x_{m_i-1}, x_{n_i-1})) - \varphi(M_s(x_{m_i-1}, x_{n_i-1})) + L\psi(N(x_{m_i-1}, x_{n_i-1})), \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} M_s(x_{m_i-1}, x_{n_i-1}) &= \max \left\{ d(x_{m_i-1}, x_{n_i-1}), d(x_{m_i-1}, fx_{m_i-1}), d(x_{n_i-1}, fx_{n_i-1}), \right. \\ &\quad \left. \frac{d(x_{m_i-1}, fx_{n_i-1}) + d(fx_{m_i-1}, x_{n_i-1})}{2s} \right\} \\ &= \max \left\{ d(x_{m_i-1}, x_{n_i-1}), d(x_{m_i-1}, x_{m_i}), d(x_{n_i-1}, x_{n_i}), \right. \\ &\quad \left. \frac{d(x_{m_i-1}, x_{n_i}) + d(x_{m_i}, x_{n_i-1})}{2s} \right\} \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} N(x_{m_i-1}, x_{n_i-1}) &= \min \{ d(x_{m_i-1}, fx_{m_i-1}), d(fx_{m_i-1}, x_{n_i-1}) \} \\ &= \min \{ d(x_{m_i-1}, x_{m_i}), d(x_{m_i}, x_{n_i-1}) \}. \end{aligned} \tag{2.15}$$

Taking the upper limit as $i \rightarrow \infty$ in (2.14) and (2.15) and using (2.7), (2.10), (2.11) and (2.12), we get

$$\begin{aligned} \frac{\varepsilon}{s^2} &= \min \left\{ \frac{\varepsilon}{s^2}, \frac{\frac{\varepsilon}{s} + \frac{\varepsilon}{s}}{2s} \right\} \\ &\leq \limsup_{i \rightarrow \infty} M_s(x_{m_i-1}, x_{n_i-1}) \\ &= \max \left\{ \limsup_{i \rightarrow \infty} d(x_{m_i-1}, x_{n_i-1}), 0, 0, \right. \end{aligned}$$

$$\left. \frac{\limsup_{i \rightarrow \infty} d(x_{m_i-1}, x_{n_i}) + \limsup_{i \rightarrow \infty} d(x_{m_i}, x_{n_i-1})}{2s} \right\} \\ \leq \max \left\{ \varepsilon s, \frac{\varepsilon s^2 + \varepsilon}{2s} \right\} = \varepsilon s.$$

So, we have

$$\frac{\varepsilon}{s^2} \leq \limsup_{i \rightarrow \infty} M_s(x_{m_i-1}, x_{n_i-1}) \leq \varepsilon s \tag{2.16}$$

and

$$\limsup_{i \rightarrow \infty} N(x_{m_i-1}, x_{n_i-1}) = 0. \tag{2.17}$$

Similarly, we can obtain

$$\frac{\varepsilon}{s^2} \leq \liminf_{i \rightarrow \infty} M_s(x_{m_i-1}, x_{n_i-1}) \leq \varepsilon s. \tag{2.18}$$

Now, taking the upper limit as $i \rightarrow \infty$ in (2.13) and using (2.8), (2.16) and (2.17), we have

$$\begin{aligned} \psi(\varepsilon s) &\leq \psi \left(s \limsup_{i \rightarrow \infty} d(x_{m_i}, x_{n_i}) \right) \\ &\leq \psi \left(\limsup_{i \rightarrow \infty} M_s(x_{m_i-1}, x_{n_i-1}) \right) - \liminf_{i \rightarrow \infty} \varphi \left(M_s(x_{m_i-1}, x_{n_i-1}) \right) \\ &\leq \psi(\varepsilon s) - \varphi \left(\liminf_{i \rightarrow \infty} M_s(x_{m_i-1}, x_{n_i-1}) \right), \end{aligned}$$

which further implies that

$$\varphi \left(\liminf_{i \rightarrow \infty} M_s(x_{m_i-1}, x_{n_i-1}) \right) = 0,$$

so $\liminf_{i \rightarrow \infty} M_s(x_{m_i-1}, x_{n_i-1}) = 0$, a contradiction to (2.18). Thus, $\{x_{n+1} = fx_n\}$ is a b -Cauchy sequence in X . As X is a b -complete space, there exists $u \in X$ such that $x_n \rightarrow u$ as $n \rightarrow \infty$, and

$$\lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} fx_n = u.$$

Now, suppose that f is continuous. Using the triangular inequality, we get

$$d(u, fu) \leq sd(u, fx_n) + sd(fx_n, fu).$$

Letting $n \rightarrow \infty$, we get

$$d(u, fu) \leq s \lim_{n \rightarrow \infty} d(u, fx_n) + s \lim_{n \rightarrow \infty} d(fx_n, fu) = 0.$$

So, we have $fu = u$. Thus, u is a fixed point of f . □

Note that the continuity of f in Theorem 3 is not necessary and can be dropped.

Theorem 4 *Under the same hypotheses of Theorem 3, without the continuity assumption of f , assume that whenever $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \rightarrow x \in X$, $x_n \preceq x$, for all $n \in \mathbb{N}$, then f has a fixed point in X .*

Proof Following similar arguments to those given in Theorem 3, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \rightarrow u$, for some $u \in X$. Using the assumption on X , we have $x_n \preceq u$, for all $n \in \mathbb{N}$. Now, we show that $fu = u$. By (2.1), we have

$$\begin{aligned} \psi(sd(x_{n+1}, fu)) &= \psi(sd(fx_n, fu)) \\ &\leq \psi(M_s(x_n, u)) - \varphi(M_s(x_n, u)) + L\psi(N(x_n, u)), \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} M_s(x_n, u) &= \max \left\{ d(x_n, u), d(x_n, fx_n), d(u, fu), \frac{d(x_n, fu) + d(fx_n, u)}{2s} \right\} \\ &= \max \left\{ d(x_n, u), d(x_n, x_{n+1}), d(u, fu), \frac{d(x_n, fu) + d(x_{n+1}, u)}{2s} \right\} \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} N(x_n, u) &= \min \{ d(x_n, fx_n), d(u, fx_n) \} \\ &= \min \{ d(x_n, x_{n+1}), d(u, x_{n+1}) \}. \end{aligned} \tag{2.21}$$

Letting $n \rightarrow \infty$ in (2.20) and (2.21) and using Lemma 1, we get

$$\begin{aligned} \frac{d(u, fu)}{2s^2} &= \min \left\{ \frac{d(u, fu)}{s}, \frac{d(u, fu)}{2s^2} \right\} \\ &\leq \limsup_{n \rightarrow \infty} M_s(x_n, u) \\ &\leq \max \left\{ d(u, fu), \frac{sd(u, fu)}{2s} \right\} = d(u, fu) \end{aligned} \tag{2.22}$$

and

$$N(x_n, u) \rightarrow 0.$$

Similarly, we can obtain

$$\frac{d(u, fu)}{2s^2} \leq \liminf_{n \rightarrow \infty} M_s(x_n, u) \leq d(u, fu). \tag{2.23}$$

Again, taking the upper limit as $n \rightarrow \infty$ in (2.19) and using Lemma 1 and (2.22) we get

$$\begin{aligned} \psi(d(u, fu)) &= \psi\left(\frac{1}{s}d(u, fu)\right) \leq \psi\left(s \limsup_{n \rightarrow \infty} d(x_{n+1}, fu)\right) \\ &\leq \psi\left(\limsup_{n \rightarrow \infty} M_s(x_n, u)\right) - \liminf_{n \rightarrow \infty} \varphi(M_s(x_n, u)) \\ &\leq \psi(d(u, fu)) - \varphi\left(\liminf_{n \rightarrow \infty} M_s(x_n, u)\right). \end{aligned}$$

Therefore, $\varphi(\liminf_{n \rightarrow \infty} M_s(x_n, u)) \leq 0$, equivalently, $\liminf_{n \rightarrow \infty} M_s(x_n, u) = 0$. Thus, from (2.23) we get $u = fu$ and hence u is a fixed point of f . \square

Corollary 1 *Let (X, \preceq) be a partially ordered set and suppose that there exists a b -metric d on X such that (X, d) is a b -complete b -metric space. Let $f : X \rightarrow X$ be a non-decreasing continuous mapping with respect to \preceq . Suppose that there exist $k \in [0, 1)$ and $L \geq 0$ such that*

$$d(fx, fy) \leq \frac{k}{s} \max \left\{ d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2s} \right\} + \frac{L}{s} \min \{ d(x, fx), d(y, fy) \}$$

for all comparable elements $x, y \in X$. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof Follows from Theorem 3 by taking $\psi(t) = t$ and $\varphi(t) = (1 - k)t$, for all $t \in [0, +\infty)$. \square

Corollary 2 *Under the hypotheses of Corollary 1, without the continuity assumption of f , for any non-decreasing sequence $\{x_n\}$ in X such that $x_n \rightarrow x \in X$, let us have $x_n \preceq x$ for all $n \in \mathbb{N}$. Then, f has a fixed point in X .*

Let (X, d) be an ordered b -metric space and let $f, g : X \rightarrow X$ be two mappings. Set

$$M_s(x, y) = \max \left\{ d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(y, fx)}{2s} \right\}$$

and

$$N(x, y) = \min \{ d(x, fx), d(y, gy), d(x, gy) \}.$$

Now, we present the following definition.

Definition 10 Let (X, d) be a partially ordered b -metric space and let ψ and φ be altering distance functions. We say that a mapping $f : X \rightarrow X$ is an almost generalized $(\psi, \varphi)_s$ -contractive mapping with respect to a mapping $g : X \rightarrow X$, if there exists $L \geq 0$ such that

$$\psi(s^4 d(fx, gy)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)) + L\psi(N(x, y)) \tag{2.24}$$

for all $x, y \in X$.

Definition 11 Let (X, \preceq) be a partially ordered set. Then two mappings $f, g : X \rightarrow X$ are said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$, for all $x \in X$.

Theorem 5 *Let (X, \preceq) be a partially ordered set and suppose that there exists a b -metric d on X such that (X, d) is a b -complete b -metric space, and let $f, g : X \rightarrow X$ be two weakly increasing mappings with respect to \preceq . Suppose that f satisfies 2.24, for all comparable elements $x, y \in X$. If either f or g is continuous, then f and g have a common fixed point.*

Proof Let us divide the proof into two parts as follows.

First part: We prove that u is a fixed point of f if and only if u is a fixed point of g . Suppose that u is a fixed point of f ; that is, $fu = u$. As $u \leq u$, by (2.24), we have

$$\begin{aligned} \psi(s^4 d(u, gu)) &= \psi(s^4 d(fu, gu)) \\ &\leq \psi\left(\max\left\{d(u, fu), d(u, gu), \frac{1}{2s}(d(u, gu) + d(u, fu))\right\}\right) \\ &\quad - \varphi\left(\max\left\{d(u, fu), d(u, gu), \frac{1}{2s}(d(u, gu) + d(u, fu))\right\}\right) \\ &\quad + L \min\{d(u, fu), d(u, gu)\} \\ &= \psi(d(u, gu)) - \varphi(d(u, gu)) \\ &\leq \psi(s^4 d(u, gu)) - \varphi(d(u, gu)). \end{aligned}$$

Thus, we have $\varphi(d(u, gu)) = 0$. Therefore, $d(u, gu) = 0$ and hence $gu = u$. Similarly, we can show that if u is a fixed point of g , then u is a fixed point of f .

Second part (construction of a sequence by iterative technique):

Let $x_0 \in X$. We construct a sequence $\{x_n\}$ in X such that $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$, for all non-negative integers. As f and g are weakly increasing with respect to \leq , we have:

$$x_1 = fx_0 \leq gfx_0 = x_2 = gx_1 \leq fgx_1 = x_3 \leq \dots \leq x_{2n+1} = fx_{2n} \leq gfx_{2n} = x_{2n+2} \leq \dots$$

If $x_{2n} = x_{2n+1}$, for some $n \in \mathbb{N}$, then $x_{2n} = fx_{2n}$. Thus x_{2n} is a fixed point of f . By the first part, we conclude that x_{2n} is also a fixed point of g .

If $x_{2n+1} = x_{2n+2}$, for some $n \in \mathbb{N}$, then $x_{2n+1} = gx_{2n+1}$. Thus, x_{2n+1} is a fixed point of g . By the first part, we conclude that x_{2n+1} is also a fixed point of f . Therefore, we assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Now, we complete the proof in the following steps.

Step 1: We will prove that

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0.$$

As x_{2n} and x_{2n+1} are comparable, by (2.24), we have

$$\begin{aligned} \psi(d(x_{2n+1}, x_{2n+2})) &\leq \psi(s^4 d(x_{2n+1}, x_{2n+2})) \\ &= \psi(s^4 d(fx_{2n}, gx_{2n+1})) \\ &\leq \psi(M_s(x_{2n}, x_{2n+1})) - \varphi(M_s(x_{2n}, x_{2n+1})) + L\psi(N(x_{2n}, x_{2n+1})), \end{aligned}$$

where

$$\begin{aligned} M_s(x_{2n}, x_{2n+1}) &= \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n}, fx_{2n}), d(x_{2n+1}, gx_{2n+1}), \right. \\ &\quad \left. \frac{d(fx_{2n}, x_{2n+1}) + d(x_{2n}, gx_{2n+1})}{2s}\right\} \\ &= \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+2})}{2s}\right\} \end{aligned}$$

$$\begin{aligned} &\leq \max \left\{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{sd(x_{2n}, x_{2n+1}) + sd(x_{2n+1}, x_{2n+2})}{2s} \right\} \\ &= \max \{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \} \end{aligned}$$

and

$$\begin{aligned} N(x_{2n}, x_{2n+1}) &= \min \{ d(x_{2n}, fx_{2n}), d(x_{2n+1}, fx_{2n}), d(x_{2n}, gx_{2n+1}) \} \\ &= \min \{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+1}), d(x_{2n}, x_{2n+2}) \} \\ &= 0. \end{aligned}$$

So, we have

$$\begin{aligned} \psi(d(x_{2n+1}, x_{2n+2})) &\leq \psi(\max \{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \}) \\ &\quad - \varphi(\max \{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \}). \end{aligned} \tag{2.25}$$

If

$$\max \{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \} = d(x_{2n+1}, x_{2n+2}),$$

then (2.25) becomes

$$\psi(d(x_{2n+1}, x_{2n+2})) \leq \psi(d(x_{2n+1}, x_{2n+2})) - \varphi(d(x_{2n+1}, x_{2n+2})) < \psi(d(x_{2n+1}, x_{2n+2})),$$

which gives a contradiction. So,

$$\max \{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \} = d(x_{2n}, x_{2n+1})$$

and hence (2.25) becomes

$$\psi(d(x_{2n+1}, x_{2n+2})) \leq \psi(d(x_{2n}, x_{2n+1})) - \varphi(d(x_{2n}, x_{2n+1})) \leq \psi(d(x_{2n}, x_{2n+1})). \tag{2.26}$$

Similarly, we can show that

$$\psi(d(x_{2n+1}, x_{2n})) \leq \psi(d(x_{2n-1}, x_{2n})) - \varphi(d(x_{2n-1}, x_{2n})) \leq \psi(d(x_{2n-1}, x_{2n})). \tag{2.27}$$

By (2.26) and (2.27), we get that $\{d(x_n, x_{n+1}); n \in \mathbb{N}\}$ is a non-increasing sequence of positive numbers. Hence there is $r \geq 0$ such that

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = r.$$

Letting $n \rightarrow \infty$ in (2.26), we get

$$\psi(r) \leq \psi(r) - \varphi(r) \leq \psi(r),$$

which implies that $\varphi(r) = 0$ and hence $r = 0$. So, we have

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \tag{2.28}$$

Step 2: We will prove that $\{x_n\}$ is a b -Cauchy sequence. It is sufficient to show that $\{x_{2n}\}$ is a b -Cauchy sequence. Suppose the contrary, that is, $\{x_{2n}\}$ is not a b -Cauchy sequence. Then there exists $\epsilon > 0$, for which we can find two subsequences of positive integers $\{x_{2m_i}\}$ and $\{x_{2n_i}\}$ such that n_i is the smallest index for which

$$n_i > m_i > i, \quad d(x_{2m_i}, x_{2n_i}) \geq \epsilon. \tag{2.29}$$

This means that

$$d(x_{2m_i}, x_{2n_i-2}) < \epsilon. \tag{2.30}$$

From (2.29), (2.30) and the triangular inequality, we get

$$\begin{aligned} d(x_{2n_i+1}, x_{2m_i}) &\leq sd(x_{2n_i+1}, x_{2n_i}) + sd(x_{2n_i}, x_{2m_i}) \\ &< sd(x_{2n_i+1}, x_{2n_i}) + s^2d(x_{2n_i}, x_{2n_i-1}) + s^2d(x_{2n_i-1}, x_{2m_i}) \\ &\leq sd(x_{2n_i+1}, x_{2n_i}) + s^2d(x_{2n_i}, x_{2n_i-1}) + s^3d(x_{2n_i-1}, x_{2n_i-2}) + s^3d(x_{2n_i-2}, x_{2m_i}) \\ &< sd(x_{2n_i+1}, x_{2n_i}) + s^2d(x_{2n_i}, x_{2n_i-1}) + s^3d(x_{2n_i-1}, x_{2n_i-2}) + \epsilon s^3. \end{aligned}$$

Taking the upper limit in the above inequality and using (2.28), we have

$$\limsup_{i \rightarrow \infty} d(x_{2n_i+1}, x_{2m_i}) \leq \epsilon s^3.$$

Again, from (2.29) and the triangular inequality, we get

$$\epsilon \leq d(x_{2m_i}, x_{2n_i}) \leq sd(x_{2m_i}, x_{2n_i+1}) + sd(x_{2n_i+1}, x_{2n_i}).$$

Taking the upper limit in the above inequality and using (2.28), we have

$$\frac{\epsilon}{s} \leq \limsup_{i \rightarrow \infty} d(x_{2n_i+1}, x_{2m_i}).$$

So, we obtain

$$\frac{\epsilon}{s} \leq \limsup_{i \rightarrow \infty} d(x_{2n_i+1}, x_{2m_i}) \leq \epsilon s^3. \tag{2.31}$$

Similarly, we can obtain

$$\begin{aligned} \frac{\epsilon}{s} &\leq \limsup_{i \rightarrow \infty} d(x_{2n_i}, x_{2m_i-1}) \leq \epsilon s^3, \\ \epsilon &\leq \limsup_{i \rightarrow \infty} d(x_{2n_i}, x_{2m_i}) \leq \epsilon s^4, \\ \frac{\epsilon}{s^2} &\leq \limsup_{i \rightarrow \infty} d(x_{2n_i+1}, x_{2m_i-1}) \leq \epsilon s^4. \end{aligned} \tag{2.32}$$

Since x_{2n_i} and x_{2m_i-1} are comparable, using (2.24) we have

$$\begin{aligned} &\psi\left(s^4 d(x_{2n_i+1}, x_{2m_i})\right) \\ &= \psi\left(s^4 d(fx_{2n_i}, gx_{2m_i-1})\right) \\ &\leq \psi\left(M_s(x_{2n_i}, x_{2m_i-1})\right) - \varphi\left(M_s(x_{2n_i}, x_{2m_i-1})\right) + L\psi\left(N(x_{2n_i}, x_{2m_i-1})\right), \end{aligned} \tag{2.33}$$

where

$$\begin{aligned} M_s(x_{2n_i}, x_{2m_i-1}) &= \max\left\{d(x_{2n_i}, x_{2m_i-1}), d(x_{2n_i}, fx_{2n_i}), d(x_{2m_i-1}, gx_{2m_i-1}), \right. \\ &\quad \left. \frac{d(x_{2n_i}, gx_{2m_i-1}) + d(x_{2m_i-1}, fx_{2n_i})}{2s}\right\} \\ &= \max\left\{d(x_{2n_i}, x_{2m_i-1}), d(x_{2n_i}, x_{2n_i+1}), d(x_{2m_i-1}, x_{2m_i}), \right. \\ &\quad \left. \frac{d(x_{2n_i}, x_{2m_i}) + d(x_{2n_i+1}, x_{2m_i-1})}{2s}\right\} \end{aligned} \tag{2.34}$$

and

$$\begin{aligned} N(x_{2n_i}, x_{2m_i-1}) &= \min\{d(x_{2n_i}, fx_{2n_i}), d(x_{2m_i-1}, fx_{2n_i}), d(x_{2n_i}, gx_{2m_i-1})\} \\ &= \min\{d(x_{2n_i}, x_{2n_i+1}), d(x_{2m_i-1}, x_{2n_i+1}), d(x_{2n_i}, x_{2m_i})\}. \end{aligned} \tag{2.35}$$

Taking the upper limit in (2.34) and (2.35) and using (2.28) and (2.32), we get

$$\begin{aligned} \frac{\varepsilon}{2s} + \frac{\varepsilon}{2s^3} &= \min\left\{\frac{\varepsilon}{s}, \frac{\varepsilon + \frac{\varepsilon}{s^2}}{2s}\right\} \\ &\leq \limsup_{i \rightarrow \infty} M_s(x_{2n_i}, x_{2m_i-1}) \\ &= \max\left\{\limsup_{i \rightarrow \infty} d(x_{2n_i}, x_{2m_i-1}), 0, 0, \right. \\ &\quad \left. \frac{\limsup_{i \rightarrow \infty} d(x_{2n_i}, x_{2m_i}) + \limsup_{i \rightarrow \infty} d(x_{2n_i+1}, x_{2m_i-1})}{2s}\right\} \\ &\leq \left\{\varepsilon s^3, \frac{\varepsilon s^2 + \varepsilon s^4}{2s}\right\} = \varepsilon s^3. \end{aligned}$$

So, we have

$$\frac{\varepsilon}{2s} + \frac{\varepsilon}{2s^3} \leq \limsup_{i \rightarrow \infty} M_s(x_{2n_i}, x_{2m_i-1}) \leq \varepsilon s^3 \tag{2.36}$$

and

$$\limsup_{i \rightarrow \infty} N(x_{2n_i}, x_{2m_i-1}) = 0. \tag{2.37}$$

Similarly, we can obtain

$$\frac{\varepsilon}{2s} + \frac{\varepsilon}{2s^3} \leq \liminf_{i \rightarrow \infty} M_s(x_{2n_i}, x_{2m_i-1}) \leq \varepsilon s^3. \tag{2.38}$$

Now, taking the upper limit as $i \rightarrow \infty$ in (2.33) and using (2.36), (2.37) and (2.38), we have

$$\begin{aligned} \psi(\varepsilon s^3) &= \psi\left(s^4 \frac{\varepsilon}{s}\right) \leq \psi\left(s^4 \limsup_{i \rightarrow \infty} d(x_{2n_i+1}, x_{2m_i})\right) \\ &\leq \psi\left(\limsup_{i \rightarrow \infty} M_s(x_{2n_i}, x_{2m_i-1})\right) - \liminf_{i \rightarrow \infty} \varphi(M_s(x_{2n_i}, x_{2m_i-1})) \\ &\leq \psi(\varepsilon s^3) - \varphi\left(\liminf_{i \rightarrow \infty} M_s(x_{2n_i}, x_{2m_i-1})\right), \end{aligned}$$

which implies that

$$\varphi\left(\liminf_{i \rightarrow \infty} M_s(x_{2n_i}, x_{2m_i-1})\right) = 0,$$

so $\liminf_{i \rightarrow \infty} M_s(x_{m_i-1}, x_{n_i-1}) = 0$, a contradiction to (2.38). Hence $\{x_n\}$ is a b -Cauchy sequence in X .

Step 3 (Existence of a common fixed point):

As $\{x_n\}$ is a b -Cauchy sequence in X which is a b -complete b -metric space, there exists $u \in X$ such that $x_n \rightarrow u$ as $n \rightarrow \infty$, and

$$\lim_{n \rightarrow \infty} x_{2n+1} = \lim_{n \rightarrow \infty} f x_{2n} = u.$$

Now, without any loss of generality, we may assume that f is continuous. Using the triangular inequality, we get

$$d(u, fu) \leq sd(u, f x_{2n}) + sd(f x_{2n}, fu).$$

Letting $n \rightarrow \infty$, we get

$$d(u, fu) \leq s \lim_{n \rightarrow \infty} d(u, f x_{2n}) + s \lim_{n \rightarrow \infty} d(f x_{2n}, fu) = 0.$$

So, we have $fu = u$. Thus, u is a fixed point of f . By the first part, we conclude that u is also a fixed point of g . □

The continuity of one of the functions f or g in Theorem 5 is not necessary.

Theorem 6 *Under the hypotheses of Theorem 5, without the continuity assumption of one of the functions f or g , for any non-decreasing sequence $\{x_n\}$ in X such that $x_n \rightarrow x \in X$, let us have $x_n \leq x$, for all $n \in \mathbb{N}$. Then, f and g have a common fixed point in X .*

Proof Reviewing the proof of Theorem 5, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \rightarrow u$, for some $u \in X$. Using the assumption on X , we have $x_n \leq u$, for all $n \in \mathbb{N}$. Now, we show that $fu = gu = u$. By (2.24), we have

$$\begin{aligned} \psi(s^4 d(x_{2n+1}, gu)) &= \psi(s^4 d(f x_{2n}, gu)) \\ &\leq \psi(M_s(x_{2n}, u)) - \varphi(M_s(x_{2n}, u)) + L\psi(N(x_{2n}, u)), \end{aligned} \tag{2.39}$$

where

$$\begin{aligned}
 M_s(x_{2n}, u) &= \max \left\{ d(x_{2n}, u), d(x_{2n}, fx_{2n}), d(u, gu), \frac{d(x_{2n}, gu) + d(fx_{2n}, u)}{2s} \right\} \\
 &= \max \left\{ d(x_{2n}, u), d(x_{2n}, x_{2n+1}), d(u, gu), \frac{d(x_{2n}, gu) + d(x_{2n+1}, u)}{2s} \right\} \tag{2.40}
 \end{aligned}$$

and

$$\begin{aligned}
 N(x_{2n}, u) &= \min \{ d(x_{2n}, fx_{2n}), d(u, fx_{2n}), d(x_{2n}, gu) \} \\
 &= \min \{ d(x_{2n}, x_{2n+1}), d(u, x_{2n+1}), d(x_{2n}, gu) \}. \tag{2.41}
 \end{aligned}$$

Letting $n \rightarrow \infty$ in (2.40) and (2.41) and using Lemma 1, we get

$$\begin{aligned}
 \frac{d(u, gu)}{s} &= \min \left\{ \frac{d(u, gu)}{s}, \frac{d(u, gu)}{2s^2} \right\} \\
 &\leq \limsup_{n \rightarrow \infty} M_s(x_{2n}, u) \leq \max \left\{ d(u, gu), \frac{sd(u, gu)}{2s} \right\} = d(u, gu) \tag{2.42}
 \end{aligned}$$

and

$$N(x_{2n}, u) \rightarrow 0.$$

Similarly, we can obtain

$$\frac{d(u, gu)}{s} \leq \liminf_{n \rightarrow \infty} M_s(x_{2n}, u) \leq d(u, gu). \tag{2.43}$$

Again, taking the upper limit as $n \rightarrow \infty$ in (2.39) and using Lemma 1 and (2.42), we get

$$\begin{aligned}
 \psi(s^3 d(u, gu)) &= \psi \left(s^4 \frac{1}{s} d(u, gu) \right) \leq \psi \left(s^4 \limsup_{n \rightarrow \infty} d(x_{2n+1}, gu) \right) \\
 &\leq \psi \left(\limsup_{n \rightarrow \infty} M_s(x_{2n}, u) \right) - \liminf_{n \rightarrow \infty} \varphi(M_s(x_{2n}, u)) \\
 &\leq \psi(d(u, gu)) - \varphi \left(\liminf_{n \rightarrow \infty} M_s(x_{2n}, u) \right) \\
 &\leq \psi(s^3 d(u, gu)) - \varphi \left(\liminf_{n \rightarrow \infty} M_s(x_{2n}, u) \right).
 \end{aligned}$$

Therefore, $\varphi(\liminf_{n \rightarrow \infty} M_s(x_{2n}, u)) \leq 0$, equivalently, $\liminf_{n \rightarrow \infty} M_s(x_{2n}, u) = 0$. Thus, from (2.43) we get $u = gu$ and hence u is a fixed point of g . On the other hand, similar to the first part of the proof of Theorem 5, we can show that $fu = u$. Hence, u is a common fixed point of f and g . □

Also, we have the following results.

Corollary 3 *Let (X, \preceq) be a partially ordered set and suppose that there exists a b-metric d on X such that (X, d) is a b-complete b-metric space. Let $f, g : X \rightarrow X$ be two weakly*

increasing mappings with respect to \preceq . Suppose that there exist $k \in [0, 1)$ and $L \geq 0$ such that

$$d(fx, gy) \leq \frac{k}{s^4} \max \left\{ d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(fx, y)}{2s} \right\} + \frac{L}{s^4} \min \{ d(x, fx), d(y, gx), d(x, gy) \}$$

for all comparable elements $x, y \in X$. If either f or g is continuous, then f and g have a common fixed point.

Corollary 4 Under the hypotheses of Corollary 3, without the continuity assumption of one of the functions f or g , assume that whenever $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \rightarrow x \in X$, then $x_n \preceq x$, for all $n \in \mathbb{N}$. Then f and g have a common fixed point in X .

Now, in order to support the usability of our results, we present the following examples.

Example 3 Let $X = [0, \infty)$ be equipped with the b -metric $d(x, y) = |x - y|^2$ for all $x, y \in X$, where $s = 2^{2-1} = 2$.

Define a relation \preceq on X by $x \preceq y$ iff $y \leq x$, the functions $f, g : X \rightarrow X$ by

$$fx = \ln \left(1 + \frac{x}{11} \right)$$

and

$$gx = \ln \left(1 + \frac{x}{7} \right),$$

and the altering distance functions $\psi, \varphi : [0, +\infty) \rightarrow [0, +\infty)$ by $\psi(t) = bt$ and $\varphi(t) = (b - 1)t$, where $1 \leq b \leq \frac{36}{16}$. Then, we have the following:

- (1) (X, \preceq) is a partially ordered set having the b -metric d , where the b -metric space (X, d) is b -complete.
- (2) f and g are weakly increasing mappings with respect to \preceq .
- (3) f and g are continuous.
- (4) f is an almost generalized $(\psi, \varphi)_s$ -contractive mapping with respect to g , that is,

$$\psi(s^4 d(fx, gy)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)) + L\psi(N(x, y))$$

for all $x, y \in X$ with $x \preceq y$ and $L \geq 0$, where

$$M_s(x, y) = \max \left\{ d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(y, fx)}{2s} \right\}$$

and

$$N(x, y) = \min \{ d(x, fx), d(y, gx), d(x, gy) \}.$$

Proof The proof of (1) is clear. To prove (2), for each $x \in X$, we know that $1 + \frac{x}{11} \leq e^{\frac{x}{11}}$ and $1 + \frac{x}{7} \leq e^{\frac{x}{7}}$. So, $fx = \ln(1 + \frac{x}{11}) \leq x$ and $gx = \ln(1 + \frac{x}{7}) \leq x$. Hence, $fgx = \ln(1 + \frac{gx}{11}) \leq gx$ and $gfx = \ln(1 + \frac{fx}{7}) \leq fx$, for each $x \in X$. Therefore, f and g are weakly increasing mappings with respect to \leq . It is easy to see that f and g are continuous.

To prove (4), let $x, y \in X$ with $x \leq y$. So, $y \leq x$. Thus, we have the following cases.

Case 1: If $\frac{y}{7} \leq \frac{x}{11}$, then we have

$$1 \leq \frac{1 + \frac{x}{11}}{1 + \frac{y}{7}} \leq \frac{1 + \frac{x}{7}}{1 + \frac{y}{7}} \implies 0 \leq \ln\left(\frac{1 + \frac{x}{11}}{1 + \frac{y}{7}}\right) \leq \ln\left(\frac{1 + \frac{x}{7}}{1 + \frac{y}{7}}\right).$$

Now, using the mean value theorem for function $\ln(1 + t)$, for $t \in [\frac{y}{7}, \frac{x}{7}]$, we have

$$\begin{aligned} \psi(s^4 d(fx, gy)) &= 16bd(fx, gy) \\ &= 16b\left(\ln\left(1 + \frac{x}{11}\right) - \ln\left(1 + \frac{y}{7}\right)\right)^2 = 16b\left(\ln\left(\frac{1 + \frac{x}{11}}{1 + \frac{y}{7}}\right)\right)^2 \\ &\leq 16b\left(\ln\left(\frac{1 + \frac{x}{7}}{1 + \frac{y}{7}}\right)\right)^2 = 16b\left(\ln\left(1 + \frac{x}{7}\right) - \ln\left(1 + \frac{y}{7}\right)\right)^2 \\ &\leq 16b\left(\frac{x}{7} - \frac{y}{7}\right)^2 \leq \frac{36}{49}(x - y)^2 \\ &\leq d(x, y) \leq M_2(x, y) = \psi(M_2(x, y)) - \varphi(M_2(x, y)), \end{aligned}$$

that is, we have

$$\psi(s^4 d(fx, fy)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)) + L\psi(N(x, y))$$

for each $L \geq 0$.

Case 2: If $\frac{x}{11} < \frac{y}{7} \leq \frac{x}{7}$, then we have

$$0 < \frac{y}{7} - \frac{x}{11} \leq \frac{y}{7} \implies \left(\frac{y}{7} - \frac{x}{11}\right)^2 \leq \frac{y^2}{49}.$$

Using the mean value theorem for function $\ln(1 + t)$, for $t \in [\frac{x}{11}, \frac{y}{7}]$, we have

$$\begin{aligned} \psi(s^4 d(fx, gy)) &= 16bd(fx, gy) \\ &= 16b\left(\ln\left(1 + \frac{x}{11}\right) - \ln\left(1 + \frac{y}{7}\right)\right)^2 \\ &\leq 16b\left(\frac{y}{7} - \frac{x}{11}\right)^2 \leq \frac{16}{49}by^2 \leq \frac{36}{49}y^2 \\ &= \left(\frac{6y}{7}\right)^2 \leq \left(y - \ln\left(1 + \frac{y}{7}\right)\right)^2 = d(y, gy) \\ &\leq M_2(x, y) = \psi(M_2(x, y)) - \varphi(M_2(x, y)). \end{aligned}$$

So, we have

$$\psi(s^4 d(fx, fy)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)) + L\psi(N(x, y))$$

for each $L \geq 0$. Combining Cases 1 and 2 together, we conclude that f is an almost generalized $(\psi, \varphi)_s$ -contractive mapping with respect to g . Thus, all the hypotheses of Theorem 5 are satisfied and hence f and g have a common fixed point. Indeed, 0 is the unique common fixed point of f and g . \square

Remark 1 A subset W of a partially ordered set X is said to be well ordered if every two elements of W are comparable [43]. Note that in Theorems 3 and 4, f has a unique fixed point provided that the fixed points of f are comparable. Also, in Theorems 5 and 6, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.

Example 4 Let $X = \{0, 1, 2, 3, 4\}$ be equipped with the following partial order \preceq :

$$\preceq := \{(0, 0), (1, 1), (1, 2), (2, 2), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 4)\}.$$

Define b -metric $d : X \times X \rightarrow \mathbb{R}^+$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ (x + y)^2 & \text{if } x \neq y. \end{cases}$$

It is easy to see that (X, d) is a b -complete b -metric space.

Define the self-maps f and g by

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 1 & 2 \end{pmatrix}, \quad g = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 1 & 1 \end{pmatrix}.$$

We see that f and g are weakly increasing mappings with respect to \preceq and f and g are continuous.

Define $\psi, \varphi : [0, \infty) \rightarrow [0, \infty)$ by $\psi(t) = \sqrt{t}$ and $\varphi(t) = \frac{t}{3}$. One can easily check that f is an almost generalized $(\psi, \varphi)_s$ -contractive mapping with respect to g , with $L \geq \frac{10}{3}$.

Thus, all the conditions of Theorem 5 are satisfied and hence f and g have a common fixed point. Indeed, 0 and 2 are two common fixed points of f and g . Note that the ordered set (X, \preceq) is not well ordered.

3 Applications

Let Φ denote the set of all functions $\phi : [0, +\infty) \rightarrow [0, +\infty)$ satisfying the following hypotheses:

1. Every $\phi \in \Phi$ is a Lebesgue integrable function on each compact subset of $[0, +\infty)$.
2. For any $\phi \in \Phi$ and any $\epsilon > 0$, $\int_0^\epsilon \phi(\tau) d\tau > 0$.

It is an easy matter to check that the mapping $\psi : [0, +\infty) \rightarrow [0, +\infty)$ defined by

$$\psi(t) = \int_0^t \phi(\tau) d\tau$$

is an altering distance function. Therefore, we have the following results.

Corollary 5 Let (X, \preceq) be a partially ordered set having a b -metric d such that the b -metric space (X, d) is b -complete. Let $f : X \rightarrow X$ be a non-decreasing continuous mapping with respect to \preceq . Suppose that there exist $k \in [0, 1)$ and $L \geq 0$ such that

$$\int_0^{d(fx, fy)} \phi(\tau) d\tau \leq \frac{k}{s} \int_0^{\max\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2s}\}} \phi(\tau) d\tau + \frac{L}{s} \int_0^{\min\{d(x, fx), d(y, fy)\}} \phi(\tau) d\tau$$

for all comparable elements $x, y \in X$. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof Follows from Theorem 3 by taking $\psi(t) = \int_0^t \phi(\tau) d\tau$ and $\varphi(t) = (1 - k)t$, for all $t \in [0, +\infty)$. □

Corollary 6 Let (X, \preceq) be a partially ordered set having a b -metric d such that the b -metric space (X, d) is b -complete. Let $f, g : X \rightarrow X$ be two weakly increasing mappings with respect to \preceq . Suppose that there exist $k \in [0, 1)$ and $L \geq 0$ such that

$$\int_0^{d(fx, gy)} \phi(\tau) d\tau \leq \frac{k}{s^4} \int_0^{\max\{d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(y, fx)}{2s}\}} \phi(\tau) d\tau + \frac{L}{s^4} \int_0^{\min\{d(x, fx), d(y, gy), d(x, gy)\}} \phi(\tau) d\tau$$

for all comparable elements $x, y \in X$. If either f or g is continuous, then f and g have a common fixed point.

Proof Follows from Theorem 5 by taking $\psi(t) = \int_0^t \phi(\tau) d\tau$ and $\varphi(t) = (1 - k)t$, for all $t \in [0, +\infty)$. □

Finally, let us finish this paper with the following remarks.

Remark 2 Theorem 2.1 of [20] is a special case of Corollary 1.

Remark 3 Theorem 2.2 of [20] is a special case of Corollary 2.

Remark 4 Theorem 2.3, Corollary 2.4 and Corollary 2.5 of [20] are special cases of Corollary 3.

Remark 5 Since a b -metric is a metric when $s = 1$, so our results can be viewed as a generalization and extension of corresponding results in [28] and several other comparable results.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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