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COMMON FIXED POINTS OF GENERALIZED ALMOST NONEXPANSIVE MAPPINGS

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Abstract

The concept of a generalized almost nonexpansive mappings is introduced and the existence of common fixed points for this new class of mappings is proved. As an application, an invariant approximation result is obtained.

1 Introduction and preliminaries

In 1968, Kannan [12] proved a fixed point theorem for a map satisfying a contractive condition that did not require continuity at each point. This paper was a genesis for a multitude of fixed point papers over the next two decades. Sessa [13] coined the term weakly commuting maps. Jungck [8] generalized the notion of weak commutativity by introducing compatible maps and then weakly compatible maps [10].

The concept of almost contraction property was extended to a pair of selfmaps as follows:

Definition 1.1. Let T and f be two selfmaps of a metric space (X, d). A map T is called an *almost f- contraction* if there exist a constant $\delta \in]0, 1[$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta\ d(fx, fy) + L\ d(fy, Tx), \tag{1}$$

for all $x, y \in X$. If we choose $f = I_X$, I_X is the identity map on X, we obtain the definition of *almost contraction,* the concept introduced by Berinde ([5], [6]).
This concept was introduced by Berinde as 'weak contraction' in [5]. But in [6], Berinde renamed 'weak contraction' as 'almost contraction' which is appropriate.
It was shown in [5] that any strict contraction, the Kannan [12] and Zamfirescu [14] mappings, as well as a large class of quasi-contractions, are all almost contractions.

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Let T and f be two selfmaps of a metric space (X, d). T is said to be *f-contraction* if there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(fx, fy)$ for all $x, y \in X$. This definition can be obtained directly from (1) if we take $L = 0$.

In 2006, Al-Thagafi and Shahzad [2] proved the following theorem which is a generalization of many known results.

Theorem 1.2 (Al-Thagafi and Shahzad ([2], Theorem 2.1)). Let E be a subset of a metric space (X, d) and f and T be selfmaps of E and $T(E) \subseteq f(E)$. Suppose that f and T are weakly compatible, T is f-contraction and $T(E)$ is complete. Then f and T have a unique common fixed point in E.

Babu, Sandhya and Kameswari [3] considered the class of mappings that satisfy 'condition (B)'.

Let (X, d) be a metric space. A map $T : X \to X$ is said to satisfy *'condition (B)'* if there exist a constant $\delta \in]0, 1[$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for all $x, y \in X$.

Recently, Berinde established the following fixed point result.

Theorem 1.3 (Berinde ([6], Theorem 3.4)). Let (X, d) be a complete metric space and $T : X \to X$ a mapping for which there exist $\alpha \in]0, 1[$ and some $L \geq 0$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \alpha M(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \quad (2)$$

where, $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$. Then

(1) T has a unique fixed point, i.e., $F(T) = \{x^*\}$;

(2) for any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by (1.1) converges to some $x^* \in F(T)$

(3) the prior estimate

$$d(x_n, x^*) \leq \frac{\alpha^n}{(1-\alpha)^2} d(x_0, x_1)$$

holds, for $n = 1, 2, \cdots$,

(4) the rate of convergence of Picard iteration is given by

$$d(x_n, x^*) \leq \theta \; d(x_{n-1}, x^*)$$

for $n = 0, 1, 2, \cdots$.

The contractive condition (2) is termed as generalized condition B. We introduce the following definition as follows:

Definition 1.4. Let T and f be two selfmaps of a metric space (X, d). A map T is called *generalized almost f- contraction* if there exists $\delta \in]0, 1[$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta M(x, y) + L \min\{d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\} \quad (3)$$

for all $x, y \in X$, where

$$M(x,y) = \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\}.$$

If $f = I_X$, then we say that T satisfies 'generalized condition (B)'.
Example 1.5. Let $X = [0,1)$ with usual metric. Define $T, f : X \to X$ as

$$T(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x < \frac{2}{3} \\ \frac{2}{3} & \text{if } \frac{2}{3} \leq x < 1 \end{cases}$$

and

$$f(x) = \begin{cases} \frac{5}{6} & \text{if } 0 \leq x < \frac{2}{3} \\ \frac{4}{3} - x & \text{if } \frac{2}{3} \leq x < 1. \end{cases}$$

Here T is generalized almost $f-$ contraction with $\delta = \frac{1}{2}$ and $L = 0$.

But, when $x \in [0, \frac{2}{3})$ and $y = \frac{2}{3}$, we have $d(Tx,Ty) = \frac{1}{6}$; and $d(fx,fy) = \frac{1}{6}$ so that for any $\alpha \in [0,1)$, T fails to be an f-contraction.

Let Y be a subset of a normed space $(X, \|.\|)$. The set $P_Y(u) = \{x \in Y : \|x - u\| = dist(u,Y)\}$ is called the *set of best approximants* to $u \in X$ out of Y, where $dist(u,Y) = inf\{\|y-u\| : y \in Y\}$. We shall use $\mathbb{N}$ to denote the set of positive integers, $cl(Y)$ to denote the closure of a set Y and $wcl(Y)$ to denote the weak closure of a set Y. Let $f : Y \to Y$ be a mapping. The set of fixed points of $T($ resp. $f)$ is denoted by $F(T)$(resp. $F(f)$). A point $x \in Y$ is a *coincidence point* (*common fixed point*) of f and T if $fx = Tx$ $(x = fx = Tx)$. The set of coincidence points of f and T is denoted by $C(f,T)$. A pair (f,T) of self-mappings on X is said to be *weakly compatible* if f and T commute at their coincidence point (i.e. $fTx = Tfx$, $x \in X$ whenever $fx = Tx$). A point $y \in X$ is called a *point of coincidence* of two self-mappings f and T on X if there exists a point $x \in X$such that $y = Tx = fx$.

The set Y is called *q-starshaped* with $q \in Y$, if the segment $[q,x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$ joining q to x is contained in Y for all $x \in Y$. For further details we refer to [4], [7], [9],[11] and referenced mentioned therein.

Definition 1.6. Let X be a normed space and M be a $q-$starshaped subset of X. Then a selfmap T of X is said to be generalized almost $f-$ nonexpansive if

$$\begin{aligned} d(Tx,Ty) \leq\ & \max\{d(fx,fy), dist(fx,[q,Tx]), dist(fy,[q,Ty]), \\ & \frac{dist(fx,[q,Ty]) + dist(fy,[q,Tx])}{2}\} \\ & + L\min\{dist(fx,[q,Tx]), dist(fy,[q,Ty]), dist(fx,[q,Ty]), \\ & dist(fy,[q,Tx])\} \end{aligned} \tag{4}$$

for all $x, y \in X$, $L \geq 0$.

Definition 1.7. Let (X, d) be a metric space, T and f be self-mappings on X, with $T(X) \subset f(X)$, and $x_0 \in X$. Choose a point x_1 in X such that $fx_1 = Tx_0$. This can be done since $T(X) \subset f(X)$. Continuing this process having chosen $x_1, \cdots, x_k$, we choose x_{k+1} in X such that

$$fx_{k+1} = Tx_k, \quad k = 0, 1, 2, \cdots.$$

The sequence $\{fx_n\}$ is called a T-*sequence* with initial point x_0.

2 Common fixed point theorems

First, we establish a result on the existence of points of coincidence and common fixed points for two weakly compatible maps. We then, apply this result to obtain common fixed point of generalized almost $f-$ nonexpansive mapping.

Theorem 2.1. Let Y be a nonempty subset of a metric space (X,d), and f and T be weakly compatible self-maps of Y. Assume that $clT(Y) \subset f(Y)$, $clT(Y)$ is complete, and T is generalized almost $f-$ contraction. Then $Y \cap F(f) \cap F(T)$ is singleton.

Proof As $T(Y) \subseteq f(Y)$, one can choose $\{fx_n\}$ which is a T-*sequence* with initial point x_0. For each n, using (3), we have

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\le \delta M(x_n, x_{n+1}) \\ &+L\min\{d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}), d(fx_n, Tx_{n+1}), d(fx_{n+1}, Tx_n)\} \quad (5) \end{aligned}$$

where

$$M(x_n, x_{n+1}) = \max\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}), \frac{d(fx_n, Tx_{n+1}) + d(fx_{n+1}, Tx_n)}{2}\}.$$

Using $Tx_n = fx_{n+1}$ in (5), we obtain

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\le \delta \max\{d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n+1}), \frac{d(Tx_{n-1}, Tx_{n+1})}{2}\} \\ &= \delta \max\{d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n+1})\}. \end{aligned}$$

If for some n, $\max\{d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n+1})\} = d(Tx_n, Tx_{n+1})$, then from above inequality we have

$$d(Tx_n, Tx_{n+1}) \le \delta d(Tx_n, Tx_{n+1}),$$

a contradiction. Therefore

$$d(Tx_n, Tx_{n+1}) \le \delta d(Tx_{n-1}, Tx_n). \quad (6)$$

From (6), we obtain

$$d(Tx_n, Tx_{n+1}) \le \delta d_0$$

where $d_0 = d(Tx_0, Tx_1)$. Thus for $m, n \in N$ with $m > n$,

$$\begin{aligned} d(Tx_n, Tx_{m+n}) &\le d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + ... + d(Tx_{n+m-1}, Tx_{m+n}) \\ &+ (\delta)^n d_0 + (\delta)^{n+1} d_0 + ... + (\delta)^{n+m-1} d_0. \end{aligned}$$

So

$$d(Tx_n, Tx_{m+n}) \le \sum_{i=n}^{n+m-1} (\delta)^i d_0.$$

Therefore $\{Tx_n\}$ is a Cauchy sequences in $T(Y)$. It follows from completeness of $clT(Y)$ that $Tx_n \to w \in clT(Y)$ and hence $fx_n \to w$ as $n \to \infty$. Consequently, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = w \in clT(Y)$. Thus $w = fy$ for some $y \in Y$. Now we show that $fy = Ty$. If not, then for $n \geq 1$, we have

$$\begin{aligned} d(w,Ty) \leq d(w,Tx_n) + d(Tx_n,Ty) \leq d(w,Tx_n) \\ + \delta M(x_n,y) + L\min\{d(fx_n,Tx_n), d(fy,Ty), d(fx_n,Ty), d(fy,Tx_n)\}, \end{aligned} \quad (7)$$

where

$$M(x_n,y) = \max\{d(fx_n,fy), d(fx_n,Tx_n), d(fy,Ty), \frac{d(fx_n,Ty)+d(fy,Tx_n)}{2}\}.$$

Letting $n \to \infty$, on both side of (7), we obtain

$$d(w,Ty) \leq \delta d(w,Ty)$$

a contradiction. Hence $Ty = w = fy$. We now show the point of coincidence is unique. Suppose that for some $z \in Y$, $fz = Tz$. Then by inequality (3), we get

$$\begin{aligned} d(fy,fz) &= d(Ty,Tz) \\ &\leq \delta M(y,z) + L\min\{d(fy,Ty), d(fz,Tz), d(fy,Tz), d(fz,Ty)\}, \end{aligned} \quad (8)$$

where

$$M(x,y) = \max\{d(fy,fz), d(fy,Ty), d(fz,Tz), \frac{d(fy,Tz)+d(fz,Ty)}{2}\}.$$

By (8), we have

$$d(fy,fz) \leq \delta d(fy,fz).$$

Hence $fz = fy = Ty$ as $\delta \in (0,1)$. This implies that the point of coincidence of f and T is unique. Since f and T are weakly compatible and $fy = Ty$, we obtain $TTy = fTy = Tfy$, thereby showing that TTy is a point of coincidence of f and T. By the uniqueness of point of coincidence, we have $TTy = fTy = Ty$; thus Ty is a common fixed point of f and T. Consequently Ty is unique common fixed point of f and T.

Lemma 2.2. Let f and T be self-maps on a nonempty q-starshaped subset Y of a normed space X , f and T are weakly compatible, and T is generalized almost $f-$nonexpansive with $clT(Y) \subset f(Y)$, define a mapping T_n on Y by

$$T_n x = (1-\mu_n)q + \mu_n Tx,$$

where $\{\mu_n\}$ is a sequence of numbers in $(0,1)$ such that $\lim_{n\to\infty} \mu_n = 1$. Then for each $n \geq 1$, T_n and f have exactly one common fixed point x_n in Y such that

$$fx_n = x_n = (1-\mu_n)q + \mu_n Tx_n,$$

provided one of the following conditions hold;
(i) $cl(T_n(Y))$ is complete for each n,
(ii) for each n, $wcl(T_n(Y))$ is complete.
Proof. By definition,

$$T_n x = (1-\mu_n)q + \mu_n Tx.$$

Note that T_n is a self mapping on Y and $clT_n(Y) \subset f(Y)$. Also by (4),

$$\begin{aligned}\|T_n x - T_n y\| &= \mu_n \|Tx - Ty\| \\ &\le \mu_n max\{\|fx - fy\|, dist(fx,[q,Tx]), dist(fy,[q,Ty]), \\ &\quad \frac{dist(fx,[q,Ty]) + dist(fy,[q,Tx])}{2}\} \\ &\quad + \mu_n L \min\{dist(fx,[q,Tx]), dist(fy,[q,Ty]), dist(fx,[q,Ty]), \\ &\quad dist(fy,[q,Tx])\} \\ &\le \mu_n max\{\|fx - fy\|, \|fx - T_n x\|, \|fy - T_n y\|, \\ &\quad \frac{\|fx - T_n y\| + \|fy - T_n x\|}{2}\} + \mu_n L\{\|fx - T_n x\|, \|fy - T_n y\|, \\ &\quad \|fx - T_n y\|, \|fy - T_n x\|\}\end{aligned}$$

for each $x, y \in Y$. By Theorem 2.1, for each $n \ge 1$, there exists a unique $x_n \in Y$ such that $x_n = fx_n = T_n x_n$. Thus for each $n \ge 1$, $F(T_n) \cap F(f) \neq \phi$.
(ii) Conclusion follows from Theorem 2.1.
Theorem 2.3. Let f and T be self-maps on a q-starshaped subset Y of a normed space X. Assume that f and T are weakly compatible, , T is a generalized almost $f-$nonexpansive mapping with $clT(Y) \subset f(Y)$. Then $F(T) \cap F(f) \neq \phi$, provided one of the following conditions holds;

(i) $cl(T(Y))$ is compact and T is continuous;

(ii) X is complete, f is weakly continuous, $wcl(T(Y))$ is weakly compact and $f-T$ is demiclosed at 0.

Proof.

(i) Define a mapping T_n on Y by

$$T_n x = (1-\mu_n)q + \mu_n Tx,$$

where $\{\mu_n\}$ is a sequence of numbers in $(0,1)$ such that $\lim_{n\to\infty} \mu_n = 1$. Notice that compactness of $cl(T(Y))$ implies that $clT_n(Y)$ is compact and thus complete. From Lemma 2.2, for each $n \ge 1$, there exists $x_n \in Y$ such that $x_n = fx_n = (1-\mu_n)q + \mu_n Tx_n$. Also,

$$\begin{aligned}\|x_n - Tx_n\| &= \|(1-\mu_n)q + \mu_n Tx_n - Tx_n\| \\ &= (1-\mu_n)\|q - Tx_n\| \to 0\end{aligned}$$

as $n \to \infty$. Since $cl(T(Y))$ is compact, there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to y$ as $m \to \infty$. Now, $x_m = (1-\mu_m)q + \mu_n Tx_m$ implies that $x_m \to y$ as $m \to \infty$. By the continuity of f and T and the fact $\|x_m - Tx_m\| \to 0$, we have $y \in F(T) \cap F(f)$. Thus $F(T) \cap F(f) \neq \phi$.

(ii) The weak compactness of $wclT(Y)$ implies that $wclT_n(Y)$ is weakly compact and hence complete due to completeness of X. From Lemma 2.2, for each $n \geq 1$, there exists $x_n \in Y$ such that $x_n = fx_n = T_n x_n = (1-\mu_n)q + \mu_n Tx_n$. The analysis in (i), implies that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. The weak compactness of $wclT(Y)$ implies that there is a subsequence $\{x_m\}$ of $\{x_n\}$ converging weakly to $y \in Y$ as $m \to \infty$. Weak continuity of f implies that $fy = y$. Also we have, $fx_m - Tx_m = x_m - Tx_m \to 0$ as $m \to \infty$. If $f - T$ is demiclosed at 0, then $fy = Ty$ and hence $F(T) \cap F(f) \neq \phi$.

Following is an invariant approximation result.

Theorem 2.4. Let Y be a subset of a normed space X and $f, T : X \to X$ be mappings such that $u \in F(f) \cap F(T)$ for some $u \in X$ and $T(\partial Y \cap Y) \subseteq Y$. Suppose that $P_Y(u)$ is nonempty and q-starshaped, f is continuous on $P_Y(u)$, $\|Tx - Tu\| \leq \|fx - fu\|$ for each $x \in P_Y(u)$ and $f(P_Y(u)) \subseteq P_Y(u)$. If T and f are weakly compatible, $F(f)$ is nonempty and q-starshaped for $q \in F(f)$, T is almost generalized $f-$nonexpansive type then $P_Y(u) \cap F(f) \cap F(T) \neq \phi$, provided one of the following conditions is satisfied;

(i) T is continuous and $cl(T(P_Y(u)))$ is compact;

(ii) X is complete, $wcl(T(P_Y(u)))$ is weakly compact, f is weakly continuous and either $f - T$ is demiclosed at **0**.

Proof. Let $x \in P_Y(u)$. Then for any $h \in (0,1)$, $\|hu + (1-h)x - u\| = (1-h)\|x - u\| < dist(u, C)$. It follows that the line segment $\{hu + (1-h)x : 0 < h < 1\}$ and the set Y are disjoint. Thus x is not in the interior of Y and so $x \in \partial Y \cap Y$. Since $T(\partial Y \cap Y) \subseteq Y$, Tx must be in Y. Also $fx \in P_Y(u)$, $u \in F(f) \cap F(T)$ and f and T satisfy $\|Tx - Tu\| \leq \|fx - fu\|$, thus we have

$$\|Tx - u\| = \|Tx - Tu\| \leq \|fx - fu\| = \|fx - u\| = dist(u, Y).$$

It further implies that $Tx \in P_Y(u)$. Therefore T is a self map of $P_Y(u)$. The result now follows from Theorem 2.3.

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