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Common fixed points of *R*-weakly commuting maps in generalized metric spaces

Mujahid Abbas[1], Safeer Hussain Khan[2*] and Talat Nazir[1]

* Correspondence: safeer@qu.edu.qa
[2]Department of Mathematics, Statistics and Physics, Qatar University, Doha 2713, Qatar
Full list of author information is available at the end of the article

Abstract

In this paper, using the setting of a generalized metric space, a unique common fixed point of four *R*-weakly commuting maps satisfying a generalized contractive condition is obtained. We also present example in support of our result.

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1 Introduction and preliminaries

The study of unique common fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity. Mustafa and Sims [1] generalized the concept of a metric, in which the real number is assigned to every triplet of an arbitrary set. Based on the notion of generalized metric spaces, Mustafa et al. [2-6] obtained some fixed point theorems for mappings satisfying different contractive conditions. Study of common fixed point theorems in generalized metric spaces was initiated by Abbas and Rhoades [7]. Abbas et al. [8] obtained some periodic point results in generalized metric spaces. While, Chugh et al. [9] obtained some fixed point results for maps satisfying property p in G-metric spaces. Saadati et al. [10] studied some fixed point results for contractive mappings in partially ordered G-metric spaces. Recently, Shatanawi [11] obtained fixed points of Φ-maps in G-metric spaces. Abbas et al. [12] gave some new results of coupled common fixed point results in two generalized metric spaces (see also [13]).

The aim of this paper is to initiate the study of unique common fixed point of four R-weakly commuting maps satisfying a generalized contractive condition in G-metric spaces.

Consistent with Mustafa and Sims [2], the following definitions and results will be needed in the sequel.

Definition 1.1. Let X be a nonempty set. Suppose that a mapping $G : X \times X \times X \to R^+$ satisfies:

$\mathbf{G}_1 : G(x, y, z) = 0$ if $x = y = z$;

$\mathbf{G}_2 : 0 < G(x, y, z)$ for all $x, y, z \in X$, with $x \neq y$;

$\mathbf{G}_3 : G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$;

$\mathbf{G}_4 : G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables); and

$\mathbf{G}_5 : G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then G is called a G-metric on X and (X, G) is called a G-metric space.

Definition 1.2. A sequence $\{x_n\}$ in a G-metric space X is:

(i) a *G-Cauchy* sequence if, for any $\varepsilon > 0$, there is an $n_0 \in N$ (the set of natural numbers) such that for all $n, m, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$,
(ii) a *G-convergent* sequence if, for any $\varepsilon > 0$, there is an $x \in X$ and an $n_0 \in N$, such that for all $n, m \geq n_0$, $G(x, x_n, x_m) < \varepsilon$.

A G-metric space on X is said to be G-complete if every G-Cauchy sequence in X is G-convergent in X. It is known that $\to 0$ as $n, m \to \infty$.

Proposition 1.3. Let X be a G-metric space. Then the following are equivalent:

(1) $\{x_n\}$ is G-convergent to x.
(2) $G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.
(3) $G(x_n, x_n, x) \to 0$ as $n \to \infty$.
(4) $G(x_n, x, x) \to 0$ as $n \to \infty$.

Definition 1.4. A G-metric on X is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Proposition 1.5. Every G-metric on X will define a metric d_G on X by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \forall x, y \in X. \tag{1.1}$$

For a symmetric G-metric,

$$d_G(x, y) = 2G(x, y, y), \quad \forall x, y \in X. \tag{1.2}$$

However, if G is non-symmetric, then the following inequality holds:

$$\frac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y), \quad \forall x, y \in X. \tag{1.3}$$

It is also obvious that

$$G(x, x, y) \leq 2G(x, y, y).$$

Now, we give an example of a non-symmetric G-metric.

Example 1.6. Let $X = \{1, 2\}$ and a mapping $G : X \times X \times X \to R^+$ be defined as

(x, y, z)	$G(x, y, z)$
(1, 1, 1), (2, 2, 2)	0
(1, 1, 2), (1, 2, 1), (2, 1, 1)	0.5
(1, 2, 2), (2, 1, 2), (2, 2, 1)	1.

Note that G satisfies all the axioms of a generalized metric but $G(x, x, y) \neq G(x, y, y)$ for distinct x, y in X. Therefore, G is a non-symmetric G-metric on X.

In 1999, Pant [14] introduced the concept of weakly commuting maps in metric spaces. We shall study R-weakly commuting and compatible mappings in the frame work of G-metric spaces.

Definition 1.7. Let X be a G-metric space and f and g be two self-mappings of X. Then f and g are called R-weakly commuting if there exists a positive real number R such that $G(fgx, fgx, gfx) \leq RG(fx, fx, gx)$ holds for each $x \in X$.

Two maps f and g are said to be compatible if, whenever $\{x_n\}$ in X such that $\{fx_n\}$ and $\{gx_n\}$ are G-convergent to some $t \in X$, then $\lim_{n\to\infty} G(fgx_n, fgx_n, gfx_n) = 0$.

Example 1.8. Let $X = [0, 2]$ with complete G-metric defined by

$$G(x, y, z) = \max\{|x-y|, |x-z|, |y-z|\}.$$

Let $f, g, S, T : X \to X$ defined by

$$fx = 1, x \geq 0,$$

$$gx = \left\{ \begin{array}{l} 1, \ x \in [0,1], \\ \frac{2-x}{2}, \ x \in (1,2], \end{array} \right| ,$$

$$Sx = \left\{ \begin{array}{l} 2-x, \ x \in [0,1], \\ x, \ x \in (1,2], \end{array} \right| ,$$

and

$$Tx = \left\{ \begin{array}{l} \frac{3-x}{2}, \ x \in [0,1], \\ \frac{x}{2}, \ x \in (1,2], \end{array} \right| .$$

Then note that the pairs $\{f, S\}$ and $\{g, T\}$ are R-weakly commuting as they commute at their coincidence points. The pair $\{f, S\}$ is continuous compatible while the pair $\{g, T\}$ is non-compatible. To see that g and T are non-compatible, consider a decreasing sequence $\{x_n\}$ in X such that $x_n \to 1$. Then $gx_n \to \frac{1}{2}$, $Tx_n \to \frac{1}{2}$. $gTx_n = \frac{4-x_n}{4} \to \frac{3}{4}$ and $Tgx_n = \frac{2-x_n}{4} \to \frac{1}{4}$. □

2 Common fixed point theorems

In this section, we obtain some unique common fixed point results for four mappings satisfying certain generalized contractive conditions in the framework of a generalized metric space. We start with the following result.

Theorem 2.1. Let X be a complete G-metric space. Suppose that $\{f, S\}$ and $\{g, T\}$ be pointwise R-weakly commuting pairs of self-mappings on X satisfying

$$\begin{aligned} G(fx, fx, gy) \leq h \max\{&G(Sx, Sx, Ty), G(fx, fx, Sx), G(gy, gy, Ty), \\ &[G(fx, fx, Ty) + G(gy, gy, Sx)]/2\} \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} G(fx, gy, gy) \leq h \max\{&G(Sx, Ty, Ty), G(fx, Sx, Sx), G(gy, Ty, Ty), \\ &[G(fx, Ty, Ty) + G(gy, Sx, Sx)]/2\} \end{aligned} \tag{2.2}$$

for all $x, y \in X$, where $h \in [0, 1)$. Suppose that $fX \subseteq TX$, $gX \subseteq SX$, and one of the pair $\{f, S\}$ or $\{g, T\}$ is compatible. If the mappings in the compatible pair are continuous, then f, g, S and T have a unique common fixed point.

Proof. Suppose that f and g satisfy the conditions (2.1) and (2.2). If G is symmetric, then by adding these, we have

$$\begin{aligned} &d_G(fx, gy) \\ &\leq \frac{h}{2} \max\{d_G(Sx, Ty), d_G(fx, Sx), d_G(gy, Ty), [d_G(fx, Ty) + d_G(gy, Sx)]/2\} \\ &\quad + \frac{h}{2} \max\{d_G(Sx, Ty), d_G(fx, Sx), d_G(gy, Ty), [d_G(fx, Ty) + d_G(gy, Sx)]/2\} \\ &= h \max\{d_G(Sx, Ty), d_G(fx, Sx), d_G(gy, Ty), [d_G(fx, Ty) + d_G(gy, Sx)]/2\}, \end{aligned}$$

for all $x, y \in X$ with $0 \leq h < 1$, the existence and uniqueness of a common fixed point follows from [14]. However, if X is non-symmetric G-metric space, then by the definition of metric d_G on X and (1.3), we obtain

$$\begin{aligned} &d_G(fx, gy) \\ &\quad = G(fx, fx, gy) + G(fx, gy, gy) \\ &\quad \leq \frac{2h}{3} \max\{d_G(Sx, Ty), d_G(fx, Sx), d_G(gy, Ty), [d_G(fx, Ty) + d_G(gy, Sx)]/2\} \\ &\quad\quad + \frac{2h}{3} \max\{d_G(Sx, Ty), d_G(fx, Sx), d_G(gy, Ty), [d_G(fx, Ty) + d_G(gy, Sx)]/2\} \\ &\quad = \frac{4h}{3} \max\{d_G(Sx, Ty), d_G(fx, Sx), d_G(gy, Ty), [d_G(fx, Ty) + d_G(gy, S_X)]/2\}, \end{aligned}$$

for all $x, y \in X$. Here, the contractivity factor $\frac{4h}{3}$ needs not be less than 1. Therefore, metric d_G gives no information. In this case, let x_0 be an arbitrary point in X. Choose x_1 and x_2 in X such that $gx_0 = Sx_1$ and $fx_1 = Tx_2$. This can be done, since the ranges of S and T contain those of g and f, respectively. Again choose x_3 and x_4 in X such that $gx_2 = Sx3$ and $fx_3 = Tx_4$. Continuing this process, having chosen x_n in X such that $gx_{2n} = Sx_{2n+1}$ and $fx_{2n+1} = Tx_{2n+2}$, $n = 0, 1, 2, \ldots$. Let

$$y_{2n} = Sx_{2n+1} = gx_{2n} \text{ and } y_{2n+1} = Tx_{2n+2} = fx_{2n+1} \text{ for all } n = 0, 1, 2, \ldots .$$

For a given $n \in \mathbf{N}$, if n is even, so $n = 2k$ for some $k \in \mathbf{N}$. Then from (2.1)

$$\begin{aligned} &G(y_{n+1}, y_{n+1}, y_n) \\ &\quad = G(y_{2k+1}, y_{2k+1}, y_{2k}) \\ &\quad = G(fx_{2k+1}, fx_{2k+1}, gx_{2k}) \\ &\quad \leq h \max\{G(Sx_{2k+1}, Sx_{2k+1}, Tx_{2k}), G(fx_{2k+1}, fx_{2k+1}, Sx_{2k+1}), \\ &\quad\quad G(gx_{2k}, gx_{2k}, Tx_{2k}), [G(fx_{2k+1}, fx_{2k+1}, Tx_{2k}) + G(gx_{2k}, gx_{2k}, Sx_{2k+1})]/2\} \\ &\quad = h \max\{G(y_{2k}, y_{2k}, y_{2k-1}), G(y_{2k+1}, y_{2k+1}, y_{2k}), \\ &G(y_{2k}, y_{2k}, y_{2k-1}), [G(y_{2k+1}, y_{2k+1}, y_{2k-1}) + G(y_{2k}, y_{2k}, y_{2k})]/2\} \\ &\quad \leq h \max\{G(y_{2k}, y_{2k}, y_{2k-1}), G(y_{2k+1}, y_{2k+1}, y_{2k}), \\ &\quad\quad [G(y_{2k+1}, y_{2k+1}, y_{2k}) + G(y_{2k}, y_{2k}, y_{2k-1})]/2\} \\ &\quad = h \max\{G(y_n, y_n, y_{n-1}), G(y_{n+1}, y_{n+1}, y_n)\}. \end{aligned}$$

This implies that

$$G(y_{n+1}, y_{n+1}, y_n) \leq hG(y_n, y_n, y_{n-1}).$$

If n is odd, then $n = 2k + 1$ for some $k \in \mathbf{N}$. In this case (2.1) gives

$$\begin{aligned} &G(y_{n+1}, y_{n+1}, y_n) \\ &\quad = G(y_{2k+2}, y_{2k+2}, y_{2k+1}) \\ &\quad = G(fx_{2k+2}, fx_{2k+2} + gx_{2k+1}) \\ &\quad \leq h \max\{G(Sx_{2k+2}, Sx_{2k+2}, Tx_{2k+1}), G(fx_{2k+2}, fx_{2k+2}, Sx_{2k+2}), \\ &\quad\quad G(gx_{2k+1}, gx_{2k+1}, Tx_{2k+1}), [G(fx_{2k+2}, fx_{2k+2}, Tx_{2k+1}) + G(gx_{2k+1}, gx_{2k+1}, Sx_{2k+2})]/2\} \\ &\quad = h \max\{G(y_{2k+1}, y_{2k+1}, y_{2k}), G(y_{2k+2}, y_{2k+2}, y_{2k+1}), \\ &\quad\quad G(y_{2k+1}, y_{2k+1}, y_{2k}), [G(y_{2k+2}, y_{2k+2}, y_{2k}) + G(y_{2k+1}, y_{2k+1}, y_{2k+1})]/2\} \\ &\quad \leq h \max\{G(y_{2k+1}, y_{2k+1}, y_{2k}), G(y_{2k+2}, y_{2k+2}, y_{2k+1}), \\ &\quad\quad [G(y_{2k+2}, y_{2k+2}, y_{2k+1}) + G(y_{2k+1}, y_{2k+1}, y_{2k})]/2\} \\ &\quad = h \max\{G(y_{2k+1}, y_{2k+1}, y_{2k}), G(y_{2k+2}, y_{2k+2}, y_{2k+1})\} \\ &\quad = h \max\{G(y_n, y_n, y_{n-1}), G(y_{n+1}, y_{n+1}, y_n)\}, \end{aligned}$$

that is,

$$G(y_{n+1}, y_{n+1}, y_n) \leq hG(y_n, y_n, y_{n-1}).$$

Continuing the above process, we have

$$G(y_{n+1}, y_{n+1}, y_n) \leq h^n G(y_1, y_1, y_0).$$

Thus, if $y_0 = y_1$, we get $G(y_n, y_{n+1}, y_{n+1}) = 0$ for each $n \in \mathbf{N}$. Hence, $y_n = y_{n+1}$ for each $n \in \mathbf{N}$. Therefore, $\{y_n\}$ is G-Cauchy. So we may assume that $y_0 \neq y_1$.

Let $n, m \in \mathbf{N}$ with $m > n$,

$$\begin{aligned} &G(y_n, y_m, y_m) \\ &\quad \leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + \cdots + G(y_{m-1}, y_m, y_m) \\ &\quad \leq h^n G(y_0, y_1, y_1) + h^{n+1} G(y_0, y_1, y_1) + \cdots + h^{m-1} G(y_0, y_1, y_1) \\ &\quad = h^n G(y_0, y_1, y_1) \sum_{i=0}^{m-n-1} h^i \\ &\quad \leq \frac{h^n}{1-h} G(y_0, y_1, y_1), \end{aligned}$$

and so $G(y_n, y_m, y_m) \to 0$ as $m, n \to \infty$. Hence $\{y_n\}$ is a Cauchy sequence in X. Since X is G-complete, there exists a point $z \in X$ such that $\lim_{n\to\infty} y_n = z$.

Consequently

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} gx_{2n} = z$$

and

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Tx_{2n+2} = \lim_{n\to\infty} fx_{2n+1} = z.$$

Let f and S be continuous compatible mappings. Compatibility of f and S implies that $\lim_{n\to\infty} G(fSx_{2n+1}, fSx_{2n+1}, Sfx_{2n+1}) = 0$, that is $G(fz, fz, Sz) = 0$ which implies that $fz = Sz$. Since $fX \subset TX$, there exists some $u \in X$ such that $fz = Tu$. Now from (2.1), we have

$$\begin{aligned} G(fz, fz, gu) \leq\ & h \max\{G(Sz, Sz, Tu), G(fz, fz, Sz), G(gu, gu, Tu), \\ & [G(fz, fz, Tu) + G(gu, gu, Sz)]/2\} \\ =\ & h \max\{G(fz, fz, fz), G(fz, fz, fz), G(gu, gu, fz), \\ & [G(fz, fz, fz) + G(gu, gu, fz)]/2\} \\ =\ & hG(fz, gu, gu). \end{aligned} \tag{2.3}$$

Also, from (2.2)

$$\begin{aligned} G(fz, gu, gu) \leq\ & h \max\{G(Sz, Tu, Tu), G(fz, Sz, Sz), G(gu, Tu, Tu), \\ & [G(fz, Tu, Tu) + G(gu, Sz, Sz)]/2\} \\ =\ & h \max\{G(fz, fz, fz), G(fz, fz, fz), G(gu, fz, fz), \\ & [G(fz, fz, fz) + G(gu, fz, fz)]/2\} \\ =\ & hG(fz, fz, gu). \end{aligned} \tag{2.4}$$

Combining above two inequalities, we get

$$G(fz, fz, gu) \leq h^2 G(fz, fz, gu).$$

Since $h < 1$, so that $fz = gu$. Hence, $fz = Sz = gu = Tu$. As the pair $\{g, T\}$ is R-weakly commuting, there exists $R > 0$ such that

$$G(gTu,\ gTu,\ Tgu) \leq RG(gu,\ gu,\ Tu) = 0,$$

that is, $gTu = Tgu$. Moreover, $ggu = gTu = Tgu = TTu$. Similarly, the pair $\{f, S\}$ is R-weakly commuting, there exists some $R > 0$ such that

$$G(fSz,\ fSz,\ Sfz) \leq RG(fz,\ fz,\ Sz) = 0,$$

so that $fSz = Sfz$ and $ffz = fSz = Sfz = SSz$.

Now by (2.1)

$$\begin{aligned} G(ffz,\ ffz,\ fz) &= G(ffz,\ ffz,\ gu) \\ &\leq h \max\{G(Sfz,\ Sfz,\ Tu),\ G(ffz,\ ffz,\ Sfz),\ G(gu,\ gu,\ Tu), \\ &\quad [G(ffz,\ ffz,\ Tu) + G(gu,\ gu,\ Sfz)]/2\} \\ &= h \max\{G(ffz,\ ffz,\ gu),\ G(ffz,\ ffz,\ ffz),\ G(gu,\ gu,\ gu), \\ &\quad [G(ffz,\ ffz,\ gu) + G(gu,\ gu,\ ffz)]/2\} \\ &= h \max\{G(ffz,\ ffz,\ fz),\ [G(ffz,\ ffz,\ fz) + G(fz,\ fz,\ ffz)]/2\} \\ &= \frac{h}{2}[G(ffz,\ ffz,\ fz) + G(fz,\ fz,\ ffz)], \end{aligned}$$

so that

$$G(ffz,\ ffz,\ fz) \leq hG(fz,\ fz,\ ffz). \tag{2.5}$$

Again from (2.2), we have

$$\begin{aligned} G(ffz,\ fz,\ fz) &= G(ffz,\ gu,\ gu) \\ &\leq h \max\{G(Sfz, Tu, Tu), G(ffz,\ Sfz,\ Sfz),\ G(gu, Tu, Tu), \\ &\quad [G(ffZ,\ Tu,\ Tu) + G(gu,\ Sfz,\ Sfz)]/2\} \\ &= h \max\{G(Sfz,\ gu,\ gu),\ G(ffz,\ ffz,\ ffz),\ G(gu,\ gu,\ gu), \\ &\quad [G(ffz,\ gu,\ gu) + G(gu,\ ffz,\ ffz)]/2\} \\ &= h \max\{G(ffz,\ fz,\ fz),\ [G(ffz,\ fz,\ fz) + G(fz,\ ffz,\ ffz)]/2\} \\ &= \frac{h}{2}[G(ffz,\ fz,\ fz) + G(ffz,\ ffz,\ fz)], \end{aligned}$$

which implies

$$G(ffz,\ fz,\ fz) \leq hG(ffz,\ ffz,\ fz). \tag{2.6}$$

From (2.5) and (2.6), we obtain

$$G(ffz,\ ffz,\ fz) \leq h^2 G(ffz,\ ffz,\ fz),$$

and since $h^2 < 1$ so that $ffz = fz$. Hence, $ffz = Sfz = fz$, and fz is the common fixed point of f and S. Since $gu = fz$, following arguments similar to those given above we conclude that fz is a common fixed point of g and T as well. Now we show the uniqueness of fixed point. For this, assume that there exists another point w in X which is the common fixed point of f, g, S and T. From (2.1), we obtain

$$\begin{aligned} G(fz, fz, w) &= G(ffz, ffz, gw) \\ &\leq h\max\{G(Sfz, Sfz, Tw), G(ffz, ffz, Sfz), G(gw, gw, Tw), \\ &\quad [G(ffz, ffz, Tw) + G(gw, gw, Sfz)]/2\} \\ &= h\max\{G(fz, fz, w), G(fz, fz, fz), G(w, w, w), \\ &\quad [G(fz, fz, w) + G(w, w, fz)]/2\} \\ &= \frac{h}{2}[G(fz, fz, w) + G(w, w, fz)], \end{aligned}$$

which implies that

$$G(fz, fz, w) \leq hG(w, w, fz). \tag{2.7}$$

From (2.2), we get

$$\begin{aligned} G(fz, w, w) &= G(ffz, gw, gw) \\ &\leq h\max\{G(Sfz, Tw, Tw), G(ffz, Sfz, Sfz), G(gw, Tw, Tw), \\ &\quad [G(ffz, Tw, Tw) + G(gw, Sfz, Sfz)]/2\} \\ &= h\max\{G(fz, w, w), G(fz, fz, fz), G(w, w, w), \\ &\quad [G(fz, w, w) + G(w, fz, fz)]/2\} \\ &= \frac{h}{2}[G(fz, w, w) + G(w, fz, fz)], \end{aligned}$$

which implies

$$G(fz, w, w) \leq hG(fz, fz, w). \tag{2.8}$$

Now (2.7) and (2.8) give

$$G(fz, fz, w) \leq h^2 G(fz, fz, w),$$

and $fz = w$. This completes the proof.

Example 2.2. Let $X = \{0, 1, 2\}$ with G-metric defined by

(x, y, z)	$G(x, y, z)$
(0, 0, 0), (1, 1, 1), (2, 2, 2),	0
(0, 0, 1), (0, 1, 0), (1, 0, 0),	
(0, 0, 2), (0, 2, 0), (2, 0, 0),	1
(0, 2, 2), (2, 0, 2), (2, 2, 0),	
(0, 1, 1), (1, 0, 1), (1, 1, 0),	
(1, 1, 2), (1, 2, 1), (2, 1, 1),	2
(1, 2, 2), (2, 1, 2), (2, 2, 1),	
(0, 1, 2), (0, 2, 1), (1, 0, 2),	2
(1, 2, 0), (2, 0, 1), (2, 1, 0),	

is a non-symmetric G-metric on X because $G(0, 0, 1) \neq G(0, 1, 1)$.

Let $f, g, S, T : X \to X$ defined by

x	$f(x)$	$g(x)$	$S(x)$	$T(x)$
0	0	0	0	0
1	0	2	2	1
2	0	0	1	1

Then $fX \subseteq TX$ and $gX \subseteq SX$, with the pairs $\{f, S\}$ and $\{g, T\}$ are R-weakly commuting as they commute at their coincidence points.

Now to get (2.1) and (2.2) satisfied, we have the following nine cases: (I) $x, y = 0$, (II) $x = 0, y = 2$, (III) $x = 1, y = 0$, (IV) $x = 1, y = 2$, (V) $x = 2, y = 0$, (VI) $x = 2, y = 2$. For all these cases, $f(x) = g(y) = 0$ implies $G(fx, fx, gy) = 0$ and (2.1) and (2.2) hold.

(VII) For $x = 0, y = 1$, then $fx = 0, gy = 2, Sx = 0, Ty = 1$.

$$\begin{aligned} &G(fx, fx, gy)\\ &\quad = G(0,0,2) = 1\\ &\quad \leq h\max\{1,0,2,1\}\\ &\quad = h\max\{G(0,0,1),\ G(0,\ 0,0),\ G(2,2,1),\ [G(0,\ 0,1)+G(2,2,0)]/2\}\\ &\quad = h\max\{G(Sx,\ Sx,\ Ty),\ G(fx,\ fx,\ Sx),\ G(gy,\ gy,\ Ty),\\ &\qquad [G(fx,\ fx,\ Ty)+G(gy,\ gy,\ Sx)]/2\}. \end{aligned}$$

Thus, (2.1) is satisfied where $h = \frac{4}{5}$.

Also

$$\begin{aligned} &G(fx, gy, gy)\\ &\quad = G(0,2,2) = 1\\ &\quad \leq h\max\{2,0,2,1.5\}\\ &\quad = h\max\{G(0,1,1),\ G(0,\ 0,0),\ G(2,1,1),\ [G(0,\ 1,1)+G(2,0,0)]/2\}\\ &\quad = h\max\{G(Sx,\ Ty,\ Ty),\ G(fx,\ Sx,\ Sx),\ G(gy,\ Ty,\ Ty),\\ &\qquad [G(fx,\ Ty,\ Ty)+G(gy,\ Sx,\ Sx)]/2\}. \end{aligned}$$

Thus, (2.2) is satisfied where $h = \frac{4}{5}$.

(VIII) Now when $x = 1, y = 1$, then $fx = 0, gy = 2, Sx = 2, Ty = 1$.

$$\begin{aligned} &G(fx, fx, gy)\\ &\quad = G(0,0,2) = 1\\ &\quad \leq h\max\{2,1,2,0.5\}\\ &\quad = h\max\{G(2,2,1),\ G(0,\ 0,2),\ G(2,2,1),\ [G(0,\ 0,1)+G(2,2,2)]/2\}\\ &\quad = h\max\{G(Sx,\ Sx,\ Ty),\ G(fx,\ fx,\ Sx),\ G(gy,\ gy,\ Ty),\\ &\qquad [G(fx,\ fx,\ Ty)+G(gy,\ gy,\ Sx)]/2\}. \end{aligned}$$

Thus, (2.1) is satisfied where $h = \frac{4}{5}$.

And

$$\begin{aligned} &G(fx, gy, gy)\\ &\quad = G(0,2,2) = 1\\ &\quad \leq h\max\{2,1,2,1\}\\ &\quad = h\max\{G(2,1,1),\ G(0,\ 2,2),\ G(2,1,1),\ [G(0,\ 1,1)+G(2,2,2)]/2\}\\ &\quad = h\max\{G(Sx,\ Ty,\ Ty),\ G(fx,\ Sx,\ Sx),\ G(gy,\ Ty,\ Ty),\\ &\qquad [G(fx,\ Ty,\ Ty)+G(gy,\ Sx,\ Sx)]/2\}. \end{aligned}$$

Thus, (2.2) is satisfied where $h = \frac{4}{5}$.

(IX) If $x = 2, y = 1$, then $fx = 0, gy = 2, Sx = 1, Ty = 1$ and

$$\begin{aligned} &G(fx, fx, gy)\\ &\quad = G(0,0,2) = 1\\ &\quad \leq h\max\{0,1,2,1.5\}\\ &\quad = h\max\{G(1,1,1),\ G(0,\ 0,1),\ G(2,2,1),\ [G(0,\ 0,1)+G(2,2,1)]/2\}\\ &\quad = h\max\{G(Sx,\ Sx,\ Ty),\ G(fx,\ fx,\ Sx),\ G(gy,\ gy,\ Ty),\\ &\qquad [G(fx,\ fx,\ Ty)+G(gy,\ gy,\ Sx)]/2\}. \end{aligned}$$

Thus, (2.1) is satisfied where $h = \frac{4}{5}$.

Also

$$\begin{aligned} &G(fx,\ gy,\ gy) \\ &\quad = G(0,2,2) = 1 \\ &\quad \le h\max\{0,2,2,2\} \\ &\quad = h\max\{G(1,1,1),\ G(0,\ 1,1),\ G(2,1,1),\ [G(0,\ 1,1) + G(2,1,1)]/2\} \\ &\quad = h\max\{G(Sx,\ Ty,\ Ty),\ G(fx,\ Sx,\ Sx),\ G(gy,\ Ty,\ Ty), \\ &\qquad [G(fx,\ Ty,\ Ty) + G(gy,\ Sx,\ Sx)]/2\}. \end{aligned}$$

Thus, (2.2) is satisfied where $h = \frac{4}{5}$.

Hence, for all $x, y \in X$, (2.1) and (2.2) are satisfied for $h = \frac{4}{5} < 1$ so that all the conditions of Theorem 2.1 are satisfied. Moreover, 0 is the unique common fixed point for all of the mappings f, g, S and T.

In *Theorem* 2.1, if we take $f = g$, then we have the following corollary.

Corollary 2.3. Let X be a complete G-metric space. Suppose that $\{f, S\}$ and $\{f, T\}$ be pointwise R-weakly commuting pairs of self-mappings on X satisfying

$$\begin{aligned} G(fx,\ fx,\ fy) \le h\max\{&G(Sx,\ Sx,\ Ty),\ G(fx,\ fx,\ Sx),\ G(fy,\ fy,\ Ty), \\ &[G(fx,\ fx,\ Ty) + G(fy,\ fy,\ Sx)]/2\} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} G(fx,\ fy,\ fy) \le h\max\{&G(Sx, Ty,\ Ty),\ G(fx, Sx,\ Sx),\ G(fy, Ty,\ Ty)\} \\ &[G(fx,\ Ty,\ Ty) + G(fy,\ Sx,\ Sx)]/2\} \end{aligned} \tag{2.10}$$

for all $x, y \in X$, where $h \in [0, 1)$. Suppose that $fX \subseteq SX \cup TX$, and one of the pairs $\{f, S\}$ or $\{f, T\}$ is compatible. If the mappings in the compatible pair are continuous, then f, S and T have a unique common fixed point.

Also, if we take $S = T$ in *Theorem 2.1*, then we get the following.

Corollary 2.4. Let X be a complete G-metric space. Suppose that $\{f, S\}$ and $\{g, S\}$ are pointwise R-weakly commuting pairs of self-maps on X and

$$\begin{aligned} G(fx,\ fx,\ gy) \le h\max\{&G(Sx, Sx,\ Sy),\ G(fx, fx,\ Sx),\ G(gy, gy,\ Sy), \\ &[G(fx,\ fx,\ Sy) + G(gy,\ gy,\ Sx)]/2\} \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} G(fx,\ gy,\ gy) \le h\max\{&G(Sx, Sy,\ Sy),\ G(fx, Sx,\ Sx),\ G(gy, Sy,\ Sy), \\ &[G(fx,\ Sy,\ Sy) + G(gy,\ Sx,\ Sx)]/2\} \end{aligned} \tag{2.12}$$

hold for all $x, y \in X$, where $h \in [0, 1)$. Suppose that $fX \cup gX \subseteq SX$ and one of the pairs $\{f, S\}$ or $\{g, S\}$ is compatible. If the mappings in the compatible pair are continuous, then f, g and S have a unique common fixed point.

Corollary 2.5. Let X be a complete G-metric space. Suppose that f and g are two self-mappings on X satisfying

$$\begin{aligned} G(fx,\ fx,\ gy) \ \le\ h\max\{&G(x,\ x,\ y),\ G(fx,\ fx,\ x),\ G(gy,\ gy,\ y), \\ &[G(fx,\ fx,\ y) + G(gy,\ gy,\ x)]/2\} \end{aligned} \tag{2.13}$$

and

$$G(fx,\ gy,gy)\ \le\ h\max\{G(x,y,y),\ G(fx,x,\ x),\ G(gy,y,y),\\ [G(fx,\ y,\ y) + G(gy,\ x,\ x)]\ /2\} \tag{2.14}$$

for all $x, y \in X$, where $h \in [0, 1)$. Suppose that one of f or g is continuous, then f and g have a unique common fixed point.

Proof. Taking S and T as identity maps on X, the result follows from *Theorem 2.1*.

Corollary 2.6. Let X be a complete G-metric space and f be a self-map on X such that

$$G(fx,\ fx,\ fy)\ \le\ h\max\{G(x,\ x,\ y),\ G(fx,\ fx,\ x),\ G(fy,\ fy,\ y),\\ [G(fx,\ fx,\ y) + G(fy,\ fy,\ x)]/2\} \tag{2.15}$$

and

$$G(fx,\ fy,\ fy)\ \le\ h\max\{G(x,\ y,\ y),\ G(fx,\ x,\ x),\ G(fy,\ y,\ y),\\ [G(fx,\ y,\ y) + G(fy,\ x,\ x)]/2\} \tag{2.16}$$

hold for all $x, y \in X$, where $h \in [0, 1)$. Then f has a unique fixed point.

Proof. If we take $f = g$, and S and T as identity maps on X, then from f has a unique fixed point by *Theorem 2.1*.

3 Application

Let $\Omega = [0, 1]$ be bounded open set in $\mathbb{R}$, $L^2(\Omega)$, the set of functions on Ω whose square is integrable on Ω. Consider an integral equation

$$p(t,\ x(t)) = \int_\Omega q(t,\ s,\ x(s))ds \tag{3.1}$$

where $p : \Omega \times \mathbb{R} \to \mathbb{R}$ and $q : \Omega \times \Omega \times \mathbb{R} \to \mathbb{R}$ be two mappings. Define $G : X \times X \times X \to \mathbb{R}_+$ by

$$G(x,\ y,\ z) = \sup_{t\in\Omega}|x(t) - y(t)| + \sup_{t\in\Omega}|y(t) - z(t)| + \sup_{t\in\Omega}|z(t) - x(t)|.$$

Then X is a G-complete metric space. We assume the following that is there exists a function $G : \Omega \times \mathbb{R} \to \mathbb{R}^+$:

(i) $p(s, v(t)) \ge \int_\Omega q(t, s, u(s))\ ds \ge G(s, v(t))$ for each $s, t \in \Omega$..
(ii) $p(s, v(t))$ - $G(s, v(t)) \le h\ |p(s, v(t))$ - $v(t)|$.

Then integral equation (3.1) has a solution in $L^2(\Omega)$.

Proof. Define $(fx)(t) = p(t, x(t))$ and $(gx)(t) = \int_\Omega q(t, s, x(s))\ ds$. Now

$$\begin{aligned} G(fx,\ fx,\ gy) &= 2\sup_{t\in\Omega}|(fx)(t) - (gy)(t)| \\ &= 2\sup_{t\in\Omega}\left|p(t,\ x(t)) - \int_\Omega q(t,\ s,\ y(t))dt\right| \\ &\le 2\sup_{t\in\Omega}|p(t,\ x(t)) - G(t,\ x(t))| \\ &\le 2h\sup_{t\in\Omega}|p(t,\ x(t)) - x(t)| \\ &= \ hG(fx,\ fx,x). \end{aligned}$$

Thus

$$G(fx,\ fx,\ gy)\ \le\ h\max\{G(x,\ x,\ y),\ G(fx,\ fx,\ x),\ G(gy,\ gy,\ y),\\ [G(fx,\ fx,\ y)+G(gy,\ gy,\ x)]/2\}$$

is satisfied. Similarly (2.14) is satisfied. Now we can apply Corollary 2.5 to obtain the solution of integral equation (3.1) in $L^2(\Omega)$.

Remark 1. Theorems 2.8-2.9 in [3] and Corollaries 2.6-2.8 in [4] are special cases of our results Theorem 2.1 and Corollaries 2.3-2.6.

Remark 2. A G-metric naturally induces a metric d_G given by $d_G(x, y) = G(x, y, y) + G(x, x, y)$. If the G-metric is not symmetric, the inequalities (2.1) and (2.2) do not reduce to any metric inequality with the metric d_G. Hence, our theorems do not reduce to fixed point problems in the corresponding metric space (X, d_G).

Author details
[1]Department of Mathematics, Lahore University of Management Sciences, 54792 Lahore, Pakistan [2]Department of Mathematics, Statistics and Physics, Qatar University, Doha 2713, Qatar

Authors' contributions
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