Communication Complexity of Combinatorial Auctions with Submodular Valuations

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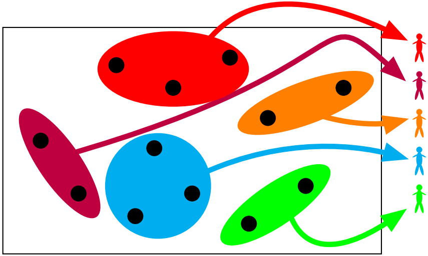
Welfare maximization in combinatorial auctions

Problem: m items are to be sold to n agents,
with monotone valuation functions $v_i : 2^M \to \mathbb{R}_+$



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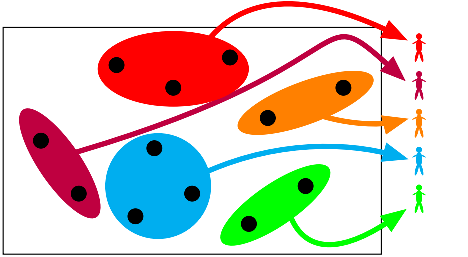

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1. The total welfare obtained by the agents is close to optimal
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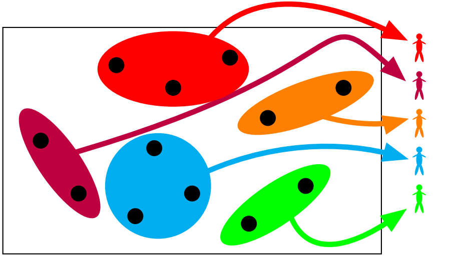

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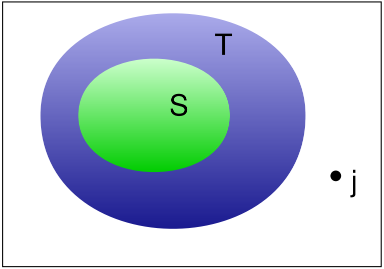
1. The total welfare obtained by the agents is close to optimal
2. The algorithm uses polynomial computation/communication
3. ~~Agents are motivated to reveal their true preferences~~
   NOT IN THIS PAPER

Submodular functions

Submodularity = property of *diminishing returns*.

Let the *marginal value* of element j be $f_S(j) = f(S+j) - f(S)$.



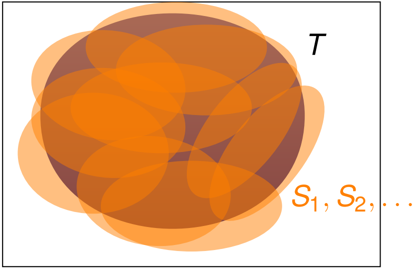
Definition: f is submodular, if
j cannot add more value to T than S.

$$f_S(j) \geq f_T(j)$$

Fractionally subadditive (XOS) functions

Fact

Submodular Functions $\subset$ Fractionally Subadditive = XOS Functions



Definition: f is fractionally subadditive,

if $f(T) \leq \sum \alpha_i f(S_i)$
whenever $\mathbf{1}_T \leq \sum \alpha_i \mathbf{1}_{S_i}$.

Definition: f is an XOS function,
if f is a maximum over linear functions: $f(S) = \max_i \sum_{j \in S} c_{ij}$

Oracle models

Input access issue:
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   - *value query*: What is your value for set S?
   - *demand query*: Which set would you buy under prices $p_1, \ldots, p_m$?

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What an algorithm can achieve may depend on the oracle model:

- For XOS valuations with demand queries, there is a $(1 - 1/e)$-approximation [Feige '06]
- For XOS valuations with value queries, there is no approximation better than $\sqrt{m}$ [Mirrokni-Schapira-V. '08]

The main question in this paper

What are the limits on approximation in combinatorial auctions, regardless of the oracle model?

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Two approaches:

- *Communication complexity:* What is the amount of information the agents have to communicate with the algorithm (or each other) to achieve a good outcome? [Nisan-Segal '01]
- *NP-hardness for simple valuations:* What can we achieve if each valuation has nonzero value only for constantly many items? [Feige '06]

Both lower bounds imply hardness for any "reasonable" oracle model as above.

Welfare maximization in CA: state of the art

Valuations	Submodular	XOS
Approximation with value queries	$1 - 1/e$ [V '08]	$m^{-1/2}$ [DNS '05]
Hardness with value queries	$1 - 1/e + \epsilon$ [KLMM '05]	$m^{-1/2+\epsilon}$ [MSV '08]
Approximation with demand queries	$1 - 1/e + 10^{-5}$ [FV '06]	$1 - 1/e$ [F '06]
Communication hardness	$1 - 1/m$ [DS '06]	$1 - 1/e + \epsilon$ [DS '06]
NP-hardness for simple valuations	$1 - \epsilon$ [FV '06]	$1 - 1/e + \epsilon$ [FV '06]

Open question: [Nisan-Segal '01]
Is it possible to achieve a PTAS for submodular valuations with polynomial communication?

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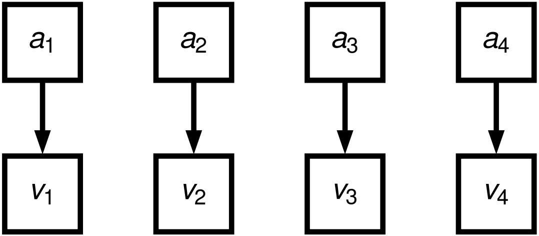
Approximation with demand queries	$1 - 1/e + 10^{-5}$ [FV '06]	$1 - 1/e$ [F '06]
Communication hardness	$1 - 1/(2e) + \epsilon$	$1 - 1/e + \epsilon$ [DS '06]
NP-hardness for simple valuations	$1 - 1/(2e) + \epsilon$	$1 - 1/e + \epsilon$ [FV '06]

New result: [Dobzinski, V.]
Any $(1 - \frac{1}{2e} + \epsilon)$-approximation submodular valuations would
(a) require exponential communication,
(b) imply $P = NP$, even for simple submodular valuations.

How do we prove this?

Recall the analysis of communication complexity for XOS valuations [Nisan '02], [Dobzinski-Schapira '06]:

Multiparty Set Disjointness [Alon-Matias-Szegedy '96]



strings $a_i \in \{0,1\}^r$

valuations $v_i : 2^M \to \mathbb{R}_+$
r relevant partitions of M

Combinatorial Auction

- YES case: all strings share a bit = 1
  $\Rightarrow$ there is a partition where all players get a Good Set
- NO case: no two strings share a bit = 1
  $\Rightarrow$ in every partition, most players get a Bad Set

New construction of submodular functions

Crucial step: How to define a function which is "high" on Good Sets and "low" on Bad Sets? Easy with XOS: $v(S) = \max_{F \in \mathcal{F}} |S \cap F|$.

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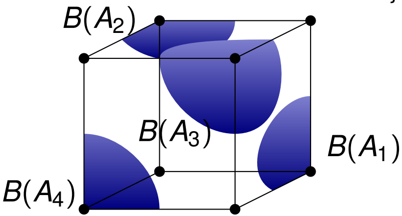
How do we do it with submodular functions?

Idea: Think of the continuous version of submodularity,

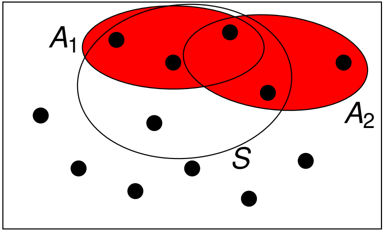
$$F : [0,1]^m \to \mathbb{R}, \qquad \frac{\partial^2 F}{\partial x_i \partial x_j} \le 0.$$



- Start with a basic function $F(x)$.
- Modify this function in *disjoint* regions $B(A)$ around Good Sets A, so that $F(\mathbf{1}_A)$ gets increased for each Good Set A.

Construction more specifically

- Partitions are chosen so that for any two Good Sets, $|A \cap A'| \leq b$.
- Call $x \in [0,1]^M$ "close to A", if $x(A) - x(\overline{A}) > b$.
- *Lemma:* x can be close to at most one Good Set A.

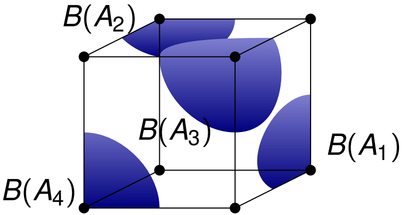


Example: $b = 1$, $x = \mathbf{1}_S$

So we can define $B(A)$ as the region of all points close to A
$\Rightarrow B(A)$ are *disjoint* for different Good Sets A.

Finishing the construction

Define disjoint regions $B(A)$ for Good Sets A:



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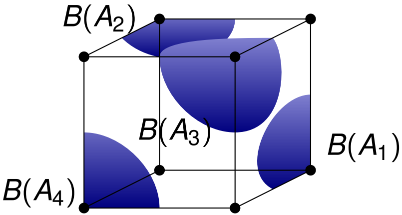

- Start with a basic function $F(x) = 1 - (1 - a\sum_{i=1}^{m} x_i)^2$.
- For each Good Set A, define a modified function $F_A(x) = 1 - \left(1 - a(2\sum_{i\in A} x_i - b)\right)_+ \left(1 - a(2\sum_{i\notin A} x_i + b)\right)_+$.
- F_A connects seamlessly to F at the boundary of B_A.

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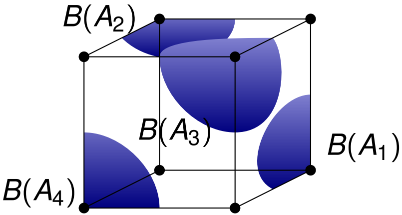

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Careful tuning of parameters:
The gap between YES/NO instances is $1 - 1/(2e) + \epsilon$.

Conclusions

- We give the first communication complexity result ruling out a PTAS for submodular combinatorial auctions, in any oracle model.
- For the demand oracle model, we narrow the gap to

$$[1 - 1/e + 10^{-5}, 1 - 1/(2e)].$$

- We prove similar results for CPP and Max-Min Allocation.

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Open questions:

- What is the optimal approximation for submodular combinatorial auctions:
  1. In the demand oracle model?
  2. Bounded only by communication complexity?
- Can we prove stronger communication complexity results for *truthful mechanisms* for combinatorial auctions?