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COMMUNITY PREFERENCES AND THE REPRESENTATIVE CONSUMER

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A representative consumer exists if market behavior corresponds to a representative income or utility level which is a function of the income distribution. Necessary and sufficient conditions are given on micro behavior and macro behavior (whether maximizing or not) for a representative consumer to exist. Nonlinear Engel curves and taste differences are permitted. If the representative income level is restricted to be mean income, we obtain the traditional linear Engel curves solution. A striking result on economy of information in the representation of a social welfare function is given.

1. INTRODUCTION

TWO NOTIONS WHICH are so closely related that one would not even put them on different sides of the same coin are that of a community indifference curve and that of a representative consumer. A representative consumer exists if the market behavior of an aggregate of different consumers is as if it were the market behavior of a number of identical hypothetical consumers, each with the same level of income. It is implicit or explicit in much of economic analysis. Community indifference curves have a long history in international trade theory. They are at least as old as Jevons. That there are problems with their existence was realized by Wicksell [**27**] and many economists since. Samuelson's 1956 paper [**22**] is still the best accessible statement of the problem and its solution.

In a classic 1953 paper, "Community Preference Fields" [**10**], Gorman established: (i) given that each consumer has sufficient income, then community preferences exist if the marginal propensity to consume for any good is the same across consumers, and (ii) given (i), income redistribution "does not matter" in that it does not affect market behavior.

These conditions imply linear expansion paths not necessarily through the origin with identical slopes across consumers.[2] Samuelson proposed an alternative

[1] This paper owes much to friends and colleagues. In rough chronological order, Angus Deaton's challenge that earlier results would be hard to generalize began it all. Early discussions with Gerald Kennally proved to be very important. At Birkbeck, Hugh Davies, Ben Fine, Sue Himmelweit, and Richard Portes made valuable contributions. Searching questions raised at a seminar at the IIES and by Peter Hammond and Jim Mirrlees were very useful in sorting out some issues. Finally, I must acknowledge my debt to Terence Gorman, a pioneer of this kind of analysis. He showed me that in an earlier version I had not reached the most general form of individual preferences. He is also responsible for suggesting the form of the direct utility function for generalized linearity.

[2] Gorman's results do not, of course, spring out of a vacuum. Samuelson [**21**] showed that the same condition is necessary in the two-commodity case to solve a related problem. This is the "transfer problem": when two consumers trade, under what conditions are the prices independent of the distribution of initial endowments? As Samuelson [**21**] points out, Keynes was already aware of the solution. Also Samuelson's 1952 paper was an extended version of an unpublished pre-war paper of his. Theil [**26**] independently dealt with linear aggregation theorems in general. However, as far as I am aware, Gorman was the first explicitly to set up and solve the community preference problem *per se*. His paper is noteworthy for the explicit way in which the restriction on the lower bound of utility is linked to convexity rather than merely to the nonnegativity of demands, and for deriving the Hicksian demand functions. Though Gorman does not point this out, adding these functions times the prices gives the cost functions.

story to justify community indifference curves. Here the government has a Bergson–Samuelson type social welfare function defined on individual utilities. Both the government and consumers are simultaneously optimizing: the former with respect to the income distribution, the latter with respect to their budget allocations.

The assumption of community preferences is standard for trade theorists and for cost benefit analysts who use as their tool the change in consumer surplus defined as the approximate money value at given prices of the change in utility which occurs for some representative consumer. This interpretation is strictly valid only if the observed market purchases in the two situations being compared are compatible with maximizing behavior and, even then, the separate question of distributional judgements arises. Although the latter is a quite separate issue, the knowledge that income distribution "does not matter" behaviorally when community preferences exist in Gorman's sense may predispose economists to neglect distributional issues.

Samuelson's alternative story for the existence of community preferences has perhaps proved even more popular among formal theorists in public economics. It underlies, for example, the elegant exposition in Diamond and McFadden [**6**]. It is even more explicitly unrealistic about distributional issues.

It is worth pointing out that this paper has only a very distant connection with the work of Scitovsky [**23**] and Arrow [**1**] who are often mentioned in connection with community preferences. Scitovsky's concept is much weaker and is discussed at the end of Section 2. Arrow's framework is much more general. But, within my specific and narrow framework of individualistic preferences and market decentralization, the negative result that without severe restrictions community preferences do not exist is related to his negative result. However, that the answer is negative here is already well known to all.

My new set of conditions for the existence of community preferences is more in the spirit of linear aggregation. I use a slightly more general framework than Gorman's, and his theorem is a special case of mine. I define my representative consumer through the representativeness of his or her *budget shares* rather than the quantities or values purchased. It turns out that this permits the Engel curves to be nonlinear. The effect of this is to re-introduce explicitly a behavioral influence for income distribution. I hope that this will make it psychologically more difficult for welfare economists to ignore distributional issues.

One might ask whether the assumption of community preferences in my sense has any ethical connotations. Does it, for example, aid in finding out whether one social situation is better than another? The answer to this is that it does not. What it does offer is an elegant and striking informational economy. This can be seen as follows: A Bergson–Samuelson social welfare function is a formalization of some ethical judgements. In particular, the implied "distributional weights" (i.e., the marginal social value of each person's income) are a convenient representation of these judgements. If under community preferences each consumer's budget shares are aggregated, weighting income by the relevant distributional weight, the resulting budget shares correspond to a *socially representative* income level. In general, more egalitarian judgements result in a lower socially representative income level. This means that the choice of the socially representative income level corresponds locally to a Bergson–Samuelson welfare function. The former, i.e., choosing one point on the income continuum is, however, much simpler and more intuitively appealing than the latter. This is perhaps the best way to grasp the informational economy for the making of welfare statements of the assumption of community preferences. This point is illustrated in Section 5.

I have two specifications of the requirements for community preferences. The weaker of the two does not require maximizing behavior but does require the absence of money illusion. The stronger specification assumes micro-maximizing behavior and requires the same of the representative consumer; i.e., it should be possible "to integrate back" from the market budget share relationships to a utility function. Recently, general equilibrium theorists have shown considerable interest in a related issue. In distinction to my problem (which is: when can we aggregate consistently?), these theorists have asked: when can we decompose consistently? Basically there are two theorems. The first, proved by Sonnenschein [**25**], says that *locally*, *any* n continuous market demand functions consistent with the budget constraint and homogeneous of degree zero in prices and aggregate income can be decomposed into the n demand functions of each of n hypothetical utility maximizing consumers. Each can have the same level of income but, in general, has different preferences. Sonnenschein says this "provides a striking indication that the (budget and homogeneity) restrictions largely exhaust the empirical implications of the utility hypothesis for market demand functions". The second theorem is a *global* one and says that a similar decomposition can be carried out for market excess demand functions. In Debreu's [**5**] version this is so for *some* distribution of initial endowments; in McFadden, et al. [**17**, Theorem 3], it is for *any* initial distribution, but decomposition works only for market excess demand functions in a neighborhood of the aggregate endowment.

Clearly, consistent aggregation is sufficient but not necessary for consistent decomposition. However, when decomposition is required for *all* initial distributions, it comes close to being the same as consistent aggregation. It is clear that there are some interesting results to be obtained in the middle ground between the two problems.

In particular, this is so for the fixed income distribution case with which I do not directly deal in this paper but which McFadden, et al. have raised as an open question. Pearce [**20**] devoted a chapter to it, but apart from some differential conditions and one very special solution did not get very far towards a general solution. The functional forms in this paper are much more general, though still not the most general solutions to this problem.

2. THE MAIN RESULTS AND THEIR INTERPRETATION

The integrability condition for an individual consumer is nothing more or less than the condition that one should be able to "integrate back" to his utility function given his market behavior, i.e., integrate back to the specification of preferences from the implicit marginal conditions for utility maximizing or cost minimizing behavior. Expressed in terms of quantities demanded, it is given by the conditions

$$(1) \qquad q_{ih} = \frac{\partial m_h(u_h, p)}{\partial p_i} \qquad (i = 1, \dots, n),$$

where q_{ih} is the purchase of good i by consumer h, p is the price vector, and $m_h(u_h, p)$ is his cost-of-utility function or expenditure function.

Equation (1) can be expressed in budget share terms:

$$(2) \qquad w_{ih} = p_i q_{ih}/y_h = \partial \log m_h(u_h, p)/\partial \log p_i$$

where $y_h = \Sigma_i p_i q_{ih}$ is the budget.[3]

Duality principles established by Hotelling [**14**], Shephard [**24**] (in a production context), and Karlin [**16**], among others, guarantee that if the standard axioms on consumer preferences hold, such an $m_h(u_h, p)$ exists and, moreover, that its properties, that m_h is concave in p, monotonic increasing in u_h and p, linear homogeneous in p, entail all the behavioral implications of these axioms. As is well known, the mathematical condition that $\partial^2 m_h/\partial p_i \partial p_j = \partial^2 m_h/\partial p_j \partial p_i$ implies the Slutsky symmetry restrictions.

I shall use the aggregate version of (2) to define Condition R, i.e., the existence of a representative consumer and the integrability conditions being satisfied by the market demands. Let y be the vector $(y_1, \dots, y_N)$. Let C be the set of y for which each of the cost functions $m_h(u_h, p)$ is concave and linear homogeneous in p, monotonic increasing and differentiable[4] in p and u_h at prices p.

CONDITION R: Given $w_{ih} = \partial \log m_h(u_h, p)/\partial \log p_i$ for $i = 1, \dots, n$ and $h = 1, \dots, N$, there exists a function $M(u_0, p)$ for all $y \in C$ so that for some u_0

$$(3) \qquad \bar{w}_i = \partial \log M(u_0, p)/\partial \log p_i \qquad (i = 1, \dots, n),$$

where $\bar{w}_i \equiv p_i \Sigma_h q_{ih}/\Sigma_h y_h \equiv \Sigma y_h w_{ih}/\Sigma_h y_h$ and $M(u_0, p)$ has concavity, etc. properties similar to those of $m_h(u_h, p)$ and where y_0 and u_0 are functions of p and of the vector y or the vector u. u_0 is interpreted as the utility level of the representative consumer. His income level $y_0 = M(u_0, p)$.

What Gorman [**10**] supplemented by Gorman [**11**][5] proved can be recast in these terms:

[3] In this paper, I shall use the words "income", "budget", and "total expenditure" synonymously. Redefining the q's as consumption in different time periods, all the aggregation theorems can be reinterpreted for savings or consumption functions defined on wealth.

[4] This is not a terribly stringent requirement.

[5] Also see his elegant notes "Duality and Its Applications" [**12**].

THEOREM 1: *Define the representative income to be* $\bar{y} = \Sigma_h y_h/N$. *Then the necessary and sufficient condition for R to be fulfilled is that*

$$M(u_0, p) = a(p) + u_0 b(p) \tag{4}$$

and that

$$m_h(u_h, p) = a_h(p) + u_h b(p) \tag{5}$$

where $a(p) = \Sigma_h a_h(p)$, $u_0 = \Sigma_h u_h/N$, *and* $a(p)$, $b(p)$, *and* $a_h(p)$, *all* h, *are concave*[6] *and linear homogeneous in* p.

It is easy to show that the requirement $y_0 = \bar{y}$ is equivalent to redefining Condition R on the quantities rather than on the budget shares. This makes clear the sense in which my requirements for consistent aggregation are weaker than Gorman's.

There is a not-necessarily-maximizing (NNM) version of Condition R as follows:

CONDITION RNNM: Given $w_{ih} = w_{ih}(y_h, p)$ for $i = 1, \ldots, n$ and $h = 1, \ldots, N$, there exists a function $y_0 = y_0(y, p)$ for all feasible y so that

$$\bar{w}_i = w_i(y_0, p) \qquad (i = 1, \ldots, n), \tag{6}$$

where $w_i(\ )$, $w_{ih}(\ )$, all h, are continuous in income and zero degree homogeneous in income and prices.

It is obvious that if R is satisfied then RNNM must be satisfied, but not conversely.

I shall prove the following theorems on the forms of macro-preferences.

THEOREM 2A: *RNNM is satisfied iff the* $w_i(\ )$ *functions satisfy the following two equivalent restrictions*:

$$\frac{\partial \bar{w}_i}{\partial y_0} \Big/ \frac{\partial \bar{w}_j}{\partial y_0} = A_{ij}(p) \qquad (i, j = 1, \ldots, n; i \neq j), \tag{7}$$

$$\bar{w}_i = v(y_0, p)A_i(p) + D_i(p) \qquad (i = 1, \ldots, n), \tag{8}$$

where $A_i/A_j = A_{ij}$, $i, j = 1, \ldots, n$, $\Sigma_i A_i = 0$, $\Sigma_i D_i = 1$, *and* v, A_i, D_i *satisfy the homogeneity restrictions.*

Equations (7) and (8) imply that any two aggregate budget share-income relationships must be related in the sense that there exists a linear transformation of one which will give the other.

[6] Then the concavity region C includes all y_h for which $u_h \geqslant 0$.

Since for $y \in C$, RNNM plus the maximization assumption is equivalent to R, it is clear that the form of the cost function corresponding to (7) and (8) will be necessary and sufficient for R.

THEOREM 3A: *The cost function*

$$M(u_0, p) = G(u_0, H(p))B(p) \tag{9}$$

where B is homogeneous of degree one, H is homogeneous of degree zero, and G is monotonic increasing in u_0, *is necessary and sufficient for R.*

Equation (9) can also be written in the form $M = \hat{G}(a(p), b(p), u)$ where $a(\ )$, $b(\ )$ are homogeneous of degree one and $a = HB$ and $b = B$. Typically we would want to assume the concavity of a and b. This imposes some restrictions on $\hat{G}$. In particular, the concavity of $\hat{G}$ in a and b is then sufficient for the concavity of $\hat{G}$ in p[7] (given some restrictions on u).

Because conditions (7) through (9) entail *linear* relationships between budget shares, given income and because they are clearly generalizations of the Gorman results, I have given them the name "generalized linearity" (GL). GL was first introduced in Muellbauer [**19**] in the context of identical preferences.

The sequence of argument is then to derive the micro-conditions corresponding to Theorem 2A.

THEOREM 2B: *RNNM is satisfied iff the* $w_{ih}(\ )$ *functions satisfy the two equivalent conditions*

$$y_h(\partial w_{ih}/\partial y_h - A_{ij}\,\partial w_{jh}/\partial y_h) + w_{ih} - A_{ij}w_{jh} - D_{ij} = 0 \tag{10}$$

and

$$w_{ih} = v_h(y_h, p)A_i(p) + D_i(p) + C_{ih}/y_h \tag{11}$$

where $A_{ij} = A_i/A_j$ *and* $D_{ij} = D_i/D_j$, *all i, j, as defined in Theorem 2A.* $\Sigma_h C_{ih} = 0$, *all i.*

Since $\bar{w}_i = \Sigma_h y_h w_{ih}/\Sigma_h y_h$, we have y_0 defined by $v(y_0, p) = \Sigma y_h v_h(y_h, p)/\Sigma y_h$.

Theorem 2B leads to the corresponding form of cost function:

THEOREM 3B: *R is satisfied iff the cost function is*

$$m_h(u_h, p) = G_h(u_h, H(p))B(p) + g_h(p) \tag{12}$$

where $\Sigma_h g_h(p) = 0$ *and H and B are as before.*[8]

Again we can write this in the form $m_h(u_h, p) = \hat{G}_h(a(p), b(p), u_h) + g_h(p)$. Since if some g_h are positive others have to be negative, the concavity requirement may somewhat restrict the ranges over which the y_h belong to the set C. Notice, in-

[7] See below for further discussion.

[8] A recent paper by Carlevaro [**2**] contains a generalization of the linear expenditure system which is a special case of this form.

cidentally, how by setting $a_h(p) = a(p) + g_h(p)$ in (5), Gorman's case is a special case of GL.

The next results are devoted to two special cases of GL. The first is the one in which y_0 is homogeneous in y. The second is the one in which y_0 is independent of p. It turns out that under both RNNM and R, the latter implies the former, but not conversely. For obvious reasons, the case where y_0 is independent of p will be given the name "price independent generalized linearity", PIGL.

THEOREM 4: *Linear homogeneity of y_0 in y implies that under RNNM (i) the macro form for $v(y_0, p)$ in (8) is either $v = y_0^{\varepsilon(p)}$ or $v = \log y_0$ and (ii) the micro form for $v_h(y_h, p)$ in (11) is either $v_h = (y_h/k_h(p))^{\varepsilon(p)}$ or $v_h = \log (y_h/k_h(p))$.*

THEOREM 5: *Independence of y_0 from p implies linear homogeneity of y_0 in y and that under RNNM (i) the macro form for $v(y_0, p)$ in (8) is either $v = y_0^{-\alpha}$ where α is a scalar constant or $v = \log y_0$ and (ii) the micro form for $v_h(y_h, p)$ in (11) is either $v_h = (y_h/k_h)^{-\alpha}$ or $v_h = \log (y_h/k_h)$ where $\alpha, k_h > 0$ are scalar constants.*

THEOREM 6: *Under R, i.e., when maximizing is assumed, the independence of y_0 from p implies: (i) the macro form of the cost function is either*

$$M(u_0, p) = ((a(p))^\alpha + u_0(b(p))^\alpha)^{1/\alpha} \tag{13}$$

or

$$M(u_0, p) = (H(p))^{u_0}B(p) \tag{14}$$

where a, b, B are linear homogeneous and H is zero degree homogeneous.[9]

(ii) *The micro form of the cost function is either*

$$m_h(u_h, p) = k_h((a(p))^\alpha + u_0(b(p))^\alpha)^{1/\alpha} \tag{15}$$

or

$$m_h(u_h, p) = k_h H(p)^{u_h}B(p) \tag{16}$$

where α, k_h are as in Theorem 5.

For cost to be increasing with utility for $\alpha < 0$ as well as $\alpha > 0$ it is enough if u is made an increasing function of α, e.g., if u is replaced by αu. An alternative way of ensuring $\partial M/\partial u > 0$ is to replace b^α by $c(p)^\alpha - a(p)^\alpha$, where $c > a$. Another advantage follows from writing (13) in the form

$$M^\alpha = a^\alpha + u(c^\alpha - a^\alpha). \tag{13'}$$

The fact that $\lim_{\alpha\to 0} (x^\alpha - 1)/\alpha = \log x$, implies that as $\alpha \to 0$ (13′) becomes $\log M = \log a + u \log (c/a)$. This has exactly the same form as (14) where $a = B$ and $(c/a) = H$. Thus it is clear how the special case (14) arises.

[9] Diewert [**9**, pp. 129–130] gives a specific indirect utility function whose cost function is a member of the class defined by (13). In the context of identical preferences, he points out that his special case aggregates consistently. A similar example is given in Diewert [**8**]. This interesting paper also shows that if the number of consumers is less than the number of goods, individual maximizing behavior does impose *some* restrictions on aggregate behavior other than adding-up. Finally, an indirect utility form, (13) belongs to the "polar-form" class arising in Gorman's [**13**] price aggregation theorem.

Again a, b ought to be concave. Equation (14) can be written as $(HB)^{u_0}B^{1-u_0}$. This is a weighted average of HB and B which suggests that (HB) and B ought both to be concave. Incidentally, since (14) implies that the budget share equations depend on $\log y$, this case ought to be entitled PIGLOG![10]

The following result about the direct form of the utility function corresponding to the GL form $M(u_0, p) = \hat{G}(a(p), b(p), u_0)$ is due to Professor Gorman. Since a, b are linear homogeneous, interpret them as the unit costs of two intermediate inputs Z_1, Z_2 "produced" from the market goods q through nonjoint constant returns production functions $Z_1 = f_1(q_1), Z_2 = f_2(q_2)$. Hence the direct utility function must have the form

$$(17) \qquad u_0 = F(f_1(q_1), f_2(q_2))$$

where $q_1 + q_2 = q$.

Following up this idea for PIGL, it turns out that $F(\ )$ has the implicit form

$$(18) \qquad (1-u)(f_1(g_1)/1-u)^{\alpha/(1-\alpha)} + u(f_2(q_2)/u)^{\alpha/(1-\alpha)} = 1$$

corresponding to (13). Similarly, in implicit form, that corresponding to (14) is

$$(19) \qquad (f_1(q_1)/1-u)^{1-u}(f_2(q_2)/u)^u = 1$$

The interpretation has two immediate consequences. One is that it suggests that a, b, being unit cost functions, ought to be concave in p. And since a, b have the interpretation of being the prices of Z_1, Z_2, this suggests that $\hat{G}(\ )$ ought to be concave in a, b.

The other consequence is to give an intuitive reason for why GL works. The reason is that if there are only two goods, both Conditions R and RNNM must hold whatever the utility function or budget share equations. This is so because if $\bar{w}_2 = \Sigma y_h w_{2h}/\Sigma y_h = w_2(y_0, p)$, then $\bar{w}_1 \equiv 1 - w_2(y_0, p) \equiv w_1(y_0, p)$. Also if $\bar{w}_2 = w_2(u_0, p)$, then $\bar{w}_1 \equiv 1 - w_2(u_0, p) \equiv w_1(u_0, p)$. Thus both in terms of utilities and incomes, there exists a representative utility or income level.

In the cost function $m_h(u_h, p) = \hat{G}_h(a(p), b(p), u_h) + g_h(p)$, g_h can be given a fixed cost or fixed endowment interpretation. But net of g_h all consumers buy the same two goods $f_1(q_1)$ and $f_2(q_2)$ at prices a and b. Since the g_h terms cancel out overall, the two-good interpretation applies precisely. Hence, a representative income level exists and so does (if consumers maximize) a representative utility level.

I conclude this presentation and interpretation of the main results by showing in detail how it is that consistent aggregation works. I do this by proving the sufficiency parts of Theorems 3B and 6.

[10] A special case of the indirect translog utility function—see Jorgensen and Lau [**15**] and Christensen, et al. [**3**]—is a member of this class.

PROOF (Sufficiency of Theorem 3B): Since $q_{ih} = \partial m_h(u_h, p)/\partial p_i$, (12) gives

$$(20) \qquad q_{ih} = \frac{\partial G_h}{\partial H}\frac{\partial H}{\partial p_i}B + G_h\frac{\partial B}{\partial p_i} + \frac{\partial g_h}{\partial p_i},$$

$$p_i \sum_h q_{ih} = \left(\sum_h \frac{\partial G_h}{\partial H}\right) B \frac{\partial H}{\partial \log p_i} + \left(\sum_h G_h\right) B \frac{\partial \log B}{\partial \log p_i} + 0.$$

Since $\Sigma_h y_h = (\Sigma_h G_h)B$,

$$\bar{w}_i = \frac{p_i \Sigma q_{ih}}{\Sigma y_h} = \left(\frac{\Sigma(\partial G_h/\partial H)}{\Sigma G_h}\right)\frac{\partial H}{\partial \log p_i} + \frac{\partial \log B}{\partial \log p_i}$$

and from (9)

$$(21) \qquad \bar{w}_i = \frac{\partial \log G(u_0, H)}{\partial H}\frac{\partial H}{\partial \log p_i} + \frac{\partial \log B}{\partial \log p_i}.$$

Since $\Sigma(\partial G_h/\partial H) = (\partial \Sigma G_h/\partial H)$, we can write

$$(22) \qquad \frac{\partial \log G(u_0, H)}{\partial H} = \frac{\partial \log\left(\sum_h G_h(u_h, H)\right)}{\partial H}.$$

Thus the aggregate budget share has the same form as that for the representative individual.

Since (20) implies

$$(23) \qquad w_{ih} = \left(\frac{\partial G_h}{\partial H}\Big/ y_h\right)\frac{\partial H}{\partial \log p_i} + \frac{\partial \log B}{\partial \log p_i} + \frac{1}{y_h}\left(\frac{\partial g_h}{\partial \log p_i} - g_h\frac{\partial \log B}{\partial \log p_i}\right)$$

$$= v_h(y_h, p)A_i(p) + D_i(p) + \frac{1}{y_h}C_{ih}.$$

Thus we see that (23) has the same form as (11) and if $A_i = \partial H/\partial \log p_i$,

$$(24) \qquad y_h v_h(y_h, p) = \partial G_h\left(V_h\left(\frac{y_h - g_h}{B}, H\right), H\right)\Big/\partial H$$

where the indirect utility function V_h is given by solving $(y_h - g_h)/B = G_h(u_h, H)$ for u_h.

Next we turn to the PIGL case.

PROOF (Sufficiency of Theorem 6): For (15), $G_h B$ can be written in the form $G_h B = k_h(H + u_h)^{1/\alpha}B$ where $H = (a/b)^\alpha$ and $B = b$. Hence, $(\partial \log G_h/\partial H) = 1/(\alpha(H + u_h)) = (1/\alpha)(y_h/k_h B)^{-\alpha}$. Hence,

$$(25) \qquad \bar{w}_i = \left(\frac{\sum_h k_h^\alpha y_h^{1-\alpha}}{\alpha \sum_h y_h}\right)B^\alpha \frac{\partial H}{\partial \log p_i} + \frac{\partial \log B}{\partial \log p_i}.$$

Hence,

$$(26) \qquad y_0 = \left(\sum_h k_h^\alpha y_h^{1-\alpha} \Big/ \sum y_h\right)^{-1/\alpha}.$$

Hence, y_0 is linear homogeneous in y and independent of prices. *Q.E.D.*

Finally, we turn to the alternative form for PIGL (16). Here

$$(27) \qquad \begin{aligned} w_{ih} &= \frac{\partial \log B}{\partial \log p_i} + u_h \frac{\partial \log H}{\partial \log p_i} \\ &= \frac{\partial \log B}{\partial \log p_i} + (\log (y_h/k_h) - \log B)\frac{\partial \log H}{\partial \log p_i}\Big/ \log H. \end{aligned}$$

Hence,

$$(28) \qquad \bar{w}_i = \frac{\partial \log B}{\partial \log p_i} + \left(\frac{\sum y_h \log (y_h/k_h)}{\sum y_h} - \log B\right)\frac{\partial \log H}{\partial \log p_i}\Big/ \log H.$$

Hence,

$$(29) \qquad y_0 = \exp\left(\frac{\sum y_h \log (y_h/k_h)}{\sum y_h}\right).$$

Hence, y_0 is linear homogeneous in y and independent of prices. *Q.E.D.*

This concludes the presentation and interpretation of the main results. Those interested in a discussion of the implications of the main features of these results for the econometric study of demand systems will find one in Muellbauer [**19**]. That paper is in the context of identical preferences[11] but the main points remain the same so that there is no point in repeating them here. However, a brief application to the optimal commodity tax problem is included as the last section of this paper.

Now that the precise sense of my notion of community preferences has been explained, it is worth briefly discussing Scitovsky's [**23**] concept of social indifference curves. As Samuelson [**22**] makes clear, Scitovsky was interested in the aggregate minimum requirements contours when each consumer is at a specified and fixed utility level. Translating into price space this can be expressed by a function $\mathcal{M}(u_1, \dots, u_N, p) \equiv \Sigma_{h=1}^N m_h(u_h, p)$. Given the u's, $\mathcal{M}$ is concave in p and is defined without restricting individual preferences. Given p, $\mathcal{M}$ defines the utility possibility locus for a given amount of aggregate money income. Although it has some theoretical uses, it is obviously a quite different concept from mine. However, given Gorman's or my form of individual preferences the utility possibility loci are very substantially restricted. For the former, $\mathcal{M} = \Sigma a_h(p) + (\Sigma u_h)b(p)$, the utility possibility contour slopes are independent of prices. For the

[11] There is also some discussion of generalization to taste differences including an approximation theorem for stochastic differences in tastes. Although forms (15) and (16) are discussed there, no *necessity* results for these forms are attempted.

latter, $\mathscr{M} = \Sigma G_h(u_h, H(p))B(p)$, these slopes depend on prices only through the scalar function $H(p)$.

3. PROOFS FOR GENERALIZED LINEARITY

PROOF OF THEOREM 2A: In order to reduce the number of subscripts, I shall temporarily adopt the notation $w_{ih} = \psi_h$, $w_{jh} = \chi_h$. Hence at given p,

$$\bar{\psi} = \sum y_h \psi_h(y_h)/\sum y_h = \psi(y_0), \tag{30}$$

$$\bar{\chi} = \sum y_h \chi_h(y_h)/\sum y_h = \chi(y_0). \tag{31}$$

We shall show $(\partial\psi/\partial y_0)/(\partial\chi/\partial y_0) = A$ (equation (7)).

Differentiate (30) and (31) with respect to y_k:

$$\begin{aligned} \psi' \frac{\partial y_0}{\partial y_k} &= \frac{(y_k \psi_k' + \psi_k)\sum y_h - \sum y_h \psi_h}{(\sum y_h)^2} \\ &= \frac{y_k \psi_k' + \psi_k - \psi}{\sum y_h}. \end{aligned} \tag{32}$$

Similarly,

$$\chi' \frac{\partial y_0}{\partial y_k} = \frac{y_k \chi_k' + \chi_k - \chi}{y_h}. \tag{33}$$

Dividing (32) by (33):

$$\begin{aligned} \frac{\psi'}{\chi'} &= \frac{y_k \psi_k' + \psi_k - \psi}{y_k \chi_k' + \chi_k - \chi} \qquad (k = 1, \ldots, n) \\ &= \frac{W_k(y_k) - \psi(y_0)}{T_k(y_k) - \chi(y_0)}, \qquad \text{say}, \qquad \text{all } k. \end{aligned} \tag{34}$$

We now show that equation (7) is valid by proving that $\psi'(y_0)/\chi'(y_0)$ is independent of y_0 and of y. In a way, this is obvious since the right-hand side of (34) is the same for all k which suggests that it is independent of y_k, all k, and hence of y_0. Formally, the result is obtained by differentiating (34) with respect to y_r, $r \neq k$:

$$\begin{aligned} \frac{\partial}{\partial y_r} \frac{\psi'(y_0)}{\chi'(y_0)} &= \frac{[(T_k - \chi)(-\psi') - (W_k - \psi)(-\chi')]}{(T_k - \chi)^2} \frac{\partial y_0}{\partial y_r} \\ &= \left[\frac{W_k - \psi}{T_k - \chi} - \frac{\psi'}{\chi'}\right] \frac{\chi'(\partial y_0/\partial y_r)}{(T_k - \chi)} \\ &= 0 \qquad \text{by (34)}. \end{aligned}$$

Thus $(\psi'(y_0))/(\chi'(y_0)) = \text{constant} = A$, say. *Q.E.D.*

Integrating gives

$$\psi(y_0) = A\chi(y_0) + D. \tag{35}$$

Letting $\chi(y_0) = v(y_0)$, we immediately see that (35) implies

$$\psi = w_i = v(y_0, p)A_i(p) + D_i(p)$$

(equation (8)) where prices are made explicit.

Since $\Sigma_i w_i = 1$, $\Sigma A_i = 0$, and $\Sigma D_i = 1$. Homogeneity implies

$$(36) \qquad 0 \equiv v' y_0 A_i + \sum_j \frac{\partial v}{\partial \log p_j} A_i + v \sum_j \frac{\partial A_i}{\partial \log p_j} + \sum_j \frac{\partial D_i}{\partial \log p_j}.$$

Several possibilities satisfy (36): e.g., D_i and v are zero homogeneous in p, and the homogeneity of v in y_0 is minus that of A_i in p. *Q.E.D.*

Theorem 3A is proved in Muellbauer [**19**]. The proof involves the fact that (7) implies $(\partial \bar{w}_i/\partial u_0)/(\partial \bar{w}_j/\partial u_0)$ is independent of u_0 and the fact that

$$\bar{w}_i = \partial \log M(u_0, p)/\partial p_i.$$

Next we turn to a proof of Theorem 2B.

Proof of Theorem 2B: (i) We want to show that equation (10) holds, i.e., $y_h(\psi'_h - A\chi'_h) + \psi_h - A\chi_h - D = 0$. Using (35) we can write (34) as

$$\frac{y_h \psi'_h + \psi_h - A\chi - D}{y_h \chi'_h + \chi_h - \chi} = A.$$

Thus, $y_h(\psi'_h - A\chi'_h) + (\psi_h - A\chi_h - D) = 0$ (equation (10)), which proves part (i).

(ii) Let $\gamma_h(y_h) = \psi_h(y_h) - A\chi_h(y_h)$. Then equation (10) becomes

$$(37) \qquad y_h \gamma'_h = D - \gamma_h.$$

But $(d/dy_h)(y_h\gamma_h) = y_h\gamma'_h + \gamma_h = D$. Integrating, $y_h\gamma_h = Dy_h + C_h$. Letting $\chi_h = v_h$, this implies $y_h(w_{ih} - A_i(p)v_h(y_h, p)) = D_i(p)y_h + C_{ih}$. Dividing by y_h gives (11). *Q.E.D.*

Since $\Sigma w_i = 1$, $\Sigma A_i = 0$, and $\Sigma D_i = 1$, and $\Sigma_i C_{ih} = 0$. Summing over h must give (8). Hence $\Sigma_h C_{ih} = 0$ and

$$(38) \qquad v(y_0, p) = \sum y_h v_h(y_h, p) \Big/ \sum y_h,$$

which defines y_0.

Now we derive the form of the micro cost functions corresponding to (11).

Proof of Theorem 3B: Rewrite (11) as

$$(39) \qquad y_h(w_{ih} - A_{ij}w_{jh}) = D_{ij}y_h + C_{ijh}.$$

Since $y_h w_{ih} = (\partial m_h(u_h, p))/\partial \log p_i$, we can rewrite (39) as

$$(40) \qquad \frac{\partial m_h}{\partial \log p_i} - A_{ij}\frac{\partial m_h}{\partial \log p_j} = D_{ij}m_h + C_{ijh}.$$

Differentiate with respect to u_h:

$$\text{(41)} \qquad \frac{\partial\left(\dfrac{\partial m_h}{\partial u_h}\right)}{\partial \log p_i} - A_{ij}\frac{\partial\left(\dfrac{\partial m_h}{\partial u_h}\right)}{\partial \log p_j} = D_{ij}\frac{\partial m_h}{\partial u_h}.$$

Let $Z_h = \log(\partial m_h/\partial u_h)$. Then (41) becomes

$$\text{(42)} \qquad \frac{\partial Z_h}{\partial \log p_i} - A_{ij}\frac{\partial Z_h}{\partial \log p_j} = D_{ij}.$$

Differentiate (42) with respect to u_h:

$$\text{(43)} \qquad \frac{\partial}{\partial \log p_i}\left(\frac{\partial Z_h}{\partial u_h}\right)\Big/\frac{\partial}{\partial \log p_j}\left(\frac{\partial Z_h}{\partial u_h}\right) = A_{ij}$$

which is independent of u_h and the same for all h. The solution to (43) is

$$\text{(44)} \qquad \frac{\partial Z_h}{\partial u_h} = \tilde{G}_h(u_h, H(p)).$$

$H(p)$ must be the same as for the macro form of preferences—see equation (9) in Theorem 3A—since by (7)

$$\text{(45)} \qquad A_{ij} = \frac{\partial w_i(u_0, p)}{\partial u_0}\Big/\frac{\partial w_j(u_0, p)}{\partial u_0} = \frac{\partial H}{\partial \log p_i}\Big/\frac{\partial H}{\partial \log p_j}.$$

However, $\tilde{G}_h$ can differ over households.

From (44),

$$\text{(46)} \qquad Z_h = \log \hat{G}_h(u_h, H(p)) + \log B_h(p).$$

Next we show that $B_h(p) = B(p)$.

To the micro form (39), there corresponds the macro form $\bar{w}_i - A_{ij}\bar{w}_j = D_{ij}$. Because this must come from the cost function (9), i.e., by (21),

$$\text{(47)} \qquad D_{ij} = \frac{\partial \log B}{\partial \log p_i} - A_{ij}\frac{\partial \log B}{\partial \log p_j}.$$

But (41) implies that

$$\text{(48)} \qquad D_{ij} = \frac{\partial \log B_h}{\partial \log p_i} - A_{ij}\frac{\partial \log B_h}{\partial \log p_j}.$$

Hence, $B_h(p) = B(p)$.

Finally, we use $Z_h = \log[(\partial m_h(u_h, p))/\partial u_h]$ to find that $\partial m_h/\partial u_h = \hat{G}_h(u_h, H(p))B(p)$. The solution is

$$m_h(u_h, p) = G_h(u_h, H(p))B(p) + g_h(p).$$

That $\Sigma_h g_h(p) = 0$ follows from the fact that $\Sigma_h C_{ijh} = 0$. Also $g_h(p)$ is zero degree homogeneous in p since $w_{ih}(\ )$ is zero degree homogeneous in y_h and p. *Q.E.D.*

4. HOMOGENEITY AND PRICE INDEPENDENCE OF y_0

In this section we prove Theorems 4, 5, and 6 which are concerned with conditions under which y_0 is (i) linear homogeneous in y and (ii) independent of p.

PROOF OF THEOREM 4: Since, omitting prices, $v(y_0) = \Sigma y_h v_h(y_h)/\Sigma y_h$, we want conditions on $v(\ )$ and $v_h(\ )$ so that

$$v(\lambda y_0) = \frac{\sum \lambda y_h v_h(\lambda y_h)}{\sum \lambda y_h} \tag{49}$$

for $\lambda > 0$ and all $y \in C$. Notice that this is a much more stringent requirement than that for *some* fixed income distribution, consistent aggregation holds for proportionate increases in incomes. It can be shown that more general micro conditions than those for which R or RNNM holds, permit consistent aggregation under a fixed income distribution rule.

First, we examine conditions on $v(\ )$. If there exists a $v(\ )$ so that (49) is true, then for that $v(\ )$ it must be true that

$$v(\lambda y_0) = \frac{\sum \lambda y_h v(\lambda y_h)}{\sum \lambda y_h} \tag{50}$$

for $\lambda > 0$ and for all y at least in a subset (not of measure zero) of C. But (50) is just the case where everyone has the same preferences. Theorem 7A in Muellbauer [**19**] shows that necessary and sufficient conditions for (50) are

$$v(y_0) = y_0^{\varepsilon(p)}, \tag{51}$$

$$v(y_0) = \log y_0, \tag{52}$$

where arbitrary constants are absorbed in A_i and B_i and where $\varepsilon(p)$ is homogeneous of degree zero in p. Thus Theorem 4, part (i) must hold.

Next we turn to the derivation of the micro form of v_h as specified in Theorem 4, part (ii).

If $v(y_0) = y_0^{\varepsilon}$, then $(y_0/v)(\partial v/\partial y_0) = \varepsilon$. Differentiating

$$v(\lambda y_0) = \frac{\sum y_h v_h(\lambda y_h)}{\sum y_h}$$

with respect to λ and setting $\lambda = 1$, we find

$$y_0 \frac{\partial v}{\partial y_0} = \frac{\sum y_h^2(\partial v_h/\partial y_h)}{\sum y_h}. \tag{53}$$

Hence,

(54) $$\varepsilon = \frac{y_0}{v}\frac{\partial v}{\partial y_0} = \sum y_h^2 \frac{\partial v_h}{\partial y_h} \bigg/ \sum y_h v_h.$$

Differentiate (54) with respect to y_r:

(55) $$0 = \frac{(y_r^2 v_r')' - \varepsilon(y_r v_r)'}{\sum y_h v_h}.$$

Hence,

(56) $$y_r^2 v_r' = \varepsilon y_r v_r + \beta_r.$$[12]

Multiplying (56) through by $y_r^{-\varepsilon-2}$, we obtain

$$y_r^{-\varepsilon} v_r' - \varepsilon y_r^{-\varepsilon-1} v_r = \frac{d}{dy_r}(v_r y_r^{-\varepsilon}) = \beta_r y_r^{-\varepsilon-2}.$$

Hence, $v_r y_r^{-\varepsilon} = (\beta_r y_r^{-\varepsilon-1}/(-\varepsilon - 1)) + k_r$. Hence,

(57) $$v_r = \frac{-\beta_r}{y_r(\varepsilon+1)} + k_r y_r^{\varepsilon}.$$

But from form (11), the term $-\beta_r/(y_r(\varepsilon + 1))$ can be absorbed in the term C_{ir}/y_r in (11) and, hence, can be taken as zero without loss of generality. Rewriting k_r as $k_r^{-\varepsilon}$, and inserting p explicitly, we have $v_h(y_h, p) = (y_h/k_h(p))^{\varepsilon(p)}$. This concludes the proof of the first part of Theorem 4(ii).

Next we derive the micro form corresponding to $v = \log y_0$. Following a similar procedure, by (53)

(58) $$y_0 \frac{\partial v}{\partial y_0} = 1 = \sum y_h^2 \frac{\partial v_h}{\partial y_h} \bigg/ \sum y_h.$$

Differentiating this with respect to y_r, $0 = ((y_r^2 v_r')' - 1)/\Sigma y_h$. Hence,

(59) $$\begin{aligned} y_r^2 v_r' &= y_r + \beta_r, \\ v_r' &= \frac{1}{y_r} + \frac{\beta_r}{y_r^2}. \end{aligned}$$

Hence,

$$\begin{aligned} v_r &= \log y_r - \frac{\beta_r}{y_r} + \alpha_r \\ &= \log(y_r/k_r) \end{aligned}$$

since, without loss of generality, the term $-\beta_r/y_r$ can be absorbed in C_{ir}/y_r in (11) and α_r can be replaced by $-\log k_r$. Putting in p explicitly, $v_h(y_h, p) = \log(y_h/k_h(p))$. This completes the proof of Theorem 4.

[12] Constants of integration must be independent of i since v_r is independent of i.

Now we turn to Theorem 5.

PROOF OF THEOREM 5: Price independence implies that v_h and v are independent of prices. If a $v(y_0)$ exists, then arguing as before, $v(y_0) = \Sigma y_h v(y_h)/\Sigma y_h$ for at least a subset not of measure zero of C. But this is the same as the price independence condition under identical preferences analyzed in Theorem 5 (a) in Muellbauer [**19**]. This tells us that $v = y_0^{-\alpha}$ where α is constant or $v = \log y_0$ which gives us part (i) of the current Theorem 5.

But since these forms are just special cases of the macro-forms in Theorem 4, we know that the micro-forms are either $v_h = (y_h/k_h(p))^{-\alpha}$ or $v_h = \log(y_h/k_h(p))$. But since $y_0^{-\alpha} = \Sigma y_h(y_h/k_h(p))^{-\alpha}/\Sigma y_h$ is independent of p, k_h must be independent of p, all h. Thus k_h is a scalar constant, all h and $k_h > 0$ can be assumed without loss of generality. The same holds for $v_h = \log(y_h/k_h(p))$. This completes the proof of Theorem 5.

I have investigated the form of the cost function corresponding to $v = y^{\varepsilon(p)}$ in Theorem 4. However, I have not been able to obtain a closed form characterization. Since this analysis is a little tedious it is relegated to the Appendix.

Finally, we turn to Theorem 6 which derives the cost functions corresponding to Theorem 5.

PROOF OF THEOREM 6: Theorem 5 in Muellbauer [**19**] implies that the macro-cost functions have either the form $M(u_0, p) = (a^\alpha(p) + u_0 b^\alpha(p))^{1/\alpha}$ or $M(u_0, p) = (H(p))^{u_0} B(p)$ with the stated homogeneity conditions.

Next, we derive the cost functions corresponding to the micro-forms for the price independent case. We know that equation (11) corresponds to the cost function $m_h(u_h, p) = G_h(u_h, H(p))B(p) + g_h(p)$. If $C_{ih} = 0$ and $g_h(p) = 0$, then we know that $v_h = (y_h/k_h)^{-\alpha}$ implies the cost function $m_h(u_h, p) = k_h(a^\alpha(p) + u_h b^\alpha(p))^{1/\alpha}$.

The most general form adds $g_h(p)$ to this but we shall show that price independence implies $g_h = 0$. Including g_h, the demand equations are

$$p_i q_{ih} = k_h\left(\frac{y_h - g_h}{k_h}\right)^{1-\alpha}\left[a^\alpha a_i + \left(\left(\frac{y_h - g_h}{k_h}\right)^\alpha - a^\alpha\right)b_i\right] + \frac{\partial g_h}{\partial p_i}$$

where $a_i = \partial \log a/\partial \log p_i$, $b_i = \partial \log b/\partial \log p_i$. Hence,

$$w_{ih} = \frac{k_h}{y_h}\left(\frac{y_h - g_h}{k_h}\right)^{1-\alpha} a^\alpha(a_i - b_i) + b_i + \frac{1}{y_h}\left(\frac{\partial g_h}{\partial p_i} - \frac{g_h b_i}{k_h}\right). \tag{60}$$

Thus,

$$y_0^{-\alpha} = \sum k_h\left(\frac{y_h - g_h}{k_h}\right)^{1-\alpha} \Big/ \sum y_h.$$

But with g_h a function of p, y_0 cannot be price independent. For $g_h \neq 0$, g_h cannot be a constant parameter since that would violate the homogeneity of the cost function. Thus $g_h = 0$, all h, for y_0 to be price independent.

Finally, that the price independence of y_0 implies the linear homogeneity in y of y_0 follows immediately from equation (26) corresponding to (15) and equation (29) corresponding to (16).

5. AN APPLICATION TO OPTIMAL COMMODITY TAX THEORY[13]

Suppose the money income distribution is fixed and the government wants to tax goods so as to maximize a Bergson-Samuelson welfare function $W = W(u_1, \ldots, u_N)$ subject to the revenue constraint $\Sigma_h \Sigma_i (p_i - r_i) q_{ih} = T$. Here T is the tax revenue objective $(< \Sigma y_h)$ and r_i is the fixed producer price of good i so that $p_i - r_i = t_i$ is the tax rate.

We know, using Roy's Lemma, that

$$\text{(61)} \qquad \frac{\partial W}{\partial p_j} = \sum_h \beta_h^w \frac{\partial y_h}{\partial u_h} \frac{\partial u_h}{\partial p_j} = \sum_h - \beta_h^w q_{jh}$$

where $\beta_h^w = \partial W/\partial y_h$, which is the social shadow value of h's income. Hence, the marginal conditions of the standard Lagrangian problem are

$$\text{(62)} \qquad -\frac{\partial W}{\partial p_j} = \sum_h \beta_h^w q_{jh} = \psi\left(\sum_i t_i \frac{\partial Q_i}{\partial p_j} + Q_j\right)$$

for $j = 1, \ldots, n$ where ψ is the Lagrangian multiplier and where $Q_i = \Sigma_h q_{ih}$. This is easy to interpret. Think of a small (unit) change in p_j. The left-hand side gives the "equity part" of the condition. The effects of the price change depend on how much of the good each consumer buys but each consumer is weighted by the social shadow value of his income which introduces the equity properties of $W(\ )$.[14] The right-hand side $= \psi(\Sigma_i t_i\, \partial Q_i/\partial p_j + Q_j)$ is the "revenue efficiency" part of the rule. The Q_j gains in revenue are partly offset by the substitution terms $t_i\, \partial Q_i/\partial p_j$.

Suppose all consumers have GL cost functions of the form $m_h(u_h, p) = G_h(u_h, H(p))B(p)$. Then the demand functions are

$$\text{((20))} \qquad q_{jh} = \frac{\partial G_h}{\partial H} \frac{\partial H}{\partial p_j} B + G_h \frac{\partial B}{\partial p_j}.$$

Hence,

$$\sum_h \beta_h^w q_{jh} = \left(\sum_h \beta_h^w \frac{\partial G_h}{\partial H}\right) \frac{\partial H}{\partial p_j} B + \left(\sum_h \beta_h^w G_h\right) \frac{\partial B}{\partial p_j}.$$

[13] In writing this section, I benefited from seeing a draft paper by Angus Deaton [**4**].

[14] See Diamond and Mirrlees [7, pp. 265–268]. Contrast the one-consumer case where

$$\lambda Q_j = \psi\left(\sum t_i\, \partial Q_i/\partial p_j + Q_j\right)$$

(see [7, p. 262]). This gives $(\Sigma\, t_i\, \partial Q_i/\partial p_j)/Q_j =$ constant or $(\Sigma\, t_i\, \partial Q_i/\partial p_j|_u)/Q_j =$ constant as two familiar ways of writing the marginal conditions.

Hence,

$$
(63)\quad \begin{aligned} p_j \sum_h \beta_h^w q_{jh} &= \left(\sum \beta_h^w G_h B\right)\left[\left(\sum \beta_h^w \frac{\partial G_h}{\partial H}\Big/\sum \beta_h^w G_h\right)\frac{\partial H}{\partial \log p_j} + \frac{\partial \log B}{\partial \log p_j}\right] \\ &= \left(\sum \beta_h^w y_h\right)\left[\frac{\partial \log G_w(u_0^w, H)}{\partial H}\,\frac{\partial H}{\partial p_j} + \frac{\partial \log B}{\partial \log p_j}\right] \end{aligned}
$$

where

$$
\frac{\partial \log G_w(u_0^w, H)}{\partial H} = \left(\sum \beta_h^w \frac{\partial G_h}{\partial H}\right)\Big/\sum \beta_h^w G_h
$$

and u_0^w is the "socially weighted representative level of utility". Recall from (22) that the market weighted representative utility level u_0 is defined by

$$
\frac{\partial \log G(u_0, H)}{\partial H} = \left(\sum \frac{\partial G_h(u_h, H)}{\partial H}\right)\Big/\sum G_h(u_h, H).
$$

If $W(\ )$ embodies egalitarian values, β_h^w will be higher for "low" u_h and lower for "high" u_h. Then $u_0^w < u_0$ with the difference greater, the more egalitarian is $W(\ )$. Since the term in square brackets in (63) is the budget share at u_0^w, we have

$$
\sum \beta_h^w q_{jh} = \left(\frac{\sum_h \beta_h^w y_h}{p_j}\right) w_j(u_0^w, p) = \left(\frac{\sum_h \beta_h^w y_h}{y_0^w}\right) q_j(y_0^w, p)
$$

where $q_j(y_0^w, p)$ is the amount purchased by an individual with the cost function $M_w(u, p) = G_w(u, H(p))B(p)$ at an income y_0^w or a utility level u_0^w. Thus the rule becomes

$$
(64)\quad \frac{\left(\sum_i t_i\, \partial Q_i/\partial p_j + Q_j\right)}{q_j(y_0^w, p)} = \text{constant}.
$$

Since $Q_i = (w_i(y_0, p)Y)/p_i$, it is clear that the numerator (the "revenue-efficiency term") is evaluated at y_0 which is the *market* weighted representative income level. The denominator (the "equity term"), on the other hand, is evaluated at y_0^w which is the *socially* weighted representative income level.

As is easy to see, the rule implies that other things (substitution effects) being similar, t_j ought to be low if $q_j(y_0^w, p)/Q_j$ is high. But this will happen precisely when j is a good which has a big weight in the consumption patterns of the poor, i.e., is a "necessity". The more egalitarian is W, the bigger is the difference between y_0^w and y_0 and hence, in general, the more will the ratio $q_j(y_0^w, p)/Q_j$ differ from unity. Thus the operational significance of the degree of egalitarianism in the $W(\ )$ function is made very clear.

This example well illustrates the striking informational economy for welfare judgements which is implied by the assumption that individual preferences are such that community preferences exist.

Contrast this case with the one-consumer case where the rule is

$$\frac{\sum_i t_i\,\partial Q_i/\partial p_j + Q_j}{Q_j} = \text{constant}. \tag{65}$$

There the implication is that goods with low substitution possibilities tend to be taxed heavily, which may well place a heavy burden on "necessities".

Finally, an observation which brings to full circle the analysis begun in my paper on the representativeness of official price indices, Muellbauer [**18**]. Suppose that, instead of being so sophisticated as to have a Bergson–Samuelson welfare function, the government just wants to minimize the inflationary impact as measured in the *official* consumer price index of its indirect taxes. If the index uses fixed quantity weights, the problem is

$$\min P = \frac{\sum q_i^0 p_i}{\sum q_i^0 p_i^0} \quad \text{subject to} \quad \sum_i (p_i - r_i)Q_i = T.$$

The optimal rule is given by

$$\frac{\sum_i t_i\,\partial Q_i/\partial p_j + Q_j}{q_j^0} = \text{constant}. \tag{66}$$

The formal similarity between (66) and (64) brings out most strikingly the point made in my earlier paper, that the choice of weights in the official price index is an important political matter and, indeed, implies value judgements just as does the choice of welfare function.

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APPENDIX

Here we analyze the implications for preferences of the form $v(y) = y^{\varepsilon(p)}$ which arises in Theorem 4. Maximizing behavior implies $y = m(u, p)$. Hence, $p_i q_i = (m(u, p))^{\varepsilon(p)+1} A_i(p) + m(u, p)D_i(p)$. By the symmetry of compensated cross-price effects, we know that $\partial q_i/\partial p_j|_u \equiv \partial q_j/\partial p_i|_u$ under maximizing behavior. Hence,

$$\begin{aligned}\text{(A1)}\qquad &\frac{1}{p_i}\left[(\varepsilon+1)m^{\varepsilon}\frac{\partial m}{\partial p_j}A_i + m^{\varepsilon+1}\frac{\partial A_i}{\partial p_j} + m^{\varepsilon}\log m\frac{\partial \varepsilon}{\partial p_j}A_i + m\frac{\partial D_i}{\partial p_j} + D_i\frac{\partial m}{\partial p_j}\right]\\ &\equiv \frac{1}{p_j}\left[(\varepsilon+1)m^{\varepsilon}\frac{\partial m}{\partial p_i}A_j + m^{\varepsilon+1}\frac{\partial A_j}{\partial p_i} + m^{\varepsilon}\log m\frac{\partial \varepsilon}{\partial p_i}A_j + m\frac{\partial D_j}{\partial p_i} + D_j\frac{\partial m}{\partial p_i}\right].\end{aligned}$$

We know that $\partial m/\partial p_j = q_j$ and $\partial m/\partial p_i = q_i$ and these do not involve any terms in $\log m$. Therefore, equating terms in $\log m$ in (A1), we find that

$$\frac{\partial \varepsilon}{\partial p_j}\frac{A_i}{p_i} \equiv \frac{\partial \varepsilon}{\partial p_i}\frac{A_j}{p_j}.$$

This is the only interesting restriction obtained by equating different terms in (A1).

Thus,

$$\text{(A2)} \qquad A_i/A_j = \frac{\partial \varepsilon}{\partial \log p_i} \bigg/ \frac{\partial \varepsilon}{\partial \log p_j}.$$

The solution to (A2) must take the form $A_i = F(B, \varepsilon)(\partial\varepsilon/\partial \log p_i)$ where $B = B(p)$. But since

$$\text{(A3)} \qquad w_i = m^{\varepsilon(p)}(u, p)A_i(p) + D_i(p),$$

$$A_i/A_j = \frac{\partial w_i}{\partial u}\bigg/\frac{\partial w_j}{\partial u} = \frac{\partial}{\partial u}\left(\frac{\partial \log m}{\partial \log p_i}\right)\bigg/\frac{\partial}{\partial u}\left(\frac{\partial \log m}{\partial \log p_j}\right).$$

Thus

$$\text{(A4)} \qquad \frac{\partial \varepsilon}{\partial \log p_i}\bigg/\frac{\partial \varepsilon}{\partial \log p_j} = \frac{\partial}{\partial \log p_i}\left(\frac{\partial \log m}{\partial u}\right)\bigg/\frac{\partial}{\partial \log p_j}\left(\frac{\partial \log m}{\partial u}\right).$$

The most general solution to (A4), can be written in the form $\partial \log m/\partial u = \hat{Z}(u, \varepsilon(p))$. Hence,

$$\text{(A5)} \qquad \log m = \log Z(u, \varepsilon(p)) \log B(p).$$

The budget share equations corresponding to (A5) are

$$\text{(A6)} \qquad \frac{\partial \log m}{\partial \log p_i} = w_i = (\partial \log Z/\partial\varepsilon)\frac{\partial \varepsilon}{\partial \log p_i} + \frac{\partial \log B}{\partial \log p_i} \equiv m^{\varepsilon(p)}(u, p)A_i(p) + D_i(p) \quad \text{from (A3).}$$

Since by (A5) $m = ZB$, $(ZB)^{\varepsilon}F(B, \varepsilon) = \partial \log Z/\partial\varepsilon$, therefore

$$\text{(A7)} \qquad (B^{\varepsilon})F(B, \varepsilon) = \frac{1}{Z^{\varepsilon}}\,\frac{\partial \log Z}{\partial \varepsilon}.$$

The left-hand side is a function of ε, B only; the right-hand side is a function of u, ε only. Hence, the left-hand side equals the right-hand side which equals $K(\varepsilon)$. Hence we need to solve

$$K(\varepsilon) = \frac{1}{Z^{\varepsilon}}\,\frac{\partial \log Z}{\partial \varepsilon} = \frac{1}{Z^{\varepsilon+1}}\,\frac{\partial Z}{\partial \varepsilon}.$$

Hence,

$$\text{(A8)} \qquad \frac{\partial Z}{\partial \varepsilon} = Z^{\varepsilon+1}K(\varepsilon).$$

Let $Z = G^{-1/\varepsilon}$. Then $Z^{\varepsilon} = 1/G$,

$$\frac{\partial \log Z}{\partial \varepsilon} = \frac{1}{\varepsilon^2}\left(\log G - \frac{\varepsilon}{G}\,\frac{\partial G}{\partial \varepsilon}\right) = \frac{K(\varepsilon)}{G}.$$

Therefore,

$$\text{(A9)} \qquad K(\varepsilon) = \frac{1}{\varepsilon^2}\left[G \log G - \frac{\partial G}{\partial \log \varepsilon}\right].$$

A substantial step towards a closed form solution to (A9) can be taken in the special case $K(\varepsilon) = \gamma/\varepsilon^2$ where γ is a constant. Then

$$\text{(A10)} \qquad G \log G - \gamma = \frac{\partial G}{\partial \log \varepsilon}.$$

Let $W(G) = \int (1/(G \log G - \gamma))\, dG = \log \varepsilon + \log L(u)$. Hence,

$$\text{(A11)} \qquad G = W^{-1}(\log \varepsilon u)$$

since $L(u)$ can be replaced by u without loss of generality.

However, I have not been able to solve the integral which defines W. Consideration of a Lipschitz condition makes it clear that a global solution to the differential equation $dW/dG = 1/(G \log G - \gamma)$ does not exist. Since $G = Z^{-\varepsilon} = (m/B)^{-\varepsilon}$, the implied form of the cost function is

$$m(u, p) = W^{-1}(\log \varepsilon(p)u)B(p). \tag{A12}$$

Because W may not exist for all u and because of possible concavity problems for some u, there are likely to be income ranges for which (A12) is not defined. Finally, I have not been able to push further the analysis of (A9) when $K(\varepsilon)$ does not have the special form γ/ε^2.

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