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COMMUTANTS AND DERIVATION RANGES

YANG HO

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Introduction. The inner derivation δ_A induced by an element A of the algebra $\mathscr{B}(\mathscr{H})$ of bounded linear transformations on a separable Hilbert space $\mathscr{H}$ is the map $X \rightarrow AX - XA$ for $X \in \mathscr{B}(\mathscr{H})$. Kleinecke [8] and Shirokov [10] showed that if T belongs to the intersection of the range $\mathscr{R}(\delta_A)$ of δ_A and the kernel $\{A\}'$ of δ_A then T is quasinilpotent. The same is true of each compact operator T in the intersection of $\{A\}'$ and the norm closure of $\mathscr{R}(\delta_A)$ [7, 5]. However, Anderson [2] shows that there are operators A for which the algebra $\mathscr{R}(\delta_A)^- \cap \{A\}'$ contains the identity operator.

In this paper we obtain some sufficient conditions for $I \notin \mathscr{R}(\delta_A)^-$ and show that the set of such operators is norm dense in $\mathscr{B}(\mathscr{H})$.

When H is finite dimensional one has $\mathscr{R}(\delta_A) \cap \{A^*\}' = \{0\}$. We show here that this also holds for certain classes of operators when $\mathscr{H}$ is infinite dimensional.

In the finite case $\mathscr{R}(\delta_B) \subset \mathscr{R}(\delta_A)$ is equivalent to the commutativity condition $B \in \{A\}''$, but this condition is not sufficient in the infinite case [12]. It is necessary if A is normal [6] or isometric [13] but in Section 3 we prove that it is not necessary in general.

1. Derivation ranges and the identity operator. The following lemma is a consequence of Cauchy's theorem and the functional calculus.

LEMMA 1. *Let A be an element of $\mathscr{B}(\mathscr{H})$ and f be an analytic function on an open set containing $\sigma(A)$. Then*

$$f'(A) = \frac{1}{2\pi i}\int_\Gamma (\lambda I - A)^{-2} f(\lambda) d\lambda$$

where Γ is any Jordan system that lies entirely in the domain of regularity of f and encloses $\sigma(A)$.

THEOREM 1. *Let $A \in \mathscr{B}(\mathscr{H})$ and suppose that there exists an analytic function f on an open set containing $\sigma(A)$ such that*

(1) $f' \not\equiv 0$

(2) $\mathscr{R}(\delta_{f(A)})^- \cap \{f(A)\}' = \{0\}$.

Then $I \notin \mathscr{R}(\delta_A)^-$.

PROOF. Suppose $AX_n - X_nA \to I$ for some sequence of operators $\{X_n\}$. If Γ is a Jordan system that lies entirely in the domain of regularity of f and encloses $\sigma(A)$, then $\|(\lambda - A)^{-1}X_n - X_n(\lambda - A)^{-1} - (\lambda - A)^{-2}\| = \|(\lambda - A)^{-1}[(A - \lambda)X_n - X_n(A - \lambda) - I](\lambda - A)^{-1}\| \leqq \|(\lambda - A)^{-1}\|^2\|AX_n - X_nA - I\| \to 0$ uniformly for $\lambda \in \Gamma$ as $n \to \infty$. Hence by Lemma 1

$$f(A)X_n - X_nf(A) - f'(A)$$
$$= \frac{1}{2\pi i}\int_\Gamma f(\lambda)[(\lambda - A)^{-1}X_n - X_n(\lambda - A)^{-1} - (\lambda - A)^{-2}]d\lambda \to 0$$

as $n \to \infty$. Therefore $f'(A) \in \mathscr{R}(\delta_{f(A)})^-$ and so (2) implies $f'(A) = 0$. Condition (1) and the spectral mapping theorem guarantee that $\sigma(A)$ is finite. Hence A is similar to an operator of the form $\sum_{i=1}^n \oplus A_i$ with $\sigma(A_i) = \{\lambda_i\}$ and $A_i - \lambda_i$ is nilpotent for $1 \leqq i \leqq n$.

To complete the proof we may therefore assume that A is nilpotent of index k. Then with $f(z) = z^k$ the above argument gives $0 = A^kX_n - X_nA^k \to kA^{k-1} \neq 0$.

COROLLARY (Stampfli [11]). *Let* A *and* f *be as in the theorem and* $f(A) = N$ *where* N *is a normal operator. Then* $1 \notin \mathscr{R}(\delta_A)^-$.

PROOF. In [1] Anderson shows that $\mathscr{R}(\delta_N)^- \cap \{N\}' = \{0\}$.

LEMMA 2. *Let* $A \in \mathscr{B}(\mathscr{H})$. *If* $\sigma(A)$ *has an isolated point* λ *for which* $A - \lambda$ *is Fredholm, then* $I \notin \mathscr{R}(\delta_A)^-$.

PROOF. Let Γ be the boundary of a disk with center at λ that contains no other points of $\sigma(A)$. If $E = (1/2\pi i)\int_\Gamma (z - A)^{-1}dz$ is the corresponding Riesz projection then $E^2 = E \neq 0$ and $EA = AE$. Suppose $I \in \mathscr{R}(\delta_A)^-$. Then $E \in \mathscr{R}(\delta_A)^- \cap \{A\}'$ and E has finite rank since $A - \lambda$ is Fredholm. But then $\sigma(E) = \{0\}$ by [7] and this is a contradiction.

THEOREM 2. *Let* $A \in \mathscr{B}(\mathscr{H})$. *For each* $\varepsilon > 0$ *there exists an operator* B *such that*

(1) $\operatorname{rank}(B) = 1$

(2) $\|B\| < \varepsilon$

(3) $I \notin \mathscr{R}(\delta_{A+B})^-$.

PROOF. A slight modification of the argument in [4] shows that there exist an operator B having the properties (1) and (2) and a complex number λ which is an isolated point of $\sigma(A + B)$ such that $A + B - \lambda$ is Fredholm. Therefore $I \notin \mathscr{R}(\delta_{A+B})^-$ by Lemma 2.

COROLLARY. *The set* $\{A \in \mathscr{B}(\mathscr{H}): I \notin \mathscr{R}(\delta_A)^-\}$ *is dense in* $\mathscr{B}(\mathscr{H})$ *in* *the norm topology.*

REMARK. (1) Let $\mathscr{K}$ be the ideal of compact operators on $\mathscr{H}$ and let $T \to \hat{T}$ be the natural homomorphism from $\mathscr{B}(\mathscr{H})$ onto the Calkin algebra $\mathscr{B}(\mathscr{H})/\mathscr{K}$. The above theorem assures the existence of an operator $C \in \mathscr{B}(\mathscr{H})$ such that $\hat{I} \in \mathscr{R}(\delta_{\hat{c}})^-$ but $I \notin \mathscr{R}(\delta_c)^-$.

(2) Each compact operator in the algebra $\mathscr{R}(\delta_A)^- \cap \{A\}'$ must be quasinilpotent [7]. For more information about this algebra see [5].

2. The set $\mathscr{R}(\delta_A) \cap \{A^*\}'$. If $\mathscr{H}$ is a finite dimensional Hilbert space $\langle X, Y \rangle = \text{trace } (XY^*)$ is an inner product on $\mathscr{B}(\mathscr{H})$ and we have the orthogonal direct sum decomposition $\mathscr{B}(\mathscr{H}) = \mathscr{R}(\delta_A) \oplus \{A^*\}'$. However when $\mathscr{H}$ is infinite dimensional we do not know whether $\mathscr{R}(\delta_A) \cap \{A^*\}' = \{0\}$ in general. In this section we obtain some sufficient conditions for this intersection to be trivial.

LEMMA 3. *Let $A \in \mathscr{B}(\mathscr{H})$. If $p(A)$ is normal for some polynomial $p(z)$ then $\mathscr{R}(\delta_A)^- \cap \{A\}'$ contains no nonzero normal operator.*

PROOF. Suppose $AX_n - X_nA \to C$ and that C is a normal operator in $\{A\}'$. If $p^{(k)}(z)$ denotes the k-th derivative of $p(z)$ then

$$p^{(k)}(A)X_n - X_n p^{(k)}(A) \to p^{(k+1)}(A)C \quad \text{as} \quad n \to \infty .$$

In particular, $p'(A)C \in \mathscr{R}(\delta_{p(A)})^- \cap \{p(A)\}'$ so that $p'(A)C = 0$ since $p(A)$ is normal [1]. Also $Cp'(A)X_n - CX_np'(A) \to p''(A)C^2$ and $p'(A)X_nC - X_np'(A)C \to p''(A)C^2$ so that $(p'(A)X_n - X_np'(A))C + \delta_c(X_np'(A)) \to 0$. Therefore $p''(A)C^2 \in \mathscr{R}(\delta_c)^- \cap \{C\}'$. Hence $p''(A)C^2 = 0$. By repeating the same argument it follows that $p^{(m)}(A)C^m = 0$ where m is the degree of $p(z)$. Thus $C^m = 0$ and so $C = 0$ since it is normal.

THEOREM 3. *If A satisfies one of the following conditions then $\mathscr{R}(\delta_A)^- \cap \{A^*\}' = 0$:*

1) *$p(A)$ is normal for some quadratic polynomial $p(z)$.*
2) *A is subnormal and has a cyclic vector.*

PROOF. (1) Suppose that $A^2 - 2\alpha A - \beta = N$ is a normal operator. Let $AX_n - X_nA \to B^* \in \mathscr{R}(\delta_A)^- \cap \{A^*\}'$. Then $(N + 2\alpha A)X_n - X_n(N + 2\alpha A) = A^2X_n - X_nA^2 \to AB^* + B^*A$. This implies that $AB^* + B^*A - 2\alpha B^* \in \mathscr{R}(\delta_N) \cap \{N\}'$ so that $AB^* + B^*A - 2\alpha B^* = 0$ by [1]. Hence $(B + B^*)(A - \alpha) = (A - \alpha)(B - B^*)$ and $(A - \alpha)B^* = -B^*(A - \alpha)$. The Putnam-Fuglede theorem then gives $(B^* + B)(A - \alpha) = (A - \alpha) \times (B^* - B)$ and $(A - \alpha)B = -B(A - \alpha)$. Combining these equations we get $(A - \alpha)(B^* + B) = 0$ and $(B^* + B)(A - \alpha) = 0$. Hence $B^*A = AB^*$. Therefore $B^*B \in \mathscr{R}(\delta_A)^- \cap \{A\}'$ so that $B = 0$ by Lemma 3.

(2) $\hat{A}$ is a normal element of $\mathscr{B}(\mathscr{H})/\mathscr{K}$ by [3]. Hence if $B^* \in \mathscr{R}(\delta_A)^- \cap \{A^*\}'$ then B^* is compact by [1]. Since A has a cyclic vector B is also subnormal, and therefore normal. Then $B^* \in \mathscr{R}(\delta_A)^- \cap \{A\}'$ by the Fuglede theorem. This implies that B^* is quasinilpotent [7] and therefore $B = 0$.

Stampfli [11] has exhibited a unilateral weighted shift A for which $A^* \in \mathscr{R}(\delta_A)^-$. We will show however, that $\mathscr{R}(\delta_A) \cap \{A^*\}' = \{0\}$ for any weighted shift with nonzero weights. First we prove two lemmas.

LEMMA 4. *Let W be a unilateral weighted shift with nonzero weights $\{\alpha_n\}$. If $A \geqq 0$ and $A = WX - XW$ for some $X \in \mathscr{B}(\mathscr{H})$ then A is a trace class operator with trace* $(A) \leqq \underline{\lim}\, |\alpha_n|\, ||X||$.

PROOF. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis for which $We_n = \alpha_n e_{n+1} (n \geqq 0)$. Then $\sum_{k=0}^{n} (Ae_k, e_k) = \sum_{k=0}^{n} ((WX - XW)e_k, e_k) = -\alpha_n(Xe_{n+1}, e_n)$. Hence $\sum_{k=0}^{\infty} ||A^{1/2}e_k||^2 < \infty$ so that $A^{1/2}$ is a Hilbert-Schmidt operator and $A = A^{1/2}A^{1/2}$ is a trace class operator.

LEMMA 5. *Let W be a unilateral shift as above. If $\underline{\lim}\, |\alpha_n| \neq 0$ then there is no nonzero Hilbert-Schmidt operator that commutes with W.*

PROOF. Assume B commutes with W and let $Be_j = \sum_{k=0}^{\infty} b_{k,j}e_k$ for $j \geqq 0$. If $B \neq 0$ then $b_{k,0} \neq 0$ for some k since e_0 is cyclic for W. Therefore there exists a smallest positive integer m for which $b_{m,0} \neq 0$. Assume $b_{m,0} = 1$. Then

$$b_{m+j,j+1} = \frac{\alpha_m \alpha_{m+1} \cdots \alpha_{m+j-1}}{\alpha_0 \alpha_1 \cdots \alpha_j} = \frac{\alpha_{j+1} \cdots \alpha_{m+j-1}}{\alpha_0 \alpha_1 \cdots \alpha_{m-1}}$$

for j large enough. Hence

$$\sum_{j=0}^{\infty} ||Be_j||^2 \geqq \sum_{j=m}^{\infty} |b_{m+j,j+1}|^2 = |\alpha_0 \alpha_1 \cdots \alpha_{m-1}|^{-2} \sum_{j=m}^{\infty} |\alpha_{j+1} \cdots \alpha_{m+j-1}|^2 .$$

Now $\underline{\lim}_j\, |\alpha_{j+1} \cdots \alpha_{m+j-1}|^2 \geqq \underline{\lim}\, |\alpha_n|^{2m} \neq 0$ so that B is not a Hilbert-Schmidt operator.

THEOREM 4. *Let W be a unilateral shift with nonzero weights $\{\alpha_n\}$. Then $\mathscr{R}(\delta_w) \cap \{W^*\}' = \{0\}$.*

PROOF. If $B^* \in \mathscr{R}(\delta_w) \cap \{W^*\}'$ then $B^*B = WX - XW$ for some $X \in \mathscr{B}(\mathscr{H})$. Lemma 4 shows that B^*B is a trace class operator with trace $(B^*B) \leqq \underline{\lim}\, |\alpha_n|\, ||X||$. This inequality and Lemma 5 imply that $B = 0$.

3. Double commutant and derivation range inclusion. When $\mathscr{H}$ is finite dimensional we have $\mathscr{B}(\mathscr{H}) = \mathscr{R}(\delta_A) \oplus \{A^*\}'$. This decomposition shows that $\mathscr{R}(\delta_B) \subset \mathscr{R}(\delta_A)$ if and only if $B \in \{A\}''$. The condition $B \in \{A\}''$

is not sufficient for $\mathscr{R}(\delta_B) \subset \mathscr{R}(\delta_A)$ when $\mathscr{H}$ is infinite dimensional [12]. It is necessary if A is a normal operator [6] or if A is an isometry [13]. The main result of this section is that it is not necessary in general, however.

THEOREM 5. *Let U be a nonunitary isometry, let $P = I - UU^*$, and let $A = \begin{pmatrix} U & 0 \\ 0 & U^* + 3 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ P & 0 \end{pmatrix}$ acting in the usual fashion on $\mathscr{H} \oplus \mathscr{H}$. Then $\mathscr{R}(\delta_B) \subset \mathscr{R}(\delta_A)$ and $BA \neq AB$.*

PROOF. Let $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$ be an element of $\mathscr{B}(\mathscr{H} \oplus \mathscr{H})$. Then $BX - XB = \begin{pmatrix} -X_2P & 0 \\ PX_1 - X_4P & PX_2 \end{pmatrix}$. Since $\sigma(U) \cap \sigma(U^* + 3) = \varnothing$, therefore there exists $Z \in \mathscr{B}(\mathscr{H})$ such that $(U^* + 3)Z - ZU = PX_1 - X_4P$ [9]. Because $P\mathscr{B}(\mathscr{H}) \subset \mathscr{R}(\delta_{U^*})$ and $\mathscr{B}(\mathscr{H})P \subset \mathscr{R}(\delta_U)$ [12], there exist Y and W such that $UY - YU = -X_2P$ and $U^*W - WU^* = PX_2$. Then

$$\begin{pmatrix} U & 0 \\ 0 & U^* + 3 \end{pmatrix}\begin{pmatrix} Y & 0 \\ Z & W \end{pmatrix} - \begin{pmatrix} Y & 0 \\ Z & W \end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & U^* + 3 \end{pmatrix} = BX - XB$$

which shows that $\mathscr{R}(\delta_B) \subset \mathscr{R}(\delta_A)$. If $BA = AB$ then $(U^* + 3)P = PU$. But since $(U^* + 3)$ and U have disjoint spectra, therefore $P = 0$ [9]. This contradicts the choice of U.

The operator A defined above has derivation range $\mathscr{R}(\delta_A)$ that contains a nonzero right ideal and a nonzero left ideal of $\mathscr{B}(\mathscr{H})$. The following result explains why this is the case.

THEOREM 6. *Let $A \in \mathscr{B}(\mathscr{H})$. The following conditions are equivalent:*

(1) *$\mathscr{R}(\delta_B) \subset \mathscr{R}(\delta_A)$ implies $B \in \{A\}''$.*

(2) *$\mathscr{R}(\delta_A)$ does not contain both a nonzero left ideal and a nonzero right ideal of $\mathscr{B}(\mathscr{H})$.*

PROOF. That (2) implies (1) can be found in [12]. Assume (1) holds and suppose f, g are unit vectors such that $(f \otimes f)\mathscr{B}(\mathscr{H}) \subset \mathscr{R}(\delta_A)$ and $\mathscr{B}(\mathscr{H})(g \otimes g) \subset \mathscr{R}(\delta_A)$. Then $\mathscr{R}(\delta_{f\otimes g}) \subset \mathscr{R}(\delta_A)$ so that $A(f \otimes g) = (f \otimes g)A$. Therefore, $(Ag, g)f = Af$. Moreover, if $f \otimes f = \delta_A(X)$ then

$$\delta_A(X)\mathscr{B}(\mathscr{H}) = (f \otimes f)\mathscr{B}(\mathscr{H}) \subset \mathscr{R}(\delta_A) .$$

An easy calculation shows that $X\mathscr{R}(\delta_A) \subset \mathscr{R}(\delta_A)$, hence $(Xf \otimes g)\mathscr{B}(\mathscr{H}) \subset \mathscr{R}(\delta_A)$ and $\mathscr{B}(\mathscr{H})(Xf \otimes g) \subset \mathscr{R}(\delta_A)$ so that $\mathscr{R}(\delta_{Xf\otimes g}) \subset \mathscr{R}(\delta_A)$. Therefore $A(Xf \otimes g)g = (Xf \otimes g)Ag$ so that $AXf = (Ag, g)Xf$. Therefore, $f = (f \otimes f)f = AXf - XAf = (Ag, g)Xf - (Ag, g)Xf = 0$.

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INDIANA UNIVERSITY
BLOOMINGTON, INDIANA
AND
TAIWAN NATIONAL NORMAL UNIVERSITY
TAIPEI, TAIWAN, REPUBLIC OF CHINA. (current address)