

Pacific Journal of Mathematics

COMMUTANTS AND THE OPERATOR EQUATIONS $AX = \lambda XA$

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Suppose A is a bounded operator on the Banach space \mathcal{B} such that A or A^* is one-to-one. In this note, we point out a relation between the commutant of A , the commutants of its powers, and operators which intertwine A and λA , where λ is a root of unity. A consequence of this relation is that the commutants of A and A^n are different if and only if there is an operator Y , not zero, that satisfies $AY = \lambda YA$, where $\lambda^n = 1$, $\lambda \neq 1$. Combining this with Rosenblum's theorem, we see that if the spectra of A and XA are disjoint, the commutant of A is the same as that of A^2 . We also use the theorem to give a counterexample to a conjecture of Deddens concerning intertwining analytic Toeplitz operators.

If A, B , and X are bounded operators on \mathcal{B} , we say X commutes with A if $XA = AX$, and we say X intertwines A and B if $XA = BX$. The set of operators that commute with A , the commutant of A , will be denoted $\{A\}'$.

LEMMA. Suppose A is an operator such that A or A^* is one-to-one, and λ is a primitive n th root of 1. If X commutes with A^n , the operators $Y_i = \sum_{j=0}^{n-1} \lambda^{ij} A^{n-j-1} XA^j$, for $i = 0, 1, \dots, n-1$, are the unique operators such that $AY_i = Y_i(\lambda^i A)$ and $nA^{n-1}X = \sum_{i=0}^{n-1} Y_i$.

Proof. Let $Y_i = \sum_{j=0}^{n-1} \lambda^{ij} A^{n-j-1} XA^j$.

Then

$$\begin{aligned} AY_i &= \sum_{j=0}^{n-1} \lambda^{ij} A^{n-j} XA^j = A^n X + \sum_{j=1}^{n-1} \lambda^{ij} A^{n-j} XA^j \\ &= XA^n + \sum_{j=1}^{n-1} \lambda^{ij} A^{n-j} XA^j \\ &= \sum_{k=0}^{n-1} \lambda^{i(k+1)} A^{n-k-1} XA^{k+1} \\ &= \left(\sum_{k=0}^{n-1} \lambda^{ik} A^{n-k-1} XA^k \right) (\lambda^i A) = Y_i(\lambda^i A). \end{aligned}$$

Since $\sum_{i=0}^{n-1} \lambda^{ij} = 0$ when $j \neq 0$, and the sum is n when $j = 0$,

$$\begin{aligned} \sum_{i=0}^{n-1} Y_i &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \lambda^{ij} A^{n-j-1} XA^j \\ &= \sum_{j=0}^{n-1} A^{n-j-1} XA^j \sum_{i=0}^{n-1} \lambda^{ij} = nA^{n-1}X. \end{aligned}$$

Now suppose Z_0, Z_1, \dots, Z_{n-1} are operators such that $nA^{n-1}X = \sum_{i=0}^{n-1} Z_i$ and $AZ_i = Z_i(\lambda^i A)$ for each i . We have

$$\begin{aligned} nA^{n-1}Y_i &= \sum_{j=0}^{n-1} \lambda^{ij} A^{n-j-1} (nA^{n-1}X) A^j \\ &= \sum_{j=0}^{n-1} \lambda^{ij} A^{n-j-1} \left(\sum_{k=0}^{n-1} Z_k \right) A^j = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \lambda^{ij} A^{n-j-1} A^j \lambda^{-kj} Z_k \\ &= \sum_{k=0}^{n-1} A^{n-1} Z_k \sum_{j=0}^{n-1} \lambda^{(i-k)j} = nA^{n-1}Z_i. \end{aligned}$$

If A is one-to-one, then $A^{n-1}Y_i = A^{n-1}Z_i$ implies $Y_i = Z_i$. If A^* is one-to-one, then A^{n-1} has dense range and $Y_i A^{n-1} = \lambda^{-i(n-1)} A^{n-1} Y_i = \lambda^{-i(n-1)} A^{n-1} Z_i = Z_i A^{n-1}$, which implies $Y_i = Z_i$.

THEOREM. *Suppose A or A^* is one-to-one and n is a positive integer. Then $\{A\}' = \{A^n\}'$ if and only if $AY = Y(\lambda A)$ for $\lambda^n = 1$ implies $\lambda = 1$ or $Y = 0$.*

Proof. (\Rightarrow) Suppose $\{A\}' = \{A^n\}'$ and for some Y we have $AY = Y(\lambda A)$ where $\lambda^n = 1$. Then $A^n Y = Y(\lambda^n A^n) = YA^n$, so $Y \in \{A^n\}' = \{A\}'$ and $AY = YA$ as well. Thus $\lambda YA = AY = YA$ and $(1 - \lambda)YA = 0$. Since A or A^* is one-to-one, this means that $(1 - \lambda)Y = 0$, so $\lambda = 1$ or $Y = 0$.

(\Leftarrow) Suppose $AY = Y(\lambda A)$ for $\lambda^n = 1$ implies $Y = 0$ or $\lambda = 1$. Let X be in $\{A^n\}'$, and let λ be a primitive n th root of 1. For $i = 1, 2, \dots, n - 1$ let $Y_i = \sum_{j=0}^{n-1} \lambda^{ij} A^{n-j-1} X A^j$. By the lemma, $A Y_i = Y_i(\lambda^i A)$ so, since $\lambda^i \neq 1$, our hypothesis says $Y_i = 0$. Thus, we have the $n - 1$ equations $\sum_{j=0}^{n-1} \lambda^{ij} A^{n-j-1} X A^j = 0$, ($i = 1, 2, \dots, n - 1$).

Consider the equations $\sum_{j=0}^{n-1} \lambda^{ij} w_j = 0$, ($i = 1, 2, \dots, n - 1$), in the indeterminates $w_0, w_1, w_2, \dots, w_{n-1}$. We notice that $w_0 = w_1 = w_2 = \dots = w_{n-1}$ is a solution of these equations, and since the $(n - 1) \times n$ coefficient matrix $(\lambda^{ij})_{\substack{j=0 \\ i=1}}^{n-1}$ has rank $n - 1$, this is the only solution. In our case, we conclude $A^{n-1}X = A^{n-2}XA = \dots = XA^{n-1}$. If A is one-to-one, $A^{n-1}X = A^{n-2}XA$ implies $AX = XA$, whereas if A^* is one-to-one, $AXA^{n-2} = XA^{n-1}$ implies $AX = XA$.

We have shown that X is in $\{A\}'$ if it is in $\{A^n\}'$. Since the reverse inclusion is automatic, we have $\{A^n\}' = \{A\}'$.

As illustrations, we prove the following corollaries.

COROLLARY 1. *If the spectrum of A and the spectrum of $-A$ are disjoint, then $\{A\}' = \{A^2\}'$.*

Proof. Since the spectra of A and $-A$ are disjoint, Rosenblum's theorem, [3], implies that the only solution of $AX = X(-A)$ is $X = 0$. Zero is not in the spectrum of A , so A is one-to-one and we

apply the theorem to conclude $\{A\}' = \{A^2\}'$.

COROLLARY 2. *If the spectrum of A is contained in the quarter plane $\{z \mid \operatorname{Re}(z) > 0 \text{ and } \operatorname{Im}(z) > 0\}$, then $\{A\}' = \{A^4\}'$.*

Proof. The spectra of A, iA, i^2A , and i^3A are disjoint, so by Rosenblum's theorem, the only solution of $AX = X(i^kA)$, for $k = 1, 2$, or 3 , is $X = 0$. Zero is not in the spectrum of A , so A is one-to-one, and we apply the theorem to conclude that $\{A\}' = \{A^4\}'$.

The theorem may also be used to refute a conjecture of Deddens concerning intertwining analytic Toeplitz operators [2, page 244]. We recall that if ϕ is a bounded analytic function on the unit disk D , the analytic Toeplitz operator, T_ϕ , is the operator on the Hardy space H^2 of multiplication by ϕ . Deddens conjectured that when ϕ and ψ are bounded analytic functions on D and 0 is the only solution of $XT_\phi = T_\psi X$, then the complex conjugate of the range of ψ is not contained in the point spectrum of T_ϕ^* . To see that this is false, let f be a Riemann map of D onto the slit disk $D \setminus (-1, 0]$. Then the corollary of Theorem 5 of [1] implies that $\{(T_{f^2})^2\}' = \{T_f\}' = \{T_{f^2}\}'$. If f^2 and $-f^2$ are the ϕ and ψ of the conjecture, we note that $\operatorname{range} \psi = D \setminus \{0\} = \text{point spectrum } T_\phi^*$. But since $\{(T_{f^2})^2\}' = \{T_{f^2}\}'$ and T_{f^2} is one-to-one, the theorem implies that 0 is the only operator which intertwines T_{f^2} and $-T_{f^2} = T_{-f^2}$. The difficulty seems to be associated with the fact that the multiplicities of f^2 and $-f^2$ are different on the real axis.

The unfortunate presence of A^{n-1} in the formula $nA^{n-1}X = \sum_{i=0}^{n-1} Y_i$ of the lemma is essential when A is not invertible; it is easy to give examples of operators X in $\{T_z^2\}'$ so that $2T_z X = Y_0 + Y_1$, as above, but $Y_0 \neq T_z B$ for any bounded operator B . On the other hand, if A is invertible, we may solve the equation for X and obtain $X = \sum_{i=0}^{n-1} \hat{Y}_i$, where $\hat{Y}_i = (nA^{n-1})^{-1} Y_i$. These operators are the unique operators that satisfy $A\hat{Y}_i = \hat{Y}_i(\lambda^i A)$ and $X = \sum_{i=0}^{n-1} \hat{Y}_i$.

In the above, we have found a relation between the commutants of A and $p(A)$ for the polynomials $p(z) = z^n$. Of course, there is an analogous result for polynomials of the form $p(z) = (z - \alpha)^n + \beta$. It would be interesting (and apparently more difficult) to obtain information about the relation between $\{A\}'$ and $\{p(A)\}'$ for more complicated polynomials.

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Received June 12, 1978. Supported in part by National Science Foundation Grant MCS 77-03650.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Older back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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