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COMMUTANTS OF ANALYTIC TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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Abstract. In this note we show that if two Toeplitz operators on a Bergman space commute and the symbol of one of them is analytic and nonconstant, then the other one is also analytic.

Let Ω be a bounded open domain in the complex plane and let dA denote area measure on Ω. The Bergman space $L_a^2(\Omega)$ is the subspace of $L^2(\Omega, dA)$ consisting of the square-integrable functions that are analytic on Ω. For a bounded measurable function φ on Ω, the Toeplitz operator T_φ with symbol φ is the operator on $L_a^2(\Omega)$ defined by

$$T_\varphi(f) = P(\varphi f),$$

where P is the orthogonal projection of $L^2(\Omega, dA)$ onto $L_a^2(\Omega)$. A Toeplitz operator is called analytic if its symbol is an analytic function on Ω. Note that if φ is a bounded analytic function on Ω, then T_φ is simply the operator of multiplication by φ on $L_a^2(\Omega)$.

The general problem that we are interested in is the following: When two Toeplitz operators commute, what is the relationship between their symbols? If we were working on the Hardy space of the circle instead of the Bergman space, then the following result would answer this question:

- If Hardy space Toeplitz operators T_φ and T_ψ commute, then either both symbols are analytic or both symbols are conjugate analytic or $a\varphi + b\psi$ is constant for some constants a, b not both 0 (Brown and Halmos [4]).

More general results concerning which operators, not necessarily Toeplitz, commute with an analytic Hardy space Toeplitz operator are due to Thompson ([10] and [11]) and Cowen [6].

On the Bergman space, the situation is more complicated. The Brown-Halmos result mentioned above fails. For example, if Ω is the unit disk, then any two Toeplitz operators whose symbols are radial functions commute (proof: an easy calculation shows that every Toeplitz operator with radial symbol has a diagonal matrix with respect to the usual orthonormal basis; any two diagonal matrices commute).

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Despite the difficulty of the general problem, we are encouraged by the partial results known when Ω is the unit disk. If Ω is the unit disk and T_φ and T_ψ commute, then the following hold:

- If $\varphi = z^n$, then ψ is analytic (Čučković [7]).
- If φ and ψ are both harmonic, then either both symbols are analytic or both symbols are conjugate analytic or $a\varphi + b\psi$ is constant for some constants a, b not both 0 (Axler and Čučković [1]).
- If φ is a radial function, then ψ is radial (Čučković and Rao [8]).
- If $\varphi = z^m \overline{z}^n$, then $\psi(re^{i\theta}) = \sum_{j=-\infty}^{\infty} \psi_j(r) e^{ij\theta}$, where $\{\psi_j\}$ are the functions (depending upon m, n) described by Čučković and Rao [8].

In this note we extend Čučković's first result above by replacing the disk with an arbitrary bounded domain and (more importantly) by replacing z^n with an arbitrary bounded analytic function. Here is our result:

Theorem. *If φ is a nonconstant bounded analytic function on Ω and ψ is a bounded measurable function on Ω such that T_φ and T_ψ commute, then ψ is analytic.*

Our proof depends on the following approximation theorem:

- Let φ be a nonconstant bounded analytic function on Ω. Then the norm closed subalgebra of $L^\infty(\Omega, dA)$ generated by $\overline{\varphi}$ and the bounded analytic functions on Ω contains $C(\overline{\Omega})$ (Bishop [3]).

There is a large literature of related approximation theorems; see, for example, Čirka [5], Axler and Shields [2], Izzo [9].

Proof of the Theorem. Suppose φ is a nonconstant bounded analytic function on Ω and ψ is a bounded measurable function on Ω such that $T_\varphi T_\psi = T_\psi T_\varphi$.

Write $\psi = f + u$ with $f \in L^2_a(\Omega)$ and $u \in L^2(\Omega) \ominus L^2_a(\Omega)$. If n is a nonnegative integer, then

$$T_{\varphi^n} T_\psi(1) = \varphi^n P(f + u) = \varphi^n f$$

and

$$T_\psi T_{\varphi^n}(1) = P(f\varphi^n + u\varphi^n) = f\varphi^n + P(u\varphi^n).$$

Our hypothesis implies that $T_{\varphi^n} T_\psi = T_\psi T_{\varphi^n}$, and thus the equations above imply that $P(u\varphi^n) = 0$. Hence if $h \in L^2_a(\Omega)$ we have

$$0 = \langle h, u\varphi^n \rangle = \int_\Omega \overline{u} h \overline{\varphi^n} dA.$$

Because the equation above holds for every bounded analytic function h on Ω and every nonnegative integer n, Bishop's result quoted above implies that

$$\int_\Omega \overline{u} w \, dA = 0$$

for every $w \in C(\overline{\Omega})$. But $C(\overline{\Omega})$ is dense in $L^2(\Omega, dA)$, and so this implies that $u = 0$. Thus $\psi = f$ and hence ψ is analytic, completing the proof.

Open problems

- If an operator S in the algebra generated by the Toeplitz operators commutes with a nonconstant analytic Toeplitz operator, then is S itself Toeplitz and hence (by our result) analytic?
- Suppose Ω is the unit disk and φ is a bounded harmonic function on the disk that is neither analytic nor conjugate analytic. If ψ is a bounded measurable function on the disk such that T_φ and T_ψ commute, must ψ be of the form $a\varphi + b$ for some constants a, b? This question would have a negative answer if the disk were replaced by an annulus centered at the origin because $T_{\log|z|}$ commutes with every Toeplitz operator with radial symbol.
- What is the situation on Bergman spaces in higher dimensions?

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