

Commutativity of Projectors and Trace Characterization on von Neumann Algebras. I

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(Submitted by D. Kh. Mushtari)

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Received June 4, 2009

Abstract—We obtain the necessary and sufficient conditions for commutativity of projectors in terms of operator inequalities. We apply these conditions for the trace characterization on von Neumann algebras in the class of all positive normal functionals.

We also propose a trace characterization on von Neumann algebras in terms of the commutation of products of projectors under the weight sign.

DOI: 10.3103/S1066369X0912010X

Key words and phrases: *Hilbert space, von Neumann algebra, spectral theorem, weight, trace, normal functional, linear bounded operator, projector, operator inequality, commutativity of operators.*

INTRODUCTION

The study of characterization of traces in the class of normal weights or functionals on von Neumann algebras was commenced in the seventies of the XXth century. The recent achievements in the theory of singular traces on ideals of compact operators and important applications of this theory in the noncommutative geometry [1] gave rise to problems connected with characterization of traces in wider classes of weights on von Neumann algebras.

In this paper we continue the study described in [2–4] and use denotations and terminology proposed therein. In [2] we prove the following unimprovable (with respect to the number of multipliers) assertion: *If a von Neumann algebra \mathcal{M} has no direct Abelian addend (respectively, it is properly infinite), then each operator $x \in \mathcal{M}$ is representable as a finite sum $x = \sum x_k$, where each x_k is the product of no more than three (respectively, two) projectors from \mathcal{M} .* In [3] one proposes the second proof of this fact with the uniform estimation of the number of addends in such representations. The least upper bound (that equals three) is connected with the existence of a nontrivial finite trace on these algebras.

In [4] one establishes a new condition for the existence of a pair of projectors in terms of their upper (lower) bound in the lattice of all projectors of the algebra and proves that each skew-Hermitian element of the properly infinite von Neumann algebra \mathcal{M} is representable as a finite sum of commutators of projectors from \mathcal{M} . In the finite-dimensional case one describes the set of operators with the null canonical trace tr in terms of finite sums of commutators of projectors.

In this paper we establish new criteria for the commutativity of projectors in terms of operator inequalities. We apply these inequalities for characterizing the trace in the class of all positive normal functionals on a von Neumann algebra. We obtain a criterion for the pairwise orthogonality of the set of projectors in terms of one operator inequality. We characterize the trace on von Neumann algebras in terms of the commutation of products of projectors under the weight sign.

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