COMMUTATIVE BANACH ALGEBRAS WITH IDEMPOTENT MAXIMAL IDEALS

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Introduction

Let $\mathfrak{A}$ be a commutative Banach algebra over the complex field C, M an ideal of $\mathfrak{A}$. Denote by M^2 the set of all finite linear combinations of products of elements from M. M will be termed idempotent if $M^2 = M$. The purpose of this paper is to investigate the structure of commutative Banach algebras in which all maximal ideals are idempotent.

The idempotence property of ideals would appear to have first been systematically studied by Nakano [12] in the case of the ring of integers of an infinite algebraic number field. (For finite number fields a theorem of Krull (see [17] p. 216) shows that no proper ideal of the ring of integers can be idempotent.) In the case of commutative Banach algebras the idempotence of maximal ideals was used by Singer and Wermer [15] who showed that a commutative semisimple Banach algebra with identity admits a non-zero derivation into a commutative semisimple extension if and only if at least one of its maximal ideals is not idempotent. Some generalizations of this are given in § 3 of the present paper.

The actual question which instigated this work was whether or not the idempotence of all maximal ideals in a commutative Banach algebra with identity is sufficient to ensure semisimplicity. In § 2 we show the answer is in the negative by constructing a counterexample. With extra hypotheses positive results can be obtained, however those used prove to be too strong in that they ensure that the algebras concerned are in fact finite dimensional.

Finally, one should note the crucial role played by the idempotents in the algebras considered.

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LEMMA 1. *Let $\mathfrak{A}$ be a ring with identity e, in which every maximal left (or right) ideal is principal, being generated by an idempotent. Then the (Jacobson) radical of $\mathfrak{A}$ is zero.*[2]

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[2] In the commutative case a stronger result is true; see [13] Theorem 3.1.

PROOF. We prove the result for the case of left ideals. If $\{0\}$ is the only proper left ideal the result is clear. Otherwise, let $\{M_j : j \in J\}$ be the set of maximal left ideals of $\mathfrak{A}$, J an index set, with $\{e_j : j \in J\}$ the set of generating idempotents; so that $M_j = \mathfrak{A}e_j$, $j \in J$. The left ideal generated by the set $\{e-e_j : j \in J\}$ is not contained in any maximal left ideal, and hence is all of $\mathfrak{A}$. Thus there is a finite subset I of J, and elements $a_i \in \mathfrak{A}$, $i \in I$ such that

$$e = \sum_{i \in I} a_i(e-e_i).$$

But if x is in the radical of $\mathfrak{A}$ so is xa_i, and hence $xa_i \in M_i$ for each $i \in I$. Thus $xa_i(e-e_i) = 0$ and so

$$x = xe = \sum_{i \in I} xa_i(e-e_i) = 0.$$

REMARK. In the commutative case the condition that a principal ideal be generated by an idempotent is equivalent to its being idempotent. For if N is generated by an idempotent it is certainly idempotent since $\mathfrak{A}$ has an identity. Conversely, if $N = \mathfrak{A}x$, $N^2 = N$ means $\mathfrak{A}x = \mathfrak{A}x^2$ and so there is $a \in \mathfrak{A}$ with $x = ax^2$. But then ax is an idempotent and

$$\mathfrak{A}x = \mathfrak{A}xax \subseteqq \mathfrak{A}ax \subseteqq \mathfrak{A}x,$$

so that $N = \mathfrak{A}ax$.

COROLLARY. *Let $\mathfrak{A}$ be a commutative ring with identity having the minimum condition. Then $\mathfrak{A}$ is semisimple if and only if every maximal ideal is idempotent.*

PROOF. If every maximal ideal is idempotent it follows easily from [3] Theorem 2.6B that every maximal ideal has an idempotent generator. Thus $\mathfrak{A}$ is semisimple by Lemma 1.

The converse is clear from [3] Theorem 4.2A.

The author is indebted to the referee for the following theorem, which is a generalization of his original result.

THEOREM 1. *Let $\mathfrak{A}$ be a normed algebra with identity e. Then the following statements are equivalent.*

(1) *Every maximal left ideal of $\mathfrak{A}$ is generated by an idempotent.*

(2) *Every maximal right ideal of $\mathfrak{A}$ is generated by an idempotent.*

(3) *$\mathfrak{A}$ is semisimple and finite dimensional.*

PROOF. (1) implies (2) and (1) implies (3).

Assuming (1), Lemma 1 shows that $\mathfrak{A}$ is semisimple.

If $\{0\}$ is the only proper left ideal then for any non-zero $x \in \mathfrak{A}$ we have $\{yx : y \in \mathfrak{A}\} = \mathfrak{A}$. It follows that $\mathfrak{A}$ is a division algebra and hence is finite dimensional by the Gelfand-Mazur theorem. Otherwise, let $\{M_j\}$, $\{e_j\}$, J be as in Lemma 1. Then for $j \in J$ the right annihilator of M_j in $\mathfrak{A}$ is $(e-e_j)\,\mathfrak{A} \neq \{0\}$, so that, since $\mathfrak{A}$ is semisimple, $\mathfrak{A}$ is a right modular annihilator algebra in the sense of [16]. But then by [16] Theorem 3.4, $\mathfrak{A}$ is a left modular annihilator algebra and so, by [16] Lemma 3.3, (1) implies (2). Since $\mathfrak{A}$ is thus a modular annihilator algebra (3) follows from [4] Proposition 6.3.

Similarly (2) implies (1) and (2) implies (3).

It follows from [3] Theorem 4.2A that (3) implies each of (1) and (2).

COROLLARY. *Let $\mathfrak{A}$ be a commutative normed algebra with identity having minimum condition. Then $\mathfrak{A}$ is semisimple and finite dimensional if and only if every maximal ideal is idempotent.*

The proof of the next result follows the same lines as the author's original proof of the commutative case of Theorem 1. We will need a preliminary lemma.

LEMMA 2. *Let $\mathfrak{A}$ be a complex commutative normed algebra with identity e, $\overline{\mathfrak{A}}$ its completion. Let M, N be maximal ideals in $\mathfrak{A}$, $\overline{\mathfrak{A}}$ respectively. Then*

(i) $\overline{N \cap \mathfrak{A}} = N$.

(ii) $\bar{M}$ *is a maximal ideal in* $\overline{\mathfrak{A}}$.

PROOF. (i) Note firstly that $N \cap \mathfrak{A}$ is a maximal ideal in $\mathfrak{A}$, maximal since it has codimension one. Let $x \in N$. Then there is a sequence $\{x_n\} \subseteq \mathfrak{A}$ converging to x. But we have the unique decomposition $x_n = y_n + \lambda_n e$ where $y_n \in N \cap \mathfrak{A}$, and so

$$\inf_{z \in N} ||\lambda_n e - z|| = \inf_{z \in N} ||x_n - x + x - y_n - z|| \leqq ||x_n - x||.$$

On the other hand, if $z \in N$, $||e-z|| \geqq 1$ since otherwise z would have an inverse. Thus

$$\inf_{z \in N} ||\lambda_n e - z|| \geqq |\lambda_n|.$$

It follows that $\lambda_n \to 0$ and so $y_n \to x$.

(ii) $\bar{M}$ is clearly an ideal in $\overline{\mathfrak{A}}$. Let $x \in \overline{\mathfrak{A}}$ and let $\{x_n\} \subseteq \mathfrak{A}$ converge to x. Then we have the decomposition $x_n = y_n + \lambda_n e$ where $y_n \in M$. As before

$$||y_n - y_m + (\lambda_n - \lambda_m)e|| \geqq |\lambda_n - \lambda_m|$$

and so $\{\lambda_n\}$ converges, to λ say, and $\{y_n\}$ converges, to y say. It follows that $x = y + \lambda e$ where $y \in \bar{M}$. Thus $\bar{M}$ is maximal.

THEOREM 2. *Let $\mathfrak{A}$ be a complex commutative semisimple normed algebra in which every maximal regular ideal has an idempotent generator. Then $\mathfrak{A}$ is finite dimensional and so must have an identity.*

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PROOF. Let $\mathfrak{B}$ be the algebra obtained by adjoining an identity e to $\mathfrak{A}$ and extending the norm by $\|x+\lambda e\| = \|x\|+|\lambda|$ for $x \in \mathfrak{A}$. We note that $\mathfrak{B}$ is also semisimple.

Now if N is a maximal ideal in $\mathfrak{B}$, different from $\mathfrak{A}$, then $M = N \cap \mathfrak{A}$ is a maximal regular ideal in $\mathfrak{A}$, and so $M = \mathfrak{A}f$ for some idempotent $f \in \mathfrak{A}$. Furthermore, if $N' \neq \mathfrak{A}$ is some other maximal ideal in $\mathfrak{B}$ then $f \notin N'$, since otherwise $M \subsetneqq N' \cap \mathfrak{A}$ which is impossible since $M \neq N' \cap \mathfrak{A}$. Thus for each maximal ideal $N \neq \mathfrak{A}$ of $\mathfrak{B}$ there is an idempotent $f_N \in \mathfrak{A} \cap N$ such that $f_N \notin N'$ for any other maximal ideal $N' \neq \mathfrak{A}$ of $\mathfrak{B}$.

We denote the completion of $\mathfrak{A}$, $\mathfrak{B}$ by $\overline{\mathfrak{A}}$, $\overline{\mathfrak{B}}$ respectively; $\overline{\mathfrak{B}}$ is just $\overline{\mathfrak{A}}$ with identity adjoined. If $N \neq \overline{\mathfrak{A}}$ is a maximal ideal in $\overline{\mathfrak{B}}$ then $N \cap \mathfrak{B}$ is a maximal ideal in $\mathfrak{B}$, denote by e_N the idempotent in $N \cap \mathfrak{B}$ determined above; it follows from Lemma 2(i) that if $N' \neq \overline{\mathfrak{A}}$ is some other maximal ideal in $\overline{\mathfrak{B}}$ then $e_N \notin N' \cap \mathfrak{B}$.

Let $\mathfrak{M}$ be the maximal ideal space of $\overline{\mathfrak{B}}$, and for $x \in \overline{\mathfrak{B}}$ write $\hat{x}$ for its Gelfand transform. If $N \in \mathfrak{M}\diagdown\{\overline{\mathfrak{A}}\}$ then $\hat{e}_N(N') = 0$ if and only if $N' = N$ or $\overline{\mathfrak{A}}$. Since $\hat{e}_N$ takes the values 0 and 1 only, and is continuous on $\mathfrak{M}$, it follows that $\{N, \overline{\mathfrak{A}}\}$ is clopen in $\mathfrak{M}$. But then $\{N, \overline{\mathfrak{A}} : N \in \mathfrak{M}, N \neq \overline{\mathfrak{A}}\}$ is an open covering of $\mathfrak{M}$. $\mathfrak{M}$ being compact, there is a finite subcover and so $\mathfrak{M}$ is finite. Thus $\overline{\mathfrak{B}}$ has only finitely many maximal ideals and so by Lemma 2(ii) the same is true for the semisimple algebra $\mathfrak{B}$. But this means that $\mathfrak{B}$, and hence $\mathfrak{A}$, is finite dimensional. That $\mathfrak{A}$ has an identity follows from [3] Corollary 4.3B.

REMARK. In the case where $\mathfrak{A}$ is not semisimple, but is a Banach algebra, the argument of Theorem 2 shows that $\mathfrak{B}/\mathfrak{R}$ is finite dimensional, where $\mathfrak{R}$ is the radical of $\mathfrak{B}$. Since $\mathfrak{R}$ is also the radical of $\mathfrak{A}$ it follows that $\mathfrak{A}/\mathfrak{R}$ is also finite dimensional and so by [6] Theorem 1 it follows that the Wedderburn principal theorem is true for $\mathfrak{A}$.

The following result, in the same vein as Theorem 2, is easily proved using [2] Theorem 3.5.

THEOREM 3. *Let $\mathfrak{A}$ be a commutative Banach algebra which is regular in the sense of von Neumann, that is, if $x \in \mathfrak{A}$ there is $y \in \mathfrak{A}$ such that $xyx = x$. Then $\mathfrak{A}$ is semisimple and finite dimensional, and so must have an identity.*

REMARK. Finite dimensionality is actually true in the non-commutative case also; see [9].

In view of these results the question immediately arises whether either of the conditions

(i) all maximal ideals principal, or
(ii) all maximal ideals idempotent

is sufficient to ensure that $\mathfrak{A}$ is either

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(iii) semisimple, or
(iv) finite dimensional.

If all maximal ideals are principal, $\mathfrak{A}$ need not be semisimple, indeed let $\mathfrak{A} = Ce \oplus Cr$ where e, r are elements satisfying $e^2 = e$, $er = r$, $r^2 = 0$. The author has been unable to determine whether or not (i) implies (iv), though a weaker form of (i), namely that each maximal ideal is the closure of a principal ideal, is not sufficient. For consider the algebra $\mathfrak{H}$ of all functions continuous on the closed unit disc Δ and analytic on its interior, with pointwise operations and the supremum norm. Then any closed ideal of $\mathfrak{H}$, so certainly any maximal ideal, is the closure of some principal ideal ([8], chapter 6).

If all maximal ideals are idempotent then the example of § 2 shows that $\mathfrak{A}$ need not be semisimple. The algebra $C([0, 1])$ shows $\mathfrak{A}$ need not be finite dimensional.

Before going on to the next result we give an analysis of the idempotence properties of the maximal ideals of the algebra $\mathfrak{H}$ defined above. Every maximal ideal is of the form

$$M_t = \{f \in \mathfrak{H} : f(t) = 0\}$$

for some $t \in \Delta$ ([11], § 11). Thus if t is in the interior of Δ, M_t has generator $(z-t)$ and so $M_t^2 \neq M_t$. By the principal of maximal modulus, and with the maximal ideal space identified with Δ, the unit circle is the Shilov boundary of $\mathfrak{H}$. Theorem 4.54 of [1] and [15], Corollary 2.1, then show that $\overline{M_t^2} = M_t$ for $|t| = 1$, the bar signifying closure in the norm topology. An alternative proof of this is given in [14], § 8.

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In this section we show that the idempotence of all maximal ideals is not sufficient for semisimplicity when $\mathfrak{A}$ is a commutative Banach algebra with identity. The condition is clearly not necessary as is shown by the algebra $\mathfrak{H}$ of § 1 and even more so by the following example. Let $\mathfrak{A}$ be the algebra of all continuously differentiable functions on $[0, 1]$ with the usual norm ([11], § 11). Then every maximal ideal M of $\mathfrak{A}$ is the set of functions in $\mathfrak{A}$ vanishing at some point $x_0 \in [0, 1]$. But if $f \in M^2$ then $f'(x_0) = 0$ so $M^2 \neq M$.

THEOREM 4. *There exists a radical Banach algebra $\mathfrak{R}$ such that $\mathfrak{R}^2 = \mathfrak{R}$. Given any cardinal $\mathfrak{m}$, there is a commutative Banach algebra with identity, having $\mathfrak{R}$ as radical, with exactly $\mathfrak{m}$ maximal ideals, all of which are idempotent.*

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PROOF. Take $\mathfrak{R}$ to be the commutative Banach algebra $L^1(0, 1)$ under the usual multiplication and norm

$$x * y(t) = \int_0^t x(t-\tau)y(\tau)d\tau$$
$$||x||_{\mathfrak{R}} = \int_0^1 |x(t)|dt$$

for $x, y \in \mathfrak{R}$, $t \in [0, 1]$.

It is shown in [7] § 4.9 that $\mathfrak{R}$ is a radical algebra. Also, it is easily seen that if we define

$$y_h(t) = \begin{cases} 1/h & 0 \leqq t \leqq h \\ 0 & h < t \leqq 1 \end{cases}$$

then $\{y_h : 0 < h \leqq 1\}$ is an approximate identity in $\mathfrak{R}$, with $||y_h||_{\mathfrak{R}} = 1$ for $0 < h \leqq 1$. But then by a result of Cohen [5], $\mathfrak{R}^2 = \mathfrak{R}$. (A result very similar to this, namely that group algebras $L^1(G)$ are idempotent for suitably restricted locally compact abelian groups G, is proved in the papers of Rudin cited in [5]. By our definition of product $\mathfrak{R}$ is not a group algebra.)

Now let A be an index set of cardinality $\mathfrak{m}$. If A is infinite, let A^* be the one point compactification of A considered as a discrete locally compact space. Denoting the adjoined point by ∞, direct A by setting $i > j$ if there are neighbourhoods U, V of ∞ with $U \subseteqq V$ and $i \in U$, $j \in V$. Limits over A are to be taken in this sense. If A is finite let $A^* = A$. Take $\mathfrak{A}$ as the set of formal sums

$$\{\sum_i \alpha_i e_i + r : r \in \mathfrak{R},\ \alpha_i \text{ complex},\ i \in A\}$$

such that

$$||\sum_i \alpha_i e_i + r|| = \sup |\alpha_i| + ||r||_{\mathfrak{R}} < \infty$$

and for which, in the case of infinite A, $\lim_i \alpha_i$ exists. With multiplication

$$(\sum_i \alpha_i e_i + r)\ (\sum_i \beta_i e_i + s) = \sum_i \alpha_i \beta_i e_i + \alpha_k s + \beta_k r + r * s,$$

where k is some fixed element of A, and with the norm $|| \cdot ||$ defined above, $\mathfrak{A}$ is a Banach algebra with identity $\sum_i 1 \cdot e_i$. The radical of $\mathfrak{A}$ clearly contains $\mathfrak{R}$, we show that it is precisely $\mathfrak{R}$. Suppose that $x = \sum_i \alpha_i e_i + r$ with x quasinilpotent. If n is a positive integer

$$x^n = \sum_i \alpha_i^n e_i + S_n$$

for some $S_n \in \mathfrak{R}$, so that

$$||x^n||^{1/n} \geqq (\sup_i |\alpha_i^n|)^{1/n}$$
$$= \sup_i |\alpha_i|.$$

It follows that $\alpha_i = 0$ for each $i \in A$ and so $x \in \mathfrak{R}$.

Since the set

$$\{\sum_i \alpha_i e_i \in \mathfrak{A}\}$$

can be identified with the space of all continuous functions on A^*, we have that the maximal ideals of $\mathfrak{A}$ are precisely the sets

$$M_j = \{\sum_i \alpha_i e_i + r : \alpha_j = 0\},$$

for $j \in A^*$, where

$$M_\infty = \{\sum_i \alpha_i e_i + r : \lim_i \alpha_i = 0\}.$$

But then if $\sum_i \alpha_i e_i + r \in M_j$

$$\begin{aligned}\sum_i \alpha_i e_i + r &= \sum_i \sqrt{\alpha_i}\sqrt{\alpha_i} e_i + 2\sqrt{\alpha_k} r + r^2 + (1-2\sqrt{\alpha_k}) r - r^2 \\ &= (\sum_i \sqrt{\alpha_i} e_i + r)^2 + (1-2\sqrt{\alpha_k}) r - r^2 \\ &\in M_j^2 + \mathfrak{R}^2 \\ &= M_j^2.\end{aligned}$$

Thus $M_j^2 = M_j$ for each $j \in A^*$. The statement about the number of maximal ideals is clear.

This result shows that an algebra may have idempotent maximal ideals and yet have an infinite dimensional radical. Whether or not $\mathfrak{A}$ may have finite dimensional radical is not known, however if the hypothesis is slightly weakened this case is possible as the following result shows.

THEOREM 5. *Let $\mathfrak{A}$ be a commutative semisimple Banach algebra in which all regular maximal ideals are idempotent. Then there is a commutative extension $\mathfrak{B}$ of $\mathfrak{A}$ in which all maximal regular ideals, except one, are idempotent, and the radical of $\mathfrak{B}$ is one dimensional.*

PROOF. Let ϕ be a (fixed) nonzero multiplicative linear functional of $\mathfrak{A}$. Define

$$\mathfrak{B} = \mathfrak{A} + Cr$$

where r satisfies

$$r^2 = 0,\ a(\lambda r) = (\lambda r)a = \lambda(ra) = \lambda\phi(a)r$$

for $\lambda \in C$, $a \in \mathfrak{A}$. If $x, y \in \mathfrak{A}$ then

$$(xy)r = \phi(xy)r = \phi(y)\phi(x)r = \phi(y)(xr) = x(\phi(y)r) = x(yr),$$

so $\mathfrak{B}$ is an associative algebra. For $z \in \mathfrak{B}$, $z = x + \lambda r$ where $x \in \mathfrak{A}$, define

$$||z||_{\mathfrak{B}} = ||x||_{\mathfrak{A}} + |\lambda|.$$

To see that this defines a norm it suffices to show it is submultiplicative. Thus let $z_i = x_i + \lambda_i r$, $i = 1, 2$, so that

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$$\begin{aligned} ||z_1 z_2||_{\mathfrak{B}} &= ||x_1 x_2||_{\mathfrak{A}} + |\lambda_1 \phi(x_2) + \lambda_2 \phi(x_1)| \\ &\leqq ||x_1||_{\mathfrak{A}} \cdot ||x_2||_{\mathfrak{A}} + |\lambda_1 \phi(x_2)| + |\lambda_2 \phi(x_1)| + |\lambda_1 \lambda_2|. \end{aligned}$$

Since ϕ is a multiplicative linear functional $||\phi|| \leqq 1$ and so

$$\begin{aligned} ||z_1 z_2||_{\mathfrak{B}} &\leqq ||x_1||_{\mathfrak{A}} \cdot ||x_2||_{\mathfrak{A}} + |\lambda_1| \cdot ||x_2||_{\mathfrak{A}} + |\lambda_2| \cdot ||x_1||_{\mathfrak{A}} + |\lambda_1| \cdot |\lambda_2| \\ &= ||z_1||_{\mathfrak{B}} \cdot ||z_2||_{\mathfrak{B}}. \end{aligned}$$

Thus $\mathfrak{B}$ is a commutative Banach algebra (with identity if $\mathfrak{A}$ has an identity), completeness being immediate from the definition of the norm.

Now let M be a maximal regular ideal in $\mathfrak{A}$ and consider the set $M+Cr$. This is easily seen to be a regular ideal in $\mathfrak{B}$, and is in fact a maximal one. For let J be a regular ideal in $\mathfrak{B}$ which properly contains $M+Cr$, and let $x \in J \setminus (M+Cr)$. We may suppose $x \in \mathfrak{A} \setminus M$, but then the ideal in $\mathfrak{A}$ generated by $M \cup \{x\}$ is a regular ideal properly containing M, and hence is all of $\mathfrak{A}$. Thus $J = \mathfrak{B}$ and $M+Cr$ is maximal.

Conversely, if N is a maximal regular ideal in $\mathfrak{B}$, then $N \cap \mathfrak{A}$ is a regular ideal in $\mathfrak{A}$. Let M be a maximal regular ideal in $\mathfrak{A}$ containing $N \cap \mathfrak{A}$. Then by the above $M+Cr$ is a maximal regular ideal in $\mathfrak{B}$ which clearly contains $N = N \cap \mathfrak{A}+Cr$, and so $N = M+Cr$ (and $M = N \cap \mathfrak{A}$).

Thus the maximal regular ideals of $\mathfrak{B}$ are precisely the sets of the form $M+Cr$ where M is a maximal regular ideal in $\mathfrak{A}$.

Now let M be a maximal regular ideal in $\mathfrak{A}$. If $M \neq \ker \phi$ then

$$(M+Cr)^2 = M^2+MCr = M+Cr.$$

However if $M = \ker \phi$,

$$(M+Cr)^2 = M^2+MCr = M \neq M+Cr.$$

It follows that $N^2 = N$ for all maximal regular ideals of $\mathfrak{B}$ except one, and in the exceptional case N^2 has codimension one in N.

3

If $\mathfrak{A}$ and $\mathfrak{B}$ are algebras over C and ϕ is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, a linear map D from $\mathfrak{A}$ into $\mathfrak{B}$ satisfying

$$D(xy) = Dx \cdot \phi y + \phi x \cdot Dy$$

for all $x, y \in \mathfrak{A}$ will be termed a ϕ-derivation. In the case where ϕ is the identity homomorphism, so that $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, the map will be termed a derivation, if $\mathfrak{B} = C$, so that ϕ is a multiplicative linear functional on $\mathfrak{A}$, D will be termed a point derivation associated with ϕ.

In this final section we investigate the relationships between the idempotence of maximal ideals in a commutative Banach algebra $\mathfrak{A}$ and the

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existence and boundedness of derivations and point derivations on $\mathfrak{A}$. We will need the following lemma, the purely algebraic case of which is due to J. B. Miller.

LEMMA 3. *Let $\mathfrak{A}$ be a commutative algebra over a field of zero characteristic, and let $\mathfrak{A}_1$ be the algebra obtained by formally adjoining an identity e to $\mathfrak{A}$. Let I be a regular ideal in $\mathfrak{A}$, I_1 the unique ideal of $\mathfrak{A}_1$ with $I = I_1 \cap \mathfrak{A}$. Then if $I^2 = I$ it follows that $I_1^2 = I_1$. If $\mathfrak{A}$ is a Banach algebra then $\overline{I^2} = \bar{I}$ implies $\overline{I_1^2} = \bar{I}_1$.*

PROOF. We prove the second result; the proof of the first follows the same lines but is slightly simpler.

Let f be an identity modulo I, that is, $x - xf \in I$ for all $x \in \mathfrak{A}$. Then by [11] § 7.4, VI,

$$I_1 = \{z \in \mathfrak{A}_1 : fz \in I\},$$

that is,

$$\begin{aligned} I_1 &= \{\lambda e + x : x \in \mathfrak{A},\ \lambda f + xf \in I\} \\ &= \{\lambda e + x : x \in \mathfrak{A},\ \lambda f + x \in I\}. \end{aligned}$$

If $z \in I_1$, then $z = \lambda e + x$ where $\lambda f + x \in \overline{I^2}$. Thus given $\varepsilon > 0$ there are elements $m_i, n_i \in I$, $i = 1, 2, \cdots, k$ such that

$$\lambda f + x = \sum_{i=1}^{k} m_i n_i + v$$

where $||v|| < \varepsilon$. Thus

$$z = \lambda(e-f) + \sum_{i=1}^{k} m_i n_i + v.$$

Now define

$$z_0 = \sum_{i=1}^{k} (e-f+m_i) \cdot \left(\frac{\lambda}{k}(e-f) + n_i\right).$$

Since $I \subseteqq I_1$ it follows that $z_0 \in I_1^2$. But

$$\begin{aligned} z_0 &= \sum_{i=1}^{k} \frac{\lambda}{k}(e-f)^2 + \sum_{i=1}^{k} \left(\frac{\lambda}{k} m_i + n_i\right)(e-f) + \sum_{i=1}^{k} m_i n_i \\ &= \lambda(e-f) + \sum_{i=1}^{k} m_i n_i - (e-f)\left(\lambda f - \sum_{i=1}^{k}\left(\frac{\lambda}{k} m_i + n_i\right)\right). \end{aligned}$$

Thus $z = z_0 + v + w(e-f)$ where

$$w = \lambda f - \sum_{i=1}^{k} \left(\frac{\lambda}{k} m_i + n_i\right).$$

But $(e-f)x \in I$ for any $x \in \mathfrak{A}$ and so $(e-f)w \in I \subseteqq \overline{I^2}$. Since $I \subseteqq I_1$ it follows that $(e-f)w \in \overline{I_1^2}$; thus $z - v \in \overline{I_1^2}$. Now this procedure can be carried out for each $\varepsilon > 0$ and so $z \in \overline{I_1^2}$, whence $I_1 \subseteqq \overline{I_1^2}$ since $z \in I_1$ was arbitrary. Thus $\overline{I_1} \subseteqq \overline{I_1^2}$. The reverse inclusion is obvious since $I_1^2 \subseteqq I_1$, so the result follows.

By the use of this lemma we obtain the following generalization of one half of the result of Singer and Wermer stated in the introduction.

THEOREM 6. *Let* $\mathfrak{A}$ *be a commutative Banach algebra in which* $M^2 = M$ ($\overline{M^2} = M$) *for each maximal regular ideal* M. *Then any non-zero (bounded) derivation of* $\mathfrak{A}$ *into a commutative extension* $\mathfrak{B}$ *maps into the radical of* $\mathfrak{B}$.

PROOF. Suppose to the contrary that there is a non-zero (bounded) derivation D of $\mathfrak{A}$ into an algebra $\mathfrak{B}$ which does not map into the radical of $\mathfrak{B}$. By [15] Theorem 2 there thus exists a non-zero (bounded) point derivation Δ on $\mathfrak{A}$, associated with a multiplicative linear functional ϕ. We show that this is impossible.

Let $\mathfrak{A}_e$ be the algebra $\mathfrak{A}$ with identity e adjoined. It is easy to see that we can extend Δ and ϕ to $\mathfrak{A}_e$ so that they retain all their original properties. Indeed, if $x \in \mathfrak{A}_e$, we have a unique decomposition

$$x = x' + \lambda e$$

where $x' \in \mathfrak{A}$; then define

$$\Delta x = \Delta x'$$
$$\phi x = \phi x' + \lambda.$$

Let $M_e = \ker \phi$ so that M_e is the unique maximal ideal of $\mathfrak{A}_e$ such that $M = M_e \cap \mathfrak{A}$. If $x \in \mathfrak{A}_e$ we have the decomposition

$$x = y + \phi x \cdot e$$

where $y \in M_e$. Since $\Delta e = 0$ and Δ is non-zero it follows that $\Delta(M_e) \neq 0$. On the other hand, if $x \in M_e^2$ we have

$$x = \sum_{i=1}^{k} m_i n_i$$

where $m_i, n_i \in M_e$ for $i = 1, 2, \cdots, k$; so that

$$\begin{aligned} \Delta x &= \sum_{i=1}^{k} (\Delta m_i \cdot \phi n_i + \phi m_i \cdot \Delta n_i) \\ &= 0. \end{aligned}$$

Thus $\Delta(M_e^2) = 0$ ($\Delta(\overline{M_e^2}) = 0$) so that $M_e^2 \neq M_e$ ($\overline{M_e^2} \neq M_e$) and it follows from Lemma 3 that $M^2 \neq M$ ($\overline{M^2} \neq M$) which contradicts the hypothesis on $\mathfrak{A}$.

REMARK. If $\mathfrak{A}$ has an identity the converse of this is true and is a generalization of one half of [15] Corollary 2.1. Indeed, let $\mathfrak{A}$ be a commutative Banach algebra with identity and suppose that any non-zero (bounded) derivation of $\mathfrak{A}$ into any commutative extension $\mathfrak{B}$ maps into the radical of $\mathfrak{B}$. Then $M^2 = M$ ($\overline{M^2} = M$) for each maximal ideal M of $\mathfrak{A}$.

PROOF. Suppose to the contrary that $M^2 \neq M$ ($\overline{M^2} \neq M$) for some maximal ideal of M of $\mathfrak{A}$. Then as in [15] $\mathfrak{A}$ admits a non-zero (bounded) point derivation associated with ϕ, where ϕ is the canonical map $\mathfrak{A} \to \mathfrak{A}/M$. But then by [15], Theorem 2, $\mathfrak{A}$ admits a non-zero (bounded) derivation into $\mathfrak{A} \oplus C$ but not into the radical of $\mathfrak{A} \oplus C$. This is a contradiction.

Finally we have the following generalization of a result given by Kaplansky in [10] for C^*-algebras.

THEOREM 7. *Let $\mathfrak{A}$ be a non-radical Banach algebra in which all maximal regular ideals are idempotent. Suppose that $\mathfrak{A}$ is contained, as a subalgebra, in the centre of a Banach algebra $\mathfrak{B}$. Then any derivation D from $\mathfrak{A}$ into $\mathfrak{B}$ maps into the radical of $\mathfrak{B}$.*

PROOF. If $\mathfrak{B}$ is a radical algebra the result is trivially true [3]. Otherwise, let $\mathfrak{B}_e$ be the algebra $\mathfrak{B}$ with an identity e adjoined if necessary; by Lemma 3 we may suppose that $e \in \mathfrak{A}$.

Now let P be a primitive ideal in $\mathfrak{B}_e$, so that $\mathfrak{B}_e/P$ is a primitive ring and hence its centre $Z(\mathfrak{B}_e/P)$ is either zero or the complex field. Also, since $\mathfrak{A}$ is contained in the centre of $\mathfrak{B}_e$,

$$\{x+P : x \in \mathfrak{A}\} \subseteqq Z(\mathfrak{B}_e/P).$$

The map from $\{x+P : x \in \mathfrak{A}\}$ to $\mathfrak{A}/\mathfrak{A} \cap P$ defined by

$$x+P \to x+\mathfrak{A} \cap P$$

is clearly an isomorphism onto and so $\mathfrak{A}/\mathfrak{A} \cap P$ is either zero or is isomorphic to the complex field. Thus $\mathfrak{A} \cap P$ is either all of $\mathfrak{A}$ or is a maximal ideal of $\mathfrak{A}$. Since $\mathfrak{A}$ contains the identity e of $\mathfrak{B}_e$ the first possibility does not arise and so by the hypothesis on $\mathfrak{A}$, $(\mathfrak{A} \cap P)^2 = \mathfrak{A} \cap P$. But then

$$\begin{aligned} D(\mathfrak{A} \cap P) &\subseteqq (\mathfrak{A} \cap P) D(\mathfrak{A} \cap P) \\ &\subseteqq P \end{aligned}$$

since $\mathfrak{A}$ is in the centre of $\mathfrak{B}_e$ and P is an ideal containing $\mathfrak{A} \cap P$. Also $\mathfrak{A} = (\mathfrak{A} \cap P) \oplus Ce$ so that $D(\mathfrak{A}) \subseteqq P$. This is true for each primitive ideal P of $\mathfrak{B}_e$ and so D maps $\mathfrak{A}$ into the radical of $\mathfrak{B}_e$. Since D maps into $\mathfrak{B}$ by hypothesis we thus have

[3] This case may arise even though $\mathfrak{A}$ is not all radical since $\mathfrak{A}$ is only algebraically embedded in the centre of $\mathfrak{B}$, and not necessarily topologically.

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$$D(\mathfrak{A}) \subseteqq (\cap M_e) \cap \mathfrak{B}$$

where the first intersection is over all maximal left ideals of $\mathfrak{B}_e$. By [11], § 7.4, VI, it follows that

$$D(\mathfrak{A}) \subseteqq \cap M$$

where the intersection is over all maximal regular left ideals of $\mathfrak{B}$.

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