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COMMUTATIVE EXTENSIONS BY CANONICAL MODULES ARE GORENSTEIN RINGS

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ABSTRACT. Reiten has demonstrated that the trivial Hochschild extension of a Cohen-Macaulay local ring by a canonical module is a Gorenstein local ring. Here it is proved that any commutative extension of a Cohen-Macaulay local ring by a canonical module is a Gorenstein ring. Also Gorenstein extensions of a local Cohen-Macaulay ring by a module are studied.

Introduction. Suppose A is a commutative ring and that M is an A-module. A *commutative extension* of A by M is an exact sequence of abelian groups

$$0 \longrightarrow M \xrightarrow{i} E \xrightarrow{\pi} A \longrightarrow 0,$$

where E is a commutative ring, the map π is a ring homomorphism and the A-module structure on M is related to (E, i, π) by the equations

$$ei(x) = i(\pi(e)x), \quad \text{for all } e \in E \text{ and all } x \in M.$$

The i identifies M with an ideal of square zero in E. (On the other hand if $\mathfrak{J}$ is an ideal of square zero in E, then $\mathfrak{J}$ is an $E/\mathfrak{J}$-module and $0 \to \mathfrak{J} \to E \to E/\mathfrak{J} \to 0$ is an extension of $E/\mathfrak{J}$ by $\mathfrak{J}$.)

The *trivial extension* of A by M is the exact sequence

$$0 \longrightarrow M \xrightarrow{i} M \times A \xrightarrow{\pi} A \longrightarrow 0$$

where i is the first coordinate map, where π is the second projection and where $M \times A$ is a ring whose underlying additive structure is the direct sum of abelian groups and whose multiplication is given elementwise by $(m, a)(m', a') = (ma' + m'a, aa')$ for all $m, m' \in M$ and all $a, a' \in A$. This extension is denoted by $A \ltimes M$.

Now suppose A is a commutative noetherian local ring with maximal ideal m. An A-module of finite type M is a *canonical module* if it has the

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three properties:

(i) The natural homomorphism $A \to \operatorname{End}_A(M)$ is a bijection.

(ii) The groups $\operatorname{Ext}_A^i(M, M)=0$ for all $i>0$.

(iii) The injective dimension $\operatorname{id}_A M<\infty$. If B is a Gorenstein local ring (see Bass [**1**]) and if $B\to A$ is a surjection then A is a Cohen-Macaulay ring if and only if $\operatorname{Ext}_B^i(A, B)=0$ for $i\neq \dim B-\dim A$. If A is Cohen-Macaulay and if $d=\dim B-\dim A$, then $\operatorname{Ext}_B^d(A, B)$ is a canonical A-module (see Grothendieck [**5**] and Sharp [**8**]). Furthermore A is Gorenstein if and only if $A\cong\operatorname{Ext}_B^d(A, B)$. As a converse to this result Reiten [**7**] shows that if the Cohen-Macaulay local ring A has a canonical module M, then the trivial extension $A\times M$ is a Gorenstein ring. A more precise statement is found in Fossum, Griffith and Reiten [**3**]: The ring $A\ltimes M$ is a Gorenstein ring if and only if the A-module M is a canonical module.

For further properties of canonical modules we refer to Sharp [**8**], Herzog and Kunz [**6**], Foxby [**4**] and Fossum, Griffith and Reiten [**3**] although the only results which are really needed are:

(a) If A has a canonical module M, then A is Cohen-Macaulay.

(b) If M is a canonical A-module and if $a\in A$, then a is regular on M if and only if it is regular on A.

(c) If M is a canonical A-module and if a is a regular element in A, then M/aM is a canonical A/aA-module. Also we note that when $B\to A$ is a surjective ring homomorphism, when B is Gorenstein and when A is Cohen-Macaulay then the spectral sequence (Cartan and Eilenberg [**2**]) with $E_2^{pq}=\operatorname{Ext}_A^p(X, \operatorname{Ext}_B^q(A, B))$ and abutment $\operatorname{Ext}_B^n(X, B)$ degenerates to natural isomorphisms $\operatorname{Ext}_A^p(X, \operatorname{Ext}_B^d(A, B))\cong\operatorname{Ext}_B^{p+d}(X, B)$ for all A-modules X, for all integers $p\geqq 0$ and for $d=\dim B-\dim A$.

(The term canonical module seems to have been introduced by Herzog and Kunz [**6**]. A more geometric term is *module of dualizing differentials*, which is very suggestive terminology but also rather cumbersome. Sharp [**8**] uses the term Gorenstein module of rank one.)

Arbitrary extensions. The main result of this paper is the generalization of Reiten's result.

THEOREM. *Suppose A is a local noetherian ring and M is a canonical A-module. If $0\to M\to^i E\to^\pi A\to 0$ is a commutative extension of A by M, then E is a Gorenstein ring.*

The proof is based on two lemmas. The first allows a reduction to the Artin case. The second handles the Artin case.

LEMMA 1. *Suppose A, M and E are as in the statement of the theorem. An element e in E is regular if and only if $\pi(e)$ is regular in A.*

PROOF. Since M is a canonical module, an element $\pi(e)$ is regular in A if and only if it is regular on M. If $e \in E$ and if e is regular then e is regular on $i(M)$. But the restriction of e to $i(M)$ is the action of $\pi(e)$ on M. Hence $\pi(e)$ is regular in A.

Suppose, on the other hand, that $\pi(e)$ is regular in A. If $e \cdot x=0$ for some $x \in E$, then $\pi(e) \cdot \pi(x)=0$. Hence $\pi(x)=0$, since $\pi(e)$ is regular in A. There is then an element $m \in M$ such that $x=i(m)$. Then $ex=i(\pi(e)m)$. Since $\pi(e)$ is regular on M and since i is an injection, the element $m=0$. Hence $x=0$. So e is regular in E. Q.E.D.

LEMMA 2. *Suppose A is an Artin ring, that M is an A-module with* $\operatorname{Ann}_A M=(0)$ *and that E is an extension of A by M. Then* $i(\operatorname{Socle}(M))=\operatorname{Socle}(E)$.

PROOF. Let $\mathfrak{n}$ be the radical of E and $\mathfrak{m}$ the radical of A (so that $\mathfrak{m}=\pi(\mathfrak{n})$). Now $e \in \operatorname{Socle}(E)$ if and only if $\mathfrak{n}e=(0)$. If $\mathfrak{n}e=(0)$, then $i(M)e=(0)$. But $i(M)e=i(M\pi(e))$. Hence $\mathfrak{n}e=(0)$ implies $M\pi(e)=(0)$. But $\operatorname{Ann}_A M=(0)$ implies that $\pi(e)=0$. Hence $e \in \operatorname{Socle}(E)$ implies $e \in i(M)$, say $e=i(m)$ for some $m \in M$. Now $\mathfrak{n} \cdot i(m)=i(\mathfrak{m} \cdot m)$ so $\mathfrak{m} \cdot m=(0)$. Hence $m \in \operatorname{Socle}(M)$ and then $e \in i(\operatorname{Socle}(M))$. Clearly $i(\operatorname{Socle}(M)) \subseteq \operatorname{Socle}(E)$. Q.E.D.

Now these two lemmas are used to prove the theorem. The proof is by induction on $\dim A$. We know that $\dim A=\dim E$ since $i(M)$ is nilpotent. By Lemma 1 we can conclude that depth $A=$depth E. For suppose e is a regular nonunit in E. Then $\pi(e)$ is a regular nonunit in A. Multiplication by e induces the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{i} & E & \xrightarrow{\pi} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi(e)} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle \pi(e)} & & \\ 0 & \longrightarrow & M & \xrightarrow{i} & E & \xrightarrow{\pi} & A & \longrightarrow & 0. \end{array}$$

We conclude that the sequence

$$0 \longrightarrow M/\pi(e)M \xrightarrow{i'} E/eE \xrightarrow{\pi'} A/\pi(e)A \longrightarrow 0$$

is exact and is an extension of the local ring $A/\pi(e)A$ by the canonical module $M/\pi(e)M$. Now depth $E=1+$depth E/eE. By induction depth $E/eE=$depth $A/\pi(e)A=-1+$depth A. Hence depth $E=$depth A.

But we can also use this reduction to show that E is Gorenstein. For E is Gorenstein if and only if E/eE is Gorenstein.

If $\dim A=0$, then the canonical module $M \cong E(A/\mathfrak{m})$, the injective envelope of the residue class field of A. Hence $\mathrm{Ann}_A\ M=(0)$ since $A\cong \mathrm{End}_A(M)$ and $\mathrm{Socle}(M)\cong A/\mathfrak{m}$. By Lemma 2 we get $\mathrm{Socle}(E)\cong E/\mathfrak{n}\cong A/\mathfrak{m}$. Hence E has a simple socle and is therefore Gorenstein (Bass [**1**]).

Suppose, for inductive purposes, that E' is Gorenstein for $\dim E' < \dim E$ whenever E' is an extension of a local ring by a canonical module. Then we conclude that E is Gorenstein since E/eE is Gorenstein. Q.E.D.

Gorenstein extensions. Now our attention is directed to the converse problem. Suppose E is an extension of A by M, that E is Gorenstein and that A is Cohen-Macaulay. What is the relationship between M and the canonical module $\mathrm{Hom}_E(A, E)$ of A? (Note that there are Gorenstein extensions of non-Cohen-Macaulay rings. If k is a field, X and Y indeterminates, then $E=k[[X, Y]]/(X^2)$ is Gorenstein. The ideal generated by the image XY is nilpotent. But $k[[X, Y]]/(X^2, XY)$ is not Cohen-Macaulay.)

Suppose E is a Gorenstein local ring and $\mathfrak{J}$ is an ideal of square zero. Let $A=E/\mathfrak{J}$. We assume that A is Cohen-Macaulay.

Let B be a Gorenstein local ring with $\dim B>0$. Let t be a regular element in B. Then B/t^nB is Gorenstein for all $n\geqq 1$. Let $E=B/t^nB$ and let $\mathfrak{J}=t^{n-1}B/t^nB$. Then $E/\mathfrak{J}=B/t^{n-1}B$ and $\mathfrak{J}^2=(0)$. Now $\mathfrak{J}$ is a canonical $E/\mathfrak{J}$-ideal if and only if $n=2$. Hence it is not the case that $\mathfrak{J}$ is the canonical module.

Let $\Omega=\mathrm{Hom}_E(A, E)$ be the canonical module. Then $\Omega\cong\{e\in E : e\mathfrak{J}=(0)\}$. Since $\mathfrak{J}^2=(0)$, we have $\mathfrak{J}\subseteq\Omega$. We get the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & 0 & & & & \mathfrak{a} & & \\ & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{J} & \longrightarrow & E & \xrightarrow{\pi} & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{=} & & \downarrow{\gamma} & & \\ 0 & \longrightarrow & \Omega & \longrightarrow & E & \xrightarrow{\pi'} & \mathrm{Hom}_E(\mathfrak{J}, E). & & \end{array}$$

The homomorphism $\gamma: A\to\mathrm{Hom}_E(\mathfrak{J}, E)$ is the composition $A\to\mathrm{Hom}_E(\mathfrak{J}, \mathfrak{J})\to\mathrm{Hom}_E(\mathfrak{J}, E)$. Thus $\mathrm{Ker}\ \gamma=\mathrm{Ann}_A\ \mathfrak{J}$.

LEMMA 3. *The natural map* $A\to\mathrm{Hom}_E(\mathfrak{J}, \mathfrak{J})$ *is a surjection and* $\mathrm{Hom}_E(\mathfrak{J}, \mathfrak{J})\to\mathrm{Hom}_E(\mathfrak{J}, E)$ *is a bijection. Thus* $\mathrm{Hom}_E(\mathfrak{J}, \mathfrak{J})$ *is identified with both* $A/\mathrm{Ann}_A\ \mathfrak{J}$ *and* E/Ω.

PROOF. Since $\mathfrak{J}^2=(0)$ and since A is Cohen-Macaulay, the group $\mathrm{Ext}_E^i(A, E)=(0)$ for $i>0$. Hence π' is a surjection. Thus γ is also a surjection. Since $\mathrm{Hom}_E(\mathfrak{J}, \mathfrak{J})\to\mathrm{Hom}_E(\mathfrak{J}, E)$ is an injection, the statements of the lemma follow. Q.E.D.

We identify $\mathrm{Coker}(\mathfrak{J}\to\Omega)$ with $\mathfrak{a}$, the annihilator of $\mathfrak{J}$. The ring $\mathrm{Hom}_E(\mathfrak{J}, \mathfrak{J})\cong\mathrm{Hom}_A(\mathfrak{J}, \mathfrak{J})$ and is denoted by A'. Then the diagram can be displayed as

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & 0 & & & & \mathfrak{a} & & \\ & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{J} & \longrightarrow & E & \xrightarrow{\pi} & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{=} & & \downarrow{\gamma} & & \\ 0 & \longrightarrow & \Omega & \longrightarrow & E & \xrightarrow{\pi'} & A' & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \\ & & \mathfrak{a} & & & & 0 & & \\ & & \downarrow & & & & & & \\ & & 0. & & & & & & \end{array}$$

LEMMA 4. *The ring A' is Cohen-Macaulay and* $\dim A=\dim A'$.

PROOF. It is enough to prove that $\mathrm{Ext}^i_E(A', E)\cong(0)$ for $i>0$. The exact sequence $0\to\Omega\to E\to A'\to 0$ gives rise to the exact sequence

$$0 \to \mathrm{Hom}_E(A', E) \to E \to \mathrm{Hom}_E(\Omega, E) \to \mathrm{Ext}^1_E(A', E) \to 0$$

and the isomorphisms $\mathrm{Ext}^i_E(\Omega, E)\cong\mathrm{Ext}^{i+1}_E(A', E)$ for $i>0$. Since A is Cohen-Macaulay we get natural isomorphisms $\mathrm{Ext}^i_E(\Omega, E)\cong\mathrm{Ext}^i_A(\Omega, \Omega)$ for all i. Hence $\mathrm{Ext}^i_E(\Omega, E)=0$ for all $i>0$ while

$$\mathrm{Hom}_E(\Omega, E) \cong \mathrm{Hom}_A(\Omega, \Omega) \cong A,$$

since Ω is a canonical A-module. (This follows from the remarks in the introduction.) Thus $E\to\mathrm{Hom}_E(\Omega, E)$ is just $\pi: E\to A$ which is a surjection. So $\mathrm{Ext}^i_E(A', E)=(0)$ for all $i>0$ while the canonical A'-module is $\mathrm{Ker}\ \pi=\mathfrak{J}\cong\mathrm{Hom}_E(A', E)\cong\{e \in E: e\Omega=0\}$. Q.E.D.

We record several other consequences. We retain the hypotheses and notation.

PROPOSITION. (a) *The sequences*

$$0 \longrightarrow \mathfrak{J} \longrightarrow E \xrightarrow{\pi} A \longrightarrow 0 \quad \textit{and} \quad 0 \longrightarrow \Omega \longrightarrow E \xrightarrow{\pi'} A' \longrightarrow 0$$

are $\mathrm{Hom}_E(\_, E)$ *dual.*

(b) *The sequences* $0\to\mathfrak{a}\to A\to A'\to 0$ *and* $0\to\mathfrak{J}\to\Omega\to\mathfrak{a}\to 0$ *are* $\mathrm{Hom}_A(\_, \Omega)$ (*which is* $\mathrm{Hom}_E(\_, E)$) *dual.*

(c) *The group* $\mathrm{Ext}^i_E(\mathfrak{a}, E)=(0)$ *for all* $i>0$.

(d) *The A-module* $\mathfrak{J}$ *is a canonical A'-module while* Ω *is a canonical A-module. Thus, in particular, the square zero ideal* $\mathfrak{J}$ *is a canonical A-module if and only if* $\operatorname{Ann}_A \mathfrak{J}=(0)$.

(e) *We have the following natural isomorphisms*:

$$\begin{aligned} A &= \operatorname{Hom}_A(\Omega, \Omega), \\ A' &= \operatorname{Hom}_A(\mathfrak{J}, \mathfrak{J}) \cong \operatorname{Hom}_E(\mathfrak{J}, E) \cong \operatorname{Hom}_A(\mathfrak{J}, \Omega), \\ \mathfrak{J} &= \operatorname{Hom}_A(A', \Omega) \cong \operatorname{Hom}_E(A', E), \text{ and} \\ \Omega &= \operatorname{Hom}_A(A, \Omega) \cong \operatorname{Hom}_E(A, E). \qquad \text{Q.E.D.} \end{aligned}$$

Note that Ω is a nilpotent ideal in E which has square zero if and only if $\Omega=\mathfrak{J}$.

Final remarks. A result which is implicit in this article, and which follows immediately is a slight generalization of the Grothendieck-Bass-Sharp result about Cohen-Macaulay factor rings of Gorenstein rings. We state it here for possible future reference.

PROPOSITION. *Suppose A is a local Cohen-Macaulay ring with a canonical module* Ω. *The factor ring A' of A is Cohen-Macaulay if and only if* $\operatorname{Ext}^i_A(A', \Omega)=(0)$ *for all* $i \neq \dim A - \dim A'$. *If A' is Cohen-Macaulay, then it has a canonical module which is just the nonzero* Ext *group.* Q.E.D.

It should be mentioned that a nontrivial extension has been exhibited at the beginning of this section. Others will arise from symmetric 2-cocycles $f: A \times A \to \Omega$ which are not coboundaries (if they exist).

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