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COMMUTATIVE SEMI-PRIMARY SEMIGROUPS

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The study of primary semigroups is initiated in [2] by M. SATYANARAYANA. Here we deal with semigroups in which every ideal is semi-primary; where in a commutative semigroup, an ideal A is called a *semi-primary* ideal if $\sqrt{A}$ is a prime ideal. We call such semigroups to be semi-primary. We consider only commutative semigroups in this paper and for the various definitions and terms involved [2] may be consulted.

Theorem 1. *Let S be a commutative semigroup. Then the following statements about S are equivalent:*

(1) *$S$is a semiprimary semigroup.*
(2) *Every principal ideal of S is semi-primary.*
(3) *Prime ideals of S are totally ordered.*

Proof. (1) implies (2) is obvious. We prove first (2) implies (3). Let P and Q be two prime ideals of S and assume further that $P \nsubseteq Q$ and $Q \nsubseteq P$. Thus we have $a \in \in P - Q$ and $b \in Q - P$, which means $ab \in P \cap Q$ and $a \notin P \cap Q$, $b \notin P \cap Q$. Let $\sqrt{(a)} = P_1$, $\sqrt{(b)} = Q_1$ and $\sqrt{(ab)} = P'$, where P_1, Q_1 and P' are prime ideals. This gives $P' = P_1 \cap Q_1$, that is, $P_1 \subseteq P'$ or $Q_1 \subseteq P'$, but both are impossible. This contradiction proves (3). Assume now (3), and let A be any ideal of S. By 1.13 of [2], $\sqrt{A} = \bigcap P_\alpha$, where intersection is over all prime ideals $P_\alpha \supseteq A$. The totally ordered nature of prime ideals yields $\sqrt{A} = P$, for some prime P, so that A is semi-primary. Therefore S is a semi-primary semigroup.

Corollary. *Let S be a commutative semi-primary semigroup. Then idempotents form a chain under natural ordering.*

Proof. Let e and f be any two idempotents of S; then $\sqrt{(eS)}$ and $\sqrt{(fS)}$ are prime ideals, so either $\sqrt{(eS)} \subseteq \sqrt{(fS)}$ or $\sqrt{(fS)} \subseteq \sqrt{(eS)}$, which proves the assertion.

Converse of this corollary is false, which can be seen by the following

Example. Consider the semigroup of all natural numbers with respect to multiplication. In this, (6) is not prime and $\sqrt{(6)} = (6)$, thus (6) is not a semi-primary ideal, so that the semigroup is not semi-primary whereas the set of idempotents (there is just one in it) trivially forms a chain.

Theorem 2. *Let S be a regular commutative semigroup. Then the following statements about S are equivalent:*

(1) *Every ideal in S is prime.*
(2) *S is a primary semigroup.*
(3) *S is a semi-primary semigroup.*
(4) *Idempotents in S form a chain under natural ordering.*
(5) *Principal ideals of S are totally ordered.*
(6) *All ideals of S are totally ordered.*

Proof. (1) ⇒ (2) and (2) ⇒ (3) are immediate. (3) ⇒ (4) in view of the above corollary. We prove the remaining one by one. Assume (4). Let a and b be any two elements of S, $(a) = (e)$, $(b) = (f)$ where e and f are idempotents in S, since S is regular. As idempotents of S form a chain, so $(a) \subseteq (b)$ or $(b) \subseteq (a)$, which proves (5). Assume now (5) and let A, B be any two ideals of S with $A \nsubseteq B$. So we have an $a \in A - B$. For any b in B, $(b) \subseteq (a)$. Therefore $B \subseteq A$. Lastly assume (6) and let A be any ideal of S. Let $ab \in A$. As ideals are totally ordered; so either $(a) \subseteq (b)$ or $(b) \subseteq (a)$. Take for the sake of definiteness, $(a) \subseteq (b)$. As S is regular, $(a) = (a)^2$ [1]. Now $a \in (a) = (a)^2 \subseteq (a)(b) \subseteq A$. Similarly when $(b) \subseteq (a)$, $b \in A$. Thus A is a prime ideal. With this the proof of the theorem is completed.

Remark. This theorem tells us that primary semigroups and semi-primary semigroups coincide if they are regular. But this is not true in general, as can be seen by the following

Example. Let $S = \{a, a^2, a^3, \ldots\} \cup e$, where $e^2 = e$, $ae = ea = a^2$. In this semigroup, $P = \{a, a^2, \ldots\}$ is a unique proper prime ideal. So by Theorem 1, S is a semi-primary semigroup; but $A = \{a^2, a^3, \ldots\}$ is not a primary ideal in it. Thus S is not a primary semigroup. This is because S is not regular.

Corollary 1. *Let S be a commutative semi-primary semigroup. Then every ideal in S is prime if and only if S is regular.*

Proof. Let every ideal in S be prime. Now for any ideal A of S, $a \in A \Rightarrow a^2 \in A^2 \Rightarrow a \in A^2$; since A^2 is prime. Therefore $A = A^2$ for every ideal A. S, then, is regular by a theorem of Iséki [1]. The other part follows from Theorem 2.

Corollary 2. *Let S be a commutative semigroup. Then every ideal in S is prime if and only if S is regular and idempotents in S form a chain.*

Proof. When every ideal in S is prime, as in Cor. 1, S is regular. Let now e and f be any two idempotents of S such that $eS \nsubseteq fS$. Then $ef \in eS \cap fS$, $e \notin eS \cap fS$ and it being a prime ideal, $f \in eS \cap fS$. So $f = ef$. And if $eS \subseteq fS$, the result is clear. Thus idempotents of S form a chain. Converse follows from Theorem 2.

References

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