

## COMMUTATORS ON THE POTENTIAL-THEORETIC ENERGY SPACES

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1. Let  $L^1(dx/(1+x^2))$  be the  $L^1$  space of functions on the real line  $\mathbf{R}$  with respect to the measure  $dx/(1+x^2)$ . Let  $C_0^\infty$  be the totality of infinitely differentiable functions on  $\mathbf{R}$  with compact support. For  $0 < \alpha < 1$ , the energy space  $E_\alpha$  with respect to the  $\alpha$ -Riesz kernel  $|x|^{\alpha-1}$  is the Banach space of functions on  $\mathbf{R}$  obtained as the completion of  $C_0^\infty$  with respect to norm

$$\|f\|_\alpha = \left\{ \int_{-\infty}^{\infty} |\xi|^\alpha |\hat{f}(\xi)|^2 d\xi \right\}^{1/2},$$

where  $\hat{f}$  is the Fourier transform of  $f$  [5, p. 352]. The Hilbert transform  $H$  is defined by

$$Hf(x) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{f(y)}{y-x} dy.$$

For  $f \in L^1(dx/(1+x^2))$ ,  $[f, H]$  is an operator defined by

$$[f, H]g(x) = H(fg)(x) - f(x)Hg(x) \left( = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{f(y) - f(x)}{y-x} g(y) dy \right).$$

In the theory of singular integrals, this operator plays an important role [1]. In this note, we shall characterize the boundedness of  $[f, H]$  as an operator from  $E_\alpha$  to itself in terms of the BMO space  $\text{BMO}_\alpha$  with respect to  $E_\alpha$ . We say that a non-negative measure  $d\mu(x, y)$  in the upper half plane  $\mathbf{C}_+ = \{(x, y); x \in \mathbf{R}, y > 0\}$  is an  $\alpha$ -Carleson measure if there exists a constant  $B$  such that, for any open set  $O \subset \mathbf{R}$  with  $\text{Cap}_\alpha(O) < \infty$ ,

$$\iint_{\hat{O}} d\mu(x, y) \leq B \text{Cap}_\alpha(O),$$

where  $\hat{O} = \cup \{I \times (0, |I|); I \text{ component of } O\}$  ( $|I|$  is the length of  $I$ ) and  $\text{Cap}_\alpha(\cdot)$  is the capacity with respect to the  $\alpha$ -Riesz kernel [5, p. 131]. The minimum of such constants is denoted by  $\|d\mu\|_{\text{Car}, \alpha}$ . Let  $\text{BMO}_\alpha$  denote the Banach space of functions  $f \in L^1(dx/(1+x^2))$ , modulo constants, with norm

$$\|f\|_{\text{BMO}_\alpha} = \| |\nabla f(x, y)|^2 y^{1-\alpha} \|_{\text{Car}, \alpha}^{1/2},$$

where  $f(x, y)$  is the Poisson extension of  $f$  to  $C_+$  and  $|\nabla f|^2 = |\partial f/\partial x|^2 + |\partial f/\partial y|^2$ . We show:

**THEOREM.** *An operator  $[f, H]$  is bounded from  $E_\alpha$  to itself if and only if  $f \in \text{BMO}_\alpha$ .*

The dual space  $E_{-\alpha}$  of  $E_\alpha$  is the Banach space of distributions on  $\mathbf{R}$  obtained as the completion of  $C_0^\infty$  with respect to norm

$$\|f\|_{-\alpha} = \left\{ \int_{-\infty}^{\infty} |\xi|^{-\alpha} |\hat{f}(\xi)|^2 d\xi \right\}^{1/2}.$$

For  $f \in L^1(dx/(1+x^2))$ , we put

$$\|f\|_{H_\alpha^1} = \inf \sum_{k=1}^{\infty} \|g_k + i\varepsilon_k Hg_k\|_\alpha \|h_k + i\varepsilon_k Hh_k\|_{-\alpha},$$

where the infimum is taken over all sequences  $\{(g_k, h_k, \varepsilon_k)\}_{k=1}^\infty$  of triples such that

$$f = \sum_{k=1}^{\infty} (g_k + i\varepsilon_k Hg_k)(h_k + i\varepsilon_k Hh_k) \quad (g_k, h_k \in C_0^\infty, \varepsilon_k \in \{-1, 1\}).$$

Let  $H_\alpha^1$  be the Banach space of distributions with respect to norm  $\|\cdot\|_{H_\alpha^1}$ . Our theorem shows that the dual space of  $H_\alpha^1$  is  $\text{BMO}_\alpha$ . This corresponds to Fefferman’s duality theorem [2, p. 145]. Hence our theorem suggests that  $\text{BMO}_\alpha$  is useful in studying singular integrals from  $E_\alpha$  to itself. The authors express their thanks to Professors Y. Meyer and S. Semmes for some comments about commutators.

2. Throughout this note, we use  $C$  for various absolute constants and for various constants depending only on  $\alpha$ . For  $f \in L^1(dx/(1+x^2))$ , we write simply by  $f(x, y)$  its Poisson extension to  $C_+$ . Let  $\mathcal{M}f$  denote the non-centered maximal function of  $f$  [4, p. 6]. The “if” part is immediately deduced from the following known inequality.

LEMMA 1 ([3]).  $\int_0^\infty \text{Cap}_\alpha(x; \mathcal{M}f(x) > \lambda) \lambda d\lambda \leq C \|f\|_\alpha^2 \quad (f \in E_\alpha).$

Let  $f \in \text{BMO}_\alpha$ . Without loss of generality, we may assume that  $f$  is real-valued. For real-valued functions  $u, v \in C_0^\infty$ , we have

$$([f, H]u, v) = (H(fu) - fHu, v) = -(f, Hu \cdot v + uHv),$$

where  $(\cdot, \cdot)$  is the inner product (with respect to  $dx$ ). Put  $U = u - iHu$  and  $V = v - iHv$ . Then  $\|U\|_\alpha \leq 2\|u\|_\alpha$ ,  $\|V\|_{-\alpha} \leq 2\|v\|_{-\alpha}$ , and  $U(x, y), V(x, y)$  are analytic in  $C_+$ . Since

$$\left\{ \iint_{C_+} |V(x, y)|^2 y^{-1+\alpha} dx dy \right\}^{1/2} = C \|V\|_{-\alpha} \leq C \|v\|_{-\alpha},$$

we have

$$\begin{aligned} |( [f, H]u, v )| &= | \text{Im}(f, \overline{UV}) | \leq |(f, \overline{UV})| \\ &= C \left| \iint_{c_+} \frac{\partial f}{\partial x}(x, y) U(x, y) V(x, y) dx dy \right| \\ &\leq C \left\{ \iint_{c_+} |\nabla f(x, y)|^2 |U(x, y)|^2 y^{1-\alpha} dx dy \right\}^{1/2} \left\{ \iint_{c_+} |V(x, y)|^2 y^{-1+\alpha} dx dy \right\}^{1/2} \\ &\leq C \|v\|_{-\alpha} \left\{ \iint_{c_+} |\nabla f(x, y)|^2 |U(x, y)|^2 y^{1-\alpha} dx dy \right\}^{1/2}. \end{aligned}$$

Let  $O_\lambda = \{x; \mathcal{M}U(x) > \lambda\}$ ,  $O'_\lambda = \{(x, y); |U(x, y)| > \lambda\}$  ( $\lambda > 0$ ). Then  $O'_\lambda \subset \widehat{O}_{\eta\lambda}$  ( $\lambda > 0$ ) for some absolute constant  $\eta$  [4, p. 85]. Lemma 1 gives that

$$\begin{aligned} \iint_{c_+} |\nabla f(x, y)|^2 |U(x, y)|^2 y^{1-\alpha} dx dy &= C \int_0^\infty \left\{ \iint_{O'_\lambda} |\nabla f(x, y)|^2 y^{1-\alpha} dx dy \right\} \lambda d\lambda \\ &\leq C \int_0^\infty \left\{ \iint_{\widehat{O}_{\eta\lambda}} |\nabla f(x, y)|^2 y^{1-\alpha} dx dy \right\} \lambda d\lambda \leq C \|f\|_{\text{BMO}_\alpha}^2 \int_0^\infty \text{Cap}_\alpha(O_{\eta\lambda}) \lambda d\lambda \\ &\leq C \|f\|_{\text{BMO}_\alpha}^2 \|U\|_\alpha^2 \leq C \|f\|_{\text{BMO}_\alpha}^2 \|u\|_\alpha^2, \end{aligned}$$

which shows that

$$|( [f, H]u, v )| \leq C \|f\|_{\text{BMO}_\alpha} \|u\|_\alpha \|v\|_{-\alpha}.$$

Thus  $\| [f, H] \|_{\alpha, \alpha} \leq C \|f\|_{\text{BMO}_\alpha}$ , where  $\| [f, H] \|_{\alpha, \alpha}$  is the norm of  $[f, H]$  from  $E_\alpha$  to itself. This completes the proof of the "if" part.

**3.** The main part of this note is the proof of the "only if" part. We see easily the following lemma.

**LEMMA 2.** For  $f \in L^1(dx/(1+x^2))$ ,  $s > 0$ , we put  $f_s(x) = f(x, s)$  and  $F_s = f_s - iHf_s$ . Then

$$\frac{1}{2} \| [F_s, H] \|_{\alpha, \alpha} \leq \| [f_s, H] \|_{\alpha, \alpha} \leq \| [f, H] \|_{\alpha, \alpha}.$$

Let BMO denote the Banach space of functions  $f$ , modulo constants, with norm

$$\|f\|_{\text{BMO}} = \sup \frac{1}{|I|} \int_I |f(x) - (f)_I| dx,$$

where  $(f)_I$  is the mean of  $f$  over  $I$  and the supremum is taken over all intervals  $I$ . We show:

**LEMMA 3.**  $\|f\|_{\text{BMO}} \leq C \| [f, H] \|_{\alpha, \alpha}$ .

**PROOF.** For an interval  $I$ ,  $\chi$  denotes its characteristic function and  $\lambda(x) = (x - x_0)\chi(x)$ , where  $x_0$  is the midpoint of  $I$ . We have

$$\begin{aligned} |I| \int_I |f(x) - (f)_I| dx &= \int_I \left| \int_I (f(x) - f(y)) dy \right| dx \\ &= \int_I \left| \int_I \frac{f(y) - f(x)}{y - x} \{(y - x_0) - (x - x_0)\} dy \right| dx \\ &= \pi \int_{-\infty}^{\infty} |\chi(x)[f, H]\lambda(x) - \lambda(x)[f, H]\chi(x)| dx . \end{aligned}$$

Note that  $\|\chi\|_{\alpha} = C|I|^{(1-\alpha)/2}$  and  $\|\lambda\|_{\alpha} = C|I|^{(3-\alpha)/2}$ . Let  $g = |[f, H]\chi|$ . Then Parseval's equality shows that

$$\begin{aligned} \|g\|_{\alpha} &= C \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|g(x) - g(y)|^2}{|x - y|^{1+\alpha}} dx dy \right\}^{1/2} \\ &\leq C \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|[f, H]\chi(x) - [f, H]\chi(y)|^2}{|x - y|^{1+\alpha}} dx dy \right\}^{1/2} \\ &= C \|[f, H]\chi\|_{\alpha} \leq C \|[f, H]\|_{\alpha, \alpha} \|\chi\|_{\alpha} \leq C \|[f, H]\|_{\alpha, \alpha} |I|^{(1-\alpha)/2} , \end{aligned}$$

and hence

$$\begin{aligned} \int_{-\infty}^{\infty} \lambda(x)g(x)dx &\leq |I| \int_{-\infty}^{\infty} \chi(x)\overline{g(x)}dx = C|I| \int_{-\infty}^{\infty} \hat{\chi}(\xi)\overline{\hat{g}(\xi)}d\xi \\ &\leq C|I| \left\{ \int_{-\infty}^{\infty} |\xi|^{-\alpha} |\hat{\chi}(\xi)|^2 d\xi \right\}^{1/2} \left\{ \int_{-\infty}^{\infty} |\xi|^{\alpha} |\hat{g}(\xi)|^2 d\xi \right\}^{1/2} \\ &= C|I|^{1+(1+\alpha)/2} \|g\|_{\alpha} \leq C \|[f, H]\|_{\alpha, \alpha} |I|^2 . \end{aligned}$$

Let  $h = |[f, H]\lambda|$ . Then, in the same manner as above,

$$\|h\|_{\alpha} \leq C \|[f, H]\|_{\alpha, \alpha} \|\lambda\|_{\alpha} \leq C \|[f, H]\|_{\alpha, \alpha} |I|^{(3-\alpha)/2} ,$$

and hence

$$\int_{-\infty}^{\infty} \chi(x)h(x)dx \leq C|I|^{(1+\alpha)/2} \|h\|_{\alpha} \leq C \|[f, H]\|_{\alpha, \alpha} |I|^2 .$$

Consequently we have

$$\frac{1}{|I|} \int_I |f(x) - (f)_I| dx \leq C \|[f, H]\|_{\alpha, \alpha} ,$$

which shows that  $\|f\|_{\text{BMO}} \leq C \|[f, H]\|_{\alpha, \alpha}$ .

q.e.d.

Let  $L^2(1 - \alpha, C)$  denote the  $L^2$  space on the complex plane  $C$  with respect to the measure  $|y|^{1-\alpha} dx dy$ . The norm is denoted by  $\|\cdot\|_{L^2(1-\alpha, C)}$ . Let  $T^*$  be an operator defined by

$$T^*u(x, y) = \sup_{\varepsilon > 0} \left| \iint_{|\zeta - z| > \varepsilon} \frac{u(s, t)}{(\zeta - z)^2} ds dt \right| \quad (\zeta = s + it, z = x + iy) .$$

The 2-dimensional (centered) maximal operator  $\tilde{\mathcal{M}}$  is defined by

$$\tilde{\mathcal{N}}u(x, y) = \sup_{\varepsilon > 0} \frac{1}{\pi\varepsilon^2} \iint_{D((x,y),\varepsilon)} |u(s, t)| dsdt ,$$

where  $D((x, y), \varepsilon)$  is the open disk of center  $(x, y)$  and of radius  $\varepsilon$ . It is well-known that  $T^*$ ,  $\tilde{\mathcal{N}}$  are bounded operators from  $L^2(1 - \alpha, C)$  to itself [4, pp. 21 and 56]. For  $\beta \in \mathbf{R}$  and a function  $u(x, y)$  in  $C_+$ , we write

$$|||u|||_\beta = \left\{ \iint_{C_+} |u(x, y)|^2 y^\beta dx dy \right\}^{1/2} .$$

We show:

LEMMA 4. Let  $g(x, y)$  be a function in  $C_+$  such that  $|||g|||_{1+\alpha} < \infty$ . Then there exists a function  $h(x, y)$  in  $C_+$  such that

$$\bar{\partial}h(x, y) = g(x, y) \quad ((x, y) \in C_+) , \quad |||h|||_{-1+\alpha} \leq C |||g|||_{1+\alpha} ,$$

where  $\bar{\partial} = (\partial/\partial x + i\partial/\partial y)/2$ .

PROOF. We put

$$h(x, y) = \frac{2i}{\pi} \iint_{C_+} \frac{g(s, t)t}{(\zeta - z)(\bar{\zeta} - \bar{z})} dsdt \quad (\zeta = s + it, z = x + iy) .$$

(This form was communicated by Professor S. Semmes; in the earlier draft, the authors did not use this form.) Then  $\bar{\partial}h(x, y) = g(x, y)$   $((x, y) \in C_+)$ . Suppose that the support of  $g$  is compact and contained in  $C_+$ . Then we have easily  $|||h|||_{-1+\alpha} < \infty$ . For  $u \in L^2(1 - \alpha, C)$ , we have

$$\left| \iint_{C_+} h(x, y)u(x, y) dx dy \right| = \left| \iint_{C_+} g(s, t)Su(s, t)tdsdt \right| \leq |||g|||_{1+\alpha} |||Su|||_{1-\alpha} ,$$

where

$$Su(s, t) = \frac{2}{\pi} \iint_{C_+} \frac{u(x, y)}{(\zeta - z)(\bar{\zeta} - \bar{z})} dx dy \quad (\zeta = s + it, z = x + iy) .$$

We have, for  $(s, t) \in C_+$ ,

$$Su(s, t) = \frac{2}{\pi} \iint_{D((s,t),t)} + \frac{2}{\pi} \iint_{C_+ - D((s,t),t)} = L_1 + L_2 ,$$

$$|L_1| \leq \frac{C}{t} \iint_{D((s,t),t)} \frac{|u(x, y)|}{|\zeta - z|} dx dy \leq C \tilde{\mathcal{N}}u(s, t)$$

and

$$|L_2| \leq \frac{2}{\pi} \left| \iint_{C_+ - D((s,t),t)} \left\{ \frac{1}{(\zeta - z)(\bar{\zeta} - \bar{z})} - \frac{1}{(\zeta - z)^2} \right\} u(x, y) dx dy \right|$$

$$+ \frac{2}{\pi} T^*u(s, t) \leq C \tilde{\mathcal{N}}u(s, t) + \frac{2}{\pi} T^*u(s, t) .$$

Hence

$$|Su(s, t)| \leq C \tilde{\mathcal{M}}u(s, t) + \frac{2}{\pi} T^*u(s, t) \quad ((s, t) \in C_+).$$

Consequently

$$(1) \quad \left| \iint_{C_+} h(x, y)u(x, y)dx dy \right| \leq C \|g\|_{1+\alpha} \{ \|\tilde{\mathcal{M}}u\|_{L^2(1-\alpha, C)} + \|T^*u\|_{L^2(1-\alpha, C)} \} \leq C \|g\|_{1+\alpha} \|u\|_{L^2(1-\alpha, C)}.$$

We now choose

$$u(x, y) = \begin{cases} \overline{h(x, y)}y^{-1+\alpha} & ((x, y) \in C_+) \\ 0 & ((x, y) \in C - C_+). \end{cases}$$

Since  $\|u\|_{L^2(1-\alpha, C)} = \|h\|_{-1+\alpha} < \infty$ , (1) yields that  $\|h\|_{-1+\alpha} \leq C \|g\|_{1+\alpha}$ .

In the general case, we restrict  $g$  to  $\{(x, y); |x| \leq n, 1/n \leq y \leq n\}$ ; say  $g_n$ . Let  $h_n$  be the function corresponding to  $g_n$ . Then

$$\|h_n\|_{-1+\alpha} \leq C \|g_n\|_{1+\alpha} \leq C \|g\|_{1+\alpha}.$$

Letting  $n$  tend to infinity, we obtain  $\|h\|_{-1+\alpha} \leq C \|g\|_{1+\alpha}$ . q.e.d.

**LEMMA 5.** *Let  $f$  be a differentiable function on  $\mathbf{R}$  satisfying  $\sup\{|f(x)| + |f'(x)|; x \in \mathbf{R}\} < \infty$ . We put*

$$D^\alpha f(x) = -i \int_0^\infty \frac{f(x+t) - f(x-t)}{t^{1+\alpha}} dt.$$

Then, for any  $u \in C_0^\infty$ ,

$$\left\{ \iint_{C_+} |(D^\alpha f)(x, y)|^2 |\nabla U(x, y)|^2 y^{1+\alpha} dx dy \right\}^{1/2} \leq C \|f\|_{\text{BMO}} \|u\|_\alpha,$$

where  $U = u - iHu$ .

**PROOF.** Since

$$\|\nabla U\|_{1-\alpha} = C \|U\|_\alpha \leq C \|u\|_\alpha,$$

it is sufficient to show that

$$(2) \quad |(D^\alpha f)(x, y)| \leq C \|f\|_{\text{BMO}}/y^\alpha \quad ((x, y) \in C_+).$$

Without loss of generality, we may assume that  $x = 0$ . We have, with  $I = (-y, y)$ ,

$$\begin{aligned} |(D^\alpha f)(0, y)| &= \frac{1}{\pi} \left| \int_0^\infty \frac{1}{t^{1+\alpha}} \left\{ \int_{-\infty}^\infty \left( \frac{y}{(s-t)^2 + y^2} - \frac{y}{(s+t)^2 + y^2} \right) (f(s) - (f)_I) ds \right\} dt \right| \\ &\leq \frac{1}{\pi} \left| \int_0^{2y} \frac{1}{t^{1+\alpha}} \left\{ \int_{-\infty}^\infty \right\} dt \right| + \frac{1}{\pi} \left| \int_{2y}^\infty \frac{1}{t^{1+\alpha}} \left\{ \int_I \right\} dt \right| + \frac{1}{\pi} \left| \int_{2y}^\infty \frac{1}{t^{1+\alpha}} \left\{ \int_{I^c} \right\} dt \right| \\ &= L_1 + L_2 + L_3. \end{aligned}$$

Since

$$\int_{-\infty}^{\infty} \frac{y}{s^2 + y^2} |f(s) - (f)_I| ds \leq C \|f\|_{\text{BMO}} \quad ([2, \text{p. 142}] ),$$

$$|L_1| \leq C \int_0^{2y} \frac{1}{t^{1+\alpha}} \left\{ \frac{t}{y} \int_{-\infty}^{\infty} \frac{y}{s^2 + y^2} |f(s) - (f)_I| ds \right\} dt$$

$$\leq C \int_0^{2y} \frac{1}{t^\alpha} dt \|f\|_{\text{BMO}}/y \leq C \|f\|_{\text{BMO}}/y^\alpha .$$

We have

$$|L_2| \leq C \int_{2y}^{\infty} \frac{1}{t^{1+\alpha}} \left\{ \frac{1}{y} \int_I |f(s) - (f)_I| ds \right\} dt \leq C \|f\|_{\text{BMO}}/y^\alpha .$$

It remains to estimate  $L_3$ . We have

$$\int_{2y}^{\infty} \frac{1}{t^{1+\alpha}} \left\{ \int_I \frac{y}{(s-t)^2 + y^2} (f(s) - (f)_I) ds \right\} dt$$

$$= \int_{2y}^{\infty} \frac{1}{t^{1+\alpha}} \left\{ \int_{-\infty}^{-y} \right\} dt + \sum_{n=1}^{\infty} \int_{2^n y}^{2^{n+1} y} \frac{1}{t^{1+\alpha}} \left\{ \int_y^{2^{n-1} y} \right\} dt$$

$$+ \sum_{n=1}^{\infty} \int_{2^n y}^{2^{n+1} y} \frac{1}{t^{1+\alpha}} \left\{ \int_{2^{n-1} y}^{2^{n+2} y} \right\} dt + \sum_{n=1}^{\infty} \int_{2^n y}^{2^{n+1} y} \frac{1}{t^{1+\alpha}} \left\{ \int_{2^{n+2} y}^{\infty} \right\} dt$$

$$= L_{31} + L_{32} + L_{33} + L_{34} .$$

We have easily  $|L_{31}| \leq C \|f\|_{\text{BMO}}/y^\alpha$ . Since

$$|(f)_I - (f)_{(y, 2^{n-1}y)}| \leq Cn \|f\|_{\text{BMO}} \quad (n \geq 2) \quad ([2, \text{p. 142}] ),$$

$$|L_{32}| \leq C \sum_{n=2}^{\infty} \int_{2^n y}^{2^{n+1} y} \frac{1}{t^{1+\alpha}} \left\{ \frac{1}{2^n y} \int_y^{2^{n-1} y} |f(s) - (f)_I| ds \right\} dt$$

$$\leq C \sum_{n=2}^{\infty} \int_{2^n y}^{2^{n+1} y} \frac{1}{t^{1+\alpha}} \left\{ \frac{1}{2^n y} \int_y^{2^{n-1} y} |f(s) - (f)_{(y, 2^{n-1}y)}| ds + n \|f\|_{\text{BMO}} \right\} dt$$

$$\leq C \|f\|_{\text{BMO}} \sum_{n=2}^{\infty} n \int_{2^n y}^{2^{n+1} y} \frac{1}{t^{1+\alpha}} dt \leq C \|f\|_{\text{BMO}}/y^\alpha .$$

We have

$$|L_{34}| \leq C \sum_{n=1}^{\infty} \int_{2^n y}^{2^{n+1} y} \frac{1}{t^{1+\alpha}} \left\{ \int_{2^{n+2} y}^{\infty} \frac{y}{s^2 + y^2} |f(s) - (f)_I| ds \right\} dt$$

$$\leq C \int_{2y}^{\infty} \frac{1}{t^{1+\alpha}} dt \int_{-\infty}^{\infty} \frac{y}{s^2 + y^2} |f(s) - (f)_I| ds \leq C \|f\|_{\text{BMO}}/y^\alpha .$$

Since  $|(f)_I - (f)_{(t-y, t+y)}| \leq Cn \|f\|_{\text{BMO}} \quad (t \in (2^n y, 2^{n+1} y))$ ,

$$\begin{aligned}
|L_{33}| &\leq \sum_{n=1}^{\infty} \int_{2^{n_y}}^{2^{n+1}_y} \frac{1}{t^{1+\alpha}} \left\{ \int_{2^{n-1}_y(s-t)^2+y^2}^{2^{n+2}_y} \frac{y}{y^2} |f(s) - (f)_{(t-y, t+y)}| ds + Cn \|f\|_{\text{BMO}} \right\} dt \\
&\leq \sum_{n=1}^{\infty} \int_{2^{n_y}}^{2^{n+1}_y} \frac{1}{t^{1+\alpha}} \left\{ \int_{-\infty}^{\infty} \frac{y}{(s-t)^2+y^2} |f(s) - (f)_{(t-y, t+y)}| ds \right\} dt + C \|f\|_{\text{BMO}}/y^\alpha \\
&\leq C \|f\|_{\text{BMO}} \left\{ \sum_{n=1}^{\infty} \int_{2^{n_y}}^{2^{n+1}_y} \frac{1}{t^{1+\alpha}} dt + 1/y^\alpha \right\} \leq C \|f\|_{\text{BMO}}/y^\alpha.
\end{aligned}$$

Consequently,

$$(3) \quad \left| \int_{2^y}^{\infty} \frac{1}{t^{1+\alpha}} \left\{ \int_I \frac{y}{(s-t)^2+y^2} (f(s) - (f)_I) ds \right\} dt \right| \leq C \|f\|_{\text{BMO}}/y^\alpha.$$

In the same manner, we have (3) with  $(s-t)^2+y^2$  replaced by  $(s+t)^2+y^2$ . Thus  $|L_3| \leq C \|f\|_{\text{BMO}}/y^\alpha$ . This completes the proof of (2). q.e.d.

We now show the main lemma in this note.

**LEMMA 6.** *Let  $F_s$  be the function in Lemma 2. Then, for any  $u \in C_0^\infty$ ,*

$$\left\{ \iint_{C_+} |(D^\alpha F_s)(x, y)|^2 |U(x, y)|^2 y^{1-\alpha} dx dy \right\}^{1/2} \leq C \|[f, H]\|_{\alpha, \alpha} \|u\|_\alpha,$$

where  $U = u - iHu$ .

**PROOF.** Let  $V$  be a function in  $L^1(dx/(1+x^2))$  such that  $V(x, y)$  is analytic in  $C_+$ . Parseval's equality and Lemma 2 show that

$$\begin{aligned}
(4) \quad &\left| \iint_{C_+} \overline{(D^\alpha F_s)(x, y)} U(x, y) V(x, y) y^{-1+\alpha} dx dy \right| = C |(F_s, UV)| \\
&= C |(F_s, H)\bar{U}, V| \leq C \|[F_s, H]\|_{\alpha, \alpha} \|\bar{U}\|_\alpha \|V\|_{-\alpha} \\
&\leq C \|[f, H]\|_{\alpha, \alpha} \|u\|_\alpha \|V\|_{-\alpha}.
\end{aligned}$$

Let  $g(x, y) = (D^\alpha F_s)(x, y) \overline{\partial U(x, y)}$  ( $\partial = (\partial/\partial x - i\partial/\partial y)/2$ ). Then Lemmas 2, 3 and 5 show that

$$\|g\|_{1+\alpha} \leq C \|F_s\|_{\text{BMO}} \|U\|_\alpha \leq C \|[F_s, H]\|_{\alpha, \alpha} \|u\|_\alpha \leq C \|[f, H]\|_{\alpha, \alpha} \|u\|_\alpha.$$

Let  $h(x, y)$  be the function associated with  $g(x, y)$  in Lemma 4. We put

$$(5) \quad V_0(x, y) = (D^\alpha F_s)(x, y) \overline{U(x, y)} - h(x, y).$$

Then  $V_0(x, y)$  is analytic in  $C_+$  ( $V_0(x, y)$  is an analytic extension of a function in  $L^1(dx/(1+x^2))$ , say  $V_0(x)$ ) and

$$\begin{aligned}
\|V_0\|_{-\alpha} &= C \|[V_0]\|_{-1+\alpha} \leq C \|[D^\alpha F_s]\bar{U}\|_{-1+\alpha} + C \|h\|_{-1+\alpha} \\
&\leq C \|[D^\alpha F_s]U\|_{-1+\alpha} + C \|[f, H]\|_{\alpha, \alpha} \|u\|_\alpha.
\end{aligned}$$

Hence, by (4),



$$\begin{aligned} \|(D^\alpha F_s)U\|_{-1+\alpha}^2 &= \left| \iint_{c_+} \overline{(D^\alpha F_s)(x, y)} U(x, y) \{V_0(x, y) + h(x, y)\} y^{-1+\alpha} dx dy \right| \\ &\leq C \|[f, H]\|_{\alpha, \alpha} \|u\|_\alpha \|V_0\|_{-\alpha} + \|(D^\alpha F_s)U\|_{-1+\alpha} \|h\|_{-1+\alpha} \\ &\leq C \{ \|(D^\alpha F_s)U\|_{-1+\alpha} \|[f, H]\|_{\alpha, \alpha} \|u\|_\alpha + \|[f, H]\|_{\alpha, \alpha}^2 \|u\|_\alpha^2 \}, \end{aligned}$$

which shows that

$$\|(D^\alpha F_s)U\|_{-1+\alpha} \leq C \|[f, H]\|_{\alpha, \alpha} \|u\|_\alpha. \quad \text{q.e.d.}$$

LEMMA 7. Let  $F_s$ ,  $u$  and  $U$  be the same as in Lemma 6. We put

$$U_{F_s}(x, y) = \int_0^\infty (U(x, y + t) - U(x, y))(D^\alpha F_s)(x, y + t)t^{-2+\alpha} dt.$$

Then

$$\|U_{F_s}\|_{1-\alpha} \leq C \|[f, H]\|_{\alpha, \alpha} \|u\|_\alpha.$$

PROOF. We may assume that  $\|[f, H]\|_{\alpha, \alpha} = 1$ . Inequality (2) combined with Lemmas 2 and 3 shows that

$$\begin{aligned} |U_{F_s}(x, y)|^2 &\leq C \left\{ \int_0^\infty \frac{|U(x, y + t) - U(x, y)|}{t^{2-\alpha}(y + t)^\alpha} dt \right\}^2 \\ &= C \left\{ \int_0^y + \int_y^\infty \right\}^2 \leq C \left\{ y^{-\alpha} \int_0^y |U(x, y + t) - U(x, y)|^2 t^{-3+\alpha} dt \right. \\ &\quad \left. + y^{-1} \int_y^\infty |U(x, y + t) - U(x, y)|^2 t^{-2} dt \right\}, \end{aligned}$$

and hence

$$\begin{aligned} \|U_{F_s}\|_{1-\alpha}^2 &\leq C \iint_{c_+} \left\{ \int_0^\infty |U(x, y + t) - U(x, y)|^2 t^{-3+\alpha} dt \right\} y^{1-2\alpha} dx dy \\ &\quad + C \iint_{c_+} \left\{ \int_0^\infty |U(x, y + t) - U(x, y)|^2 t^{-2} dt \right\} y^{-\alpha} dx dy \\ &= C \int_0^\infty |\hat{U}(\xi)|^2 \left( \int_0^\infty e^{-2\xi y} y^{1-2\alpha} dy \right) \left( \int_0^\infty |e^{-\xi t} - 1|^2 t^{-3+\alpha} dt \right) d\xi \\ &\quad + C \int_0^\infty |\hat{U}(\xi)|^2 \left( \int_0^\infty e^{-2\xi y} y^{-\alpha} dy \right) \left( \int_0^\infty |e^{-\xi t} - 1|^2 t^{-2} dt \right) d\xi \\ &= C \int_0^\infty \xi^\alpha |\hat{U}(\xi)|^2 d\xi = C \|U\|_\alpha^2 \leq C \|u\|_\alpha^2. \quad \text{q.e.d.} \end{aligned}$$

4. We now show the “only if” part. Let  $f \in L^1(dx/(1 + x^2))$  satisfy  $\|[f, H]\|_{\alpha, \alpha} < \infty$ . We may assume that  $f$  is real-valued. Let  $F_s$ ,  $u$ ,  $U$ ,  $V_0$  be the functions in Lemma 6 and (5). Then

$$L = \left| \iint_{c_+} \overline{\left(\frac{\partial}{\partial x} F_s\right)(x, y)} U(x, y) V_0(x, y) dx dy \right|$$

$$\begin{aligned}
 &= C|(F_s, UV_0)| = C|([F_s, H]\bar{U}, V_0)| \\
 &\leq C\|[f, H]\|_{\alpha, \alpha} \|u\|_{\alpha} \|V_0\|_{-\alpha} \\
 &\leq C\|[f, H]\|_{\alpha, \alpha} \|u\|_{\alpha} \{ \|(D^\alpha F_s)U\|_{-1+\alpha} + \|[f, H]\|_{\alpha, \alpha} \|u\|_{\alpha} \} \\
 &\leq C\|[f, H]\|_{\alpha, \alpha}^2 \|u\|_{\alpha}^2
 \end{aligned}$$

and

$$\begin{aligned}
 L &\geq \left| \iint_{c_+} \overline{\left(\frac{\partial}{\partial x} F_s\right)(x, y) U(x, y) (D^\alpha F_s)(x, y) \bar{U}(x, y)} dx dy \right| \\
 &\quad - \left\| \left(\frac{\partial}{\partial x} F_s\right) U \right\|_{1-\alpha} \|h\|_{-1+\alpha} \\
 &\geq \left| \iint_{c_+} \overline{\left(\frac{\partial}{\partial x} F_s\right)(x, y) U(x, y) U(x, y) (D^\alpha F_s)(x, y)} dx dy \right| \\
 &\quad - C \left\| \left(\frac{\partial}{\partial x} F_s\right) U \right\|_{1-\alpha} \|[f, H]\|_{\alpha, \alpha} \|u\|_{\alpha} .
 \end{aligned}$$

We put  $G(x, y) = U(x, y)(D^\alpha F_s)(x, y)$  and

$$\tilde{D}^{1-\alpha} G(x, y) = - \int_0^\infty (G(x, y+t) - G(x, y)) t^{-2+\alpha} dt .$$

Since

$$\begin{aligned}
 \tilde{D}^{1-\alpha} G(x, y) &= -U(x, y) \int_0^\infty ((D^\alpha F_s)(x, y+t) - (D^\alpha F_s)(x, y)) t^{-2+\alpha} dt \\
 &\quad - U_{F_s}(x, y) = CU(x, y) \left(\frac{\partial}{\partial x} F_s\right)(x, y) - U_{F_s}(x, y) ,
 \end{aligned}$$

we have, by Lemma 7,

$$\begin{aligned}
 &\left| \iint_{c_+} \overline{\left(\frac{\partial}{\partial x} F_s\right)(x, y) U(x, y) U(x, y) (D^\alpha F_s)(x, y)} dx dy \right| \\
 &= \left| \iint_{c_+} \overline{\left(\frac{\partial}{\partial x} F_s\right)(x, y) U(x, y) \tilde{D}^{1-\alpha} G(x, y) y^{1-\alpha}} dx dy \right| \\
 &= \left| C \iint_{c_+} \left| \left(\frac{\partial}{\partial x} F_s\right)(x, y) \right|^2 |U(x, y)|^2 y^{1-\alpha} dx dy \right. \\
 &\quad \left. - \iint_{c_+} \overline{\left(\frac{\partial}{\partial x} F_s\right)(x, y) U(x, y) U_{F_s}(x, y) y^{1-\alpha}} dx dy \right| \\
 &\geq C \left\| \left(\frac{\partial}{\partial x} F_s\right) U \right\|_{1-\alpha}^2 - \left\| \left(\frac{\partial}{\partial x} F_s\right) U \right\|_{1-\alpha} \|U_{F_s}\|_{1-\alpha} \\
 &\geq C \|\nabla F_s|U\|_{1-\alpha}^2 - C \|\nabla F_s|U\|_{1-\alpha} \|[f, H]\|_{\alpha, \alpha} \|u\|_{\alpha} .
 \end{aligned}$$

Thus

$||| |\nabla F_s| U |||_{1-\alpha}^2 \leq C ||| |\nabla F_s| U |||_{1-\alpha} \| [f, H] \|_{\alpha, \alpha} \| u \|_{\alpha} + C \| [f, H] \|_{\alpha, \alpha}^2 \| u \|_{\alpha}^2$ ,  
 which gives that

$$(6) \quad \iint_{c_+} |\nabla F_s(x, y)|^2 |U(x, y)|^2 y^{1-\alpha} dx dy \leq C \| [f, H] \|_{\alpha, \alpha}^2 \| u \|_{\alpha}^2 \quad (u \in C_0^\infty).$$

The standard argument shows that (6) holds for any  $u \in E_\alpha$ .

Let  $O$  be an open set in  $\mathbf{R}$  with  $\text{Cap}_\alpha(O) < \infty$ . Then there exists a non-negative function  $u_o \in E_\alpha$  such that  $\| u_o \|_{\alpha}^2 = \text{Cap}_\alpha(O)$  and  $u_o(x) \geq C$  on  $O$  [5, p. 138]. Let  $U_o = u_o - iHu_o$ . Then  $|U_o(x, y)| \geq C$  on  $\hat{O}$ . Hence (6) shows that

$$\begin{aligned} \iint_{\hat{O}} |\nabla F_s(x, y)|^2 y^{1-\alpha} dx dy &\leq C ||| |\nabla F_s| U_o |||_{1-\alpha}^2 \\ &\leq C \| [f, H] \|_{\alpha, \alpha}^2 \| u_o \|_{\alpha}^2 = C \| [f, H] \|_{\alpha, \alpha}^2 \text{Cap}_\alpha(O). \end{aligned}$$

Thus

$$\begin{aligned} \iint_{\hat{O}} |\nabla f(x, y + s)|^2 y^{1-\alpha} dx dy &= \iint_{\hat{O}} |\nabla f_s(x, y)|^2 y^{1-\alpha} dx dy \\ &\leq \iint_{\hat{O}} |\nabla F_s(x, y)|^2 y^{1-\alpha} dx dy \leq C \| [f, H] \|_{\alpha, \alpha}^2 \text{Cap}_\alpha(O). \end{aligned}$$

Letting  $s$  tend to 0,

$$\iint_{\hat{O}} |\nabla f(x, y)|^2 y^{1-\alpha} dx dy \leq C \| [f, H] \|_{\alpha, \alpha}^2 \text{Cap}_\alpha(O).$$

Since  $O$  is arbitrary, we have  $\| f \|_{\text{BMO}_\alpha} \leq C \| [f, H] \|_{\alpha, \alpha}$ . This completes the proof of the "only if" part.

REFERENCES

[1] R. R. COIFMAN, R. ROCHBERG AND G. WEISS, Factorization theorems for Hardy spaces in several variables, *Ann. of Math.* 103 (1976), 611-625.  
 [2] C. FEFFERMAN AND E. M. STEIN,  $H^p$  spaces of several variables, *Acta Math.* 129 (1972), 137-193.  
 [3] K. HANSSON, Imbedding theorems of Sobolev type in potential theory, *Math. Scand.* 45 (1979), 77-102.  
 [4] J. L. JOURNÉ, Calderón-Zygmund operators, pseudo-differential operators and the Cauchy integral of Calderón, *Lecture Notes in Math.*, 994, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.  
 [5] N. S. LANDKOF, Foundations of modern potential theory, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1972.  
 [6] D. A. STEGENGA, Multipliers of the Dirichlet space, *Illinois J. of Math.* 24 (1980), 113-119.

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