COMMUTING MAPS ON LIE IDEALS

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Abstract. Let R be a ring and let A be a subset of R. A map $f : A \to R$ is commuting on A if $[f(x), x] = 0$ for all $x \in A$ where $[x, y] = xy - yx$. Suppose that R is a prime ring of characteristic $\neq 2$ with extended centroid C. If L is a noncommutative Lie ideal of R and $f : L \to R$ an additive commuting map, then there is $\lambda \in C$ and an additive map $\xi : L \to C$ such that $f(v) = \lambda v + \xi(v)$ for all $v \in L$.

1. Introduction

Let R be a ring with centre Z and let A be a subset of R. A map $f : A \to R$ is *centralizing* on A if $[f(x), x] \in Z$ for all $x \in A$. (Here

$[x, y] = xy - yx$ is the Lie bracket.) In the special case where $[f(x), x] = 0$ for all $x \in A$ we say that f is *commuting* on A. The study of commuting maps was initiated by Posner [9] who proved that the existence of a nonzero commuting derivation on a prime ring R implies that R is commutative. Other authors have extended this theorem to various maps f and subsets A. (See [1].)

In this paper we characterize additive commuting maps $f : L \to R$ where R is a prime ring of characteristic $\neq 2$, and L is a noncommutative Lie ideal. Recall that a *Lie ideal* is an additive subgroup, L, of R such that if $v \in L$ and $x \in R$ then $[v, x] \in L$. Our main result is as follows:

Let R be a prime ring of characteristic $\neq 2$ with extended centroid C. If L is a noncommutative Lie ideal of R and $f : L \to R$ is an additive commuting map, then there is $\lambda \in C$ and an additive map $\xi : L \to C$ such that $f(v) = \lambda v + \xi(v)$ for all $v \in L$.

This result extends the result of [1] from prime rings to Lie ideals of prime rings. The methods used here are quite different.

Our approach to this problem is to deal separately with the case when R satisfies a generalized polynomial identity (we say R is GPI) and when it does not. In the case that R is not GPI we prove:

Let R be a prime ring of characteristic $\neq 2$. If $f : R \to R$ is additive and $d \neq 0$ an inner derivation such that $[f(x), d(x)] = 0$ for all $x \in R$ then there is $\lambda \in C$ and an additive map $\xi : R \to C$ such that $f(x) = \lambda\, d(x) + \xi(x)$ for all $x \in R$.

The non-GPI result for commuting maps on Lie ideals follows quickly from this theorem. It is a curiosity that we have not been able to prove this result in the GPI case even when $R = M_n(F)$, the $n \times n$ matrices over a field.

In the GPI case we can assume that R is a primitive algebra over a field with non-zero socle. We then use a well-known approach of Jacobson to such rings to prove the result for commuting maps on Lie ideals.

2. Preliminaries

In all that follows R will be a prime ring. By $Q_\ell(R)$ we denote the *left Martindale ring of quotients* of R. $Q_\ell(R)$ can be characterized by the following four properties [8]:

(i) $R \subseteq Q_\ell(R)$,

(ii) for $q \in Q_\ell(R)$ there exists a nonzero ideal I of R such that $Iq \subseteq R$,

(iii) if $q \in Q_\ell(R)$ and $Iq = 0$ for some nonzero ideal I of R then $q = 0$,

(iv) if I is a nonzero ideal of R and $\phi : I \to R$ is a left R-module map, then there is $q \in Q_\ell(R)$ such that $\phi(x) = xq$ for all $x \in I$.

One can, of course, characterize $Q_r(R)$, the *right Martindale ring of quotients* of R in a similar manner. The ring

$$Q_s(R) = \{q \in Q_\ell(R) \mid qI \subseteq R \text{ for some nonzero ideal } I \text{ of R}\}$$

$$\simeq \{q \in Q_r(R) \mid Iq \subseteq R \text{ for some nonzero ideal } I \text{ of } R\}$$

is called the *symmetric Martindale ring of quotients of R*. The centre C of $Q_s(R)$ is a field and it is the centre of both $Q_\ell(R)$ and $Q_r(R)$. It is called the *extended centroid* of R. The ring $R_C = RC + C$ is called the *central closure* of R. Note that $R \subseteq R_C \subseteq Q_s(R)$. It is easy to see that $q_1 R q_2 = 0$ where $q_1, q_2 \in Q_\ell(R)$ or $q_1, q_2 \in Q_r(R)$, implies that $q_1 = 0$ or $q_2 = 0$. In particular this shows that all of R_C, $Q_s(R)$, $Q_\ell(R)$, and $Q_r(R)$ are prime rings so that one can construct the (left, right, symmetric) Martindale ring of quotients and the central closure of each of these rings. It is well-known that R_C is equal to its central closure. In general a prime ring is *closed over a field F* if F is both the centroid and extended centroid of R, or equivalently, R is an algebra over F where F is the extended centroid. In particular the central closure of R is closed over C.

Let $R_C *_C C\{X\}$ be the free product over C of R_C and the free algebra over C on an infinite set, X, of indeterminates. A typical element in $R_C *_C C\{X\}$ is a sum of monomials of the form $\lambda a_{i_0} x_{j_1} a_{i_1} x_{j_2} \cdots x_{j_n} a_{i_n}$ where $\lambda \in C$, $a_{i_k} \in R_C$ and $x_{j_k} \in X$. R is said to satisfy a *generalized polynomial identity* over C (or R is GPI) if there exists a nonzero polynomial $p(x_1, x_2, \cdots, x_n) \in R_C *_C C\{X\}$ such that $p(r_1, r_2, \cdots, r_n) = 0$ for all $r_1, r_2, \cdots, r_n \in R$. A well-known theorem of Martindale [7, Th. 3] states that

> *A prime ring R satisfies a* GPI *if and only if R_C is a primitive ring with nonzero socle and eR_Ce is a finite dimensional division algebra over C for each primitive idempotent e in R_C.*

Given a prime ring R, it is frequently useful to construct the ring $\tilde{R} = R_C \otimes_C F$ where F is an algebraic closure of C. In this case it is known that $\tilde{R}$ is a closed prime algebra over F [3]. Moreover, if R is GPI, so is $\tilde{R}$ and $e\tilde{R}e \simeq F$.

Finally we have need of the following result:

Theorem 2.1 [2, Main Theorem]. *Let R be a prime ring and let n, m, k, ℓ be positive integers. Suppose that*

$$\sum_{i=1}^{n} F_i(y)xa_i + \sum_{i=1}^{m} G_i(x)yb_i + \sum_{i=1}^{k} c_iyH_i(x) + \sum_{i=1}^{\ell} d_ixK_i(y) = 0$$

for all $x, y \in R$, where $F_i, G_i, H_i, K_i : R \to R_C$ are additive maps and $\{a_1, \cdots, a_n\}$, $\{b_1, \cdots, b_m\}$, $\{c_1, \cdots, c_k\}$, $\{d_1, \cdots, d_\ell\}$ are C-independent subsets of R. Then one of the following two possibilities holds:

(i) *R_C is a primitive ring with nonzero socle and eR_Ce is a finite dimensional division algebra over C for each primitive idempotent e in R_C (that is, R is a* GPI *ring),*

(ii) *there exist elements* $q_{ij} \in Q_s(R_C)$, $i = 1, \cdots, \ell$, $j = 1, \cdots, m$, $p_{ij} \in Q_s(R_C)$, $i = 1, \cdots, k$, $j = 1, \cdots, n$, *and additive maps* $\lambda_{ij} : R \to C$, $i = 1, \cdots, \ell$, $j = 1, \cdots, n$, $\mu_{ij} : R \to C$, $i = 1, \cdots, m$, $j = 1, \cdots, k$, *such that*

$$F_i(y) = \sum_{j=1}^{k} c_j y p_{ji} + \sum_{j=1}^{\ell} \lambda_{ji}(y) d_j \textit{ for all } y \in R, \quad i = 1, \cdots, n,$$

$$G_i(x) = \sum_{j=1}^{\ell} d_j x q_{ji} - \sum_{j=1}^{k} \mu_{ij}(x) c_j \textit{ for all } x \in R, \quad i = 1, \cdots, m,$$

$$H_i(x) = -\sum_{j=1}^{n} p_{ij} x a_j + \sum_{j=1}^{m} \mu_{ji}(x) b_j \textit{ for all } x \in R, \quad i = 1, \cdots, k,$$

$$K_i(y) = -\sum_{j=1}^{m} q_{ij} y b_j - \sum_{j=1}^{n} \lambda_{ij}(y) a_j \textit{ for all } y \in R, \quad i = 1, \cdots, \ell.$$

In all that follows R will be a prime ring of characteristic $\neq 2$.

3. Maps Commuting with Derivations – The Non-GPI Case

Lemma 3.1. *Suppose R is not* GPI. *Let $p_i, q_i \in Q_s(R_C)$, $i = 1, \cdots, n$, and let $f(x) = \sum_{i=1}^{n} p_i x q_i$. The following conditions are equivalent:*

(i) *There is a finite dimensional subspace V of R_C such that $f(x) \in V$ for every $x \in R$.*

(ii) *There is an ideal $J \neq \{0\}$ of R_C such that $f(x) = 0$ for all $x \in J$.*

(iii) $\sum_{i=1}^{n} p_i \otimes q_i = 0$ *as an element of* $Q_s(R_C) \otimes_C Q_s(R_C)$.

(iv) $f(x) = 0$ *as an element of* $Q_s(R_C) *_C C[x]$.

Proof. [6, Lemma 1] tells us that (ii), (iii), and (iv) are equivalent whether or not R is GPI. Of course (iv) implies (i). Thus we need show only that (i) implies (ii).

Assuming (i), we clearly have $f(x) \in V$ for every $x \in R_C$ as well. Pick an ideal $J_0 \neq \{0\}$ of R_C such that $p_i J_0, J_0 q_i \subseteq R_C$ and let $J = J_0 R_C J_0$.

Then J is a nonzero ideal of R_C. Fix $u, v \in J_0$. For each $r \in R_C$ we have $\sum_{i=1}^{n}(p_i u) r(vq_i) = f(urv) \in V$. If $\{p_i u \mid i = 1, \cdots, n\} \neq \{0\}$ choose a maximal independent subset which, by renumbering, we assume is $\{p_i u \mid i = 1, \cdots, m\}$. We then rewrite

$$f(urv) = \sum_{i=1}^{m}(p_i u) r(vq_i').$$

If $vq_i' \neq 0$ for some i, then R is GPI by [7, Th. 2]. Hence $f(urv) = 0$ and it follows that $f(J) = 0$.

Theorem 3.2. *Suppose that R is not GPI and char $R \neq 2$. Let $f : R \to R$ be an additive map and $d \neq 0$ an inner derivation of R. If $[f(x), d(x)] = 0$ for all $x \in R$, then there is $\lambda \in C$ and an additive map $\xi : R \to C$ such that $f(x) = \lambda d(x) + \xi(x)$ for $x \in R$.*

Proof. Suppose $d(x) = [a, x]$, $a \notin Z$. We have $f(x)[a, x] = [a, x]f(x)$ for all x. Replacing x by $x + y$ we see that

$$f(x)[a, y] + f(y)[a, x] = [a, x]f(y) + [a, y]f(x).$$

That is,

$$\big(-f(y)xa + f(y)ax\big) + \big(-f(x)\,ya + f(x)\,ay\big) + \big(-ay\,f(x) + ya\,f(x)\big)$$
$$+ \big(-ax\,f(y) + xa\,f(y)\big) = 0 \quad \text{for all} \quad x, y \in R.$$

Now Theorem 2.1 can be applied with $F_1(y) = -f(y)$, $a_1 = a$, $F_2(y) = f(y)\,a$, $a_2 = 1$, $G_1(x) = -f(x)$, $b_1 = a$, $G_2(x) = f(x)\,a$, $b_2 = 1$, $H_1(x) = -f(x)$, $c_1 = a$, $H_2(x) = af(x)$, $c_2 = 1$, $K_1(y) = -f(y)$, $d_1 = a$, $K_2(y) = af(y)$, $d_2 = 1$. It is clear from the proof of Theorem 2.1 that a_2, b_2, c_2 and d_2 may be taken formally to be 1 even if R does not possess a unit. Since R is not GPI we have $F_1(y) = -f(y) = ayp_{11} + yp_{21} + \lambda_{11}(y)a + \lambda_{21}(y)$ for all $y \in R$, and $H_1(x) = -f(x) = -p_{11}xa - p_{12}x + \mu_{11}(x)a + \mu_{21}(x)$

for all $x \in R$ where $p_{ij} \in Q_s(R_C)$, $\lambda_{ij}, \mu_{ij} : R \to C$. Comparing we get $p_{11}xa + axp_{11} + p_{12}x + xp_{21} = (\mu_{11} - \lambda_{11})(x)a + (\mu_{21} - \lambda_{21})(x)$ for all $x \in R$. That is, the map $x \mapsto p_{11}xa + axp_{11} + p_{12}x + xp_{21}$ maps R into the linear span of 1 and a. By Lemma 3.1 it follows that $p_{11} \otimes a + a \otimes p_{11} + p_{12} \otimes 1 + 1 \otimes p_{21} = 0$. Straightforward, but tedious, tensor product computations show that there exist $\lambda, \mu \in C$ such that $p_{11} = \lambda$, $p_{12} = -\lambda a + \mu$, $p_{21} = -\lambda a - \mu$. Thus

$$\begin{aligned} f(x) &= p_{11}xa + p_{12}x - \mu_{11}(x)a - \mu_{21}(x) \\ &= \lambda xa + (-\lambda a + \mu)x - \mu_{11}(x)a - \mu_{21}(x) \\ &= -\lambda[a, x] - \mu_{21}(x) + \mu x - \mu_{11}(x)a. \end{aligned}$$

Our goal is to show that $\mu x - \mu_{11}(x)a = 0$. We know that

$$[\mu x - \mu_{11}(x)a, d(x)] = [f(x) + \lambda[a, x] + \mu_{21}(x), d(x)] = 0.$$

If $\mu = 0$, then $\mu_{11}(x)[a, d(x)] = \mu_{11}(x)d^2(x) = 0$. Since a group cannot be the union of two proper subgroups, either $\mu_{11}(x) = 0$ for all x, or $d^2(x) = 0$ for all x. In the latter case $d(x) = 0$ by [9, Theorem 1]. Now assume $\mu \neq 0$, so that there is no loss of generality in assuming $\mu = 1$. We have (setting $\alpha(x) = \mu_{11}(x)$)

$$[x, d(x)] = \alpha(x)[a, d(x)] = \alpha(x)d^2(x). \tag{1}$$

This implies

$$[x, d(y)] + [y, d(x)] = \alpha(x)d^2(y) + \alpha(y)d^2(x). \tag{2}$$

Replacing y by yx we see

$$\begin{aligned} [x, d(y)x + yd(x)] + [yx, d(x)] &= \alpha(x)d^2(y)x + 2\alpha(x)d(y)d(x) \\ &\quad + \alpha(x)yd^2(x) + \alpha(yx)d^2(x). \end{aligned}$$

That is,

$$\begin{aligned} &[x, d(y)]x + [x, y]d(x) + 2y[x, d(x)] + [y, d(x)]x \\ &= \alpha(x)d^2(y)x + 2\alpha(x)d(y)d(x) + \alpha(x)yd^2(x) + \alpha(yx)d^2(x). \end{aligned} \tag{3}$$

Using (2) we see that (3) can be written as

$$\alpha(y)d^2(x)x + [x,y]d(x) + 2y[x,d(x)]$$
$$= 2\alpha(x)d(y)d(x) + \alpha(x)yd^2(x) + \alpha(yx)d^2(x). \tag{4}$$

Now replace y by xy in (4) to get

$$\alpha(xy)d^2(x)x + x[x,y]d(x) + 2xy[x,d(x)]$$
$$= 2\alpha(x)d(x)yd(x) + 2\alpha(x)xd(y)d(x) + \alpha(x)xyd^2(x)$$
$$+ \alpha(xyx)d^2(x).$$

Using (4), this can be written as

$$\alpha(yx)xd^2(x) - \alpha(y)xd^2(x)x + \alpha(xy)d^2(x)x - \alpha(xyx)d^2(x)$$
$$= 2\alpha(x)d(x)yd(x).$$

Fixing $x \in R$, we see that the map $y \mapsto \big(\alpha(x)d(x)\big)yd(x)$ has its range in the finite dimensional subspace of R_C spanned by

$$\{xd^2(x), xd^2(x)x, d^2(x)x, d^2(x)\}.$$

It follows from Lemma 3.1 that $\alpha(x)d(x) = 0$ for each x. Thus for each $x \in R$, either $\alpha(x) = 0$ or $d(x) = 0$. As before, either $\alpha(x) = 0$ for all x or $d(x) = 0$ for all x. Since $d \neq 0$ we have $\alpha(x) = 0$ for all x contradicting [9, Th. 2] in view of (1). Thus we have contradicted the assumption that $\mu \neq 0$ and the theorem is proved.

4. Commuting Additive Maps on Lie Ideals

Henceforth, L will be a noncommutative Lie ideal of a prime ring R of characteristic $\neq 2$, and $f : L \to R$ will be an additive commuting mapping. It is our intention to show that $f(v) = \lambda v + \xi(v)$ for some $\lambda \in C$ and $\xi : L \to C$ an additive map. We first dispose of the non-GPI case.

Lemma 4.1. *If R is not* GPI *then $f(v) = \lambda v + \xi(v)$ for some $\lambda \in C$ and an additive map $\xi : L \to C$.*

Proof. Pick $u \in L$ such that $u \notin Z$. We have $\big[f([u,x]), [u,x]\big] = 0$ for all $x \in R$. Thus the maps $x \mapsto f([u,x])$, $x \mapsto [u,x]$ satisfy the requirements of Theorem 3.2. Hence there is $\lambda \in C$ such that $f([u,x]) - \lambda[u,x] \in C$ for each $x \in R$. Since $[f(v), v] = 0$ for each $v \in L$ we have $[f(v), w] = [v, f(w)]$ for $v, w \in L$. With $w = [u,x]$, $u \in L$, $x \in R$ we have, for $v \in L$,

$$\big[f(v), [u,x]\big] = \big[v, f([u,x])\big] = \lambda\big[v, [u,x]\big].$$

That is, $\big[f(v) - \lambda v, [u,x]\big] = 0$ for $v \in L$, $x \in R$. Since $u \notin Z$, it follows from [9, Th. 1], that $f(v) - \lambda v \in C$ for $v \in L$. This proves the lemma.

Before we turn to the GPI case we prove two preliminary lemmas. By L_C we will denote the Lie ideal CL of R_C.

Lemma 4.2. *There is a bilinear map $B : L_C \times L_C \to R_C$ satisfying:*

(i) *$y \mapsto B(x,y)$ is an inner derivation for each $x \in L_C$,*

(ii) *$B(x,x) = 0$ for all $x \in L_C$,*

(iii) *$B(u,v) = [f(u), v]$ for all $u, v \in L$.*

Proof. Define

$$B\left(\sum_{i=1}^{n} \lambda_i x_i, y\right) = \sum_{i=1}^{n} \lambda_i [f(x_i), y]$$

for $\lambda_i \in C$, $x_i \in L$, $y \in L_C$. Suppose $\sum_{i=1}^{n} \lambda_i x_i = 0$. We may assume $\lambda_1 \neq 0$ so that $x_1 = \mu_2 x_2 + \cdots + \mu_n x_n$ where $\mu_i = -\lambda_1^{-1}\lambda_i$. For any $u \in L$ we have

$$[f(x_1), u] = [x_1, f(u)] = \left[\sum_{i=2}^{n} \mu_i x_i, f(u)\right]$$

$$= \sum_{i=2}^{n} \mu_i [x_i, f(u)] = \sum_{i=2}^{n} \mu_i [f(x_i), u].$$

Hence $[f(x_1) - \sum_{i=2}^{n} \mu_i f(x_i), u] = 0$ for all $u \in L$. This implies

$$\left[\sum_{i=1}^{n} \lambda_i f(x_i), u\right] = 0$$

for all $u \in L$ which, in turn, implies

$$\left[\sum_{i=1}^{n} \lambda_i f(x_i), y\right] = 0$$

for all $y \in L_C$. Hence B is well defined. It is clear that B is bilinear. Also (i) and (iii) are obvious from the definition. As for (ii),

$$\begin{aligned} B\left(\sum_{i=1}^{n} \lambda_i x_i, \sum_{i=1}^{n} \lambda_i x_i\right) &= \sum_{i=1}^{n} \lambda_i \left[f(x_i), \sum_{j=1}^{n} \lambda_j x_j\right] \\ &= \sum_{i,j=1}^{n} \lambda_i \lambda_j [f(x_i), x_j] \\ &= 0 \end{aligned}$$

since $[f(x), x] = 0$ and $[f(x), y] + [f(y), x] = 0$ for $x, y \in L$.

Remark 4.3. We deal with bilinear maps because f does not seem to have a natural linear extension to L_C.

Now let F be an algebraic closure of C and set $\tilde{R} = R_C \otimes_C F$, $\tilde{L} = L_C \otimes_C F$. Then $\tilde{L}$ is a noncommutative Lie ideal of $\tilde{R}$.

Lemma 4.4. *There is an F-bilinear map $\tilde{B} : \tilde{L} \times \tilde{L} \to \tilde{R}$ satisfying:*

(i) *$y \mapsto \tilde{B}(x, y)$ is an inner derivation for each $x \in \tilde{L}$,*

(ii) *$\tilde{B}(x, x) = 0$ for each $x \in \tilde{L}$,*

(iii) *$\tilde{B}(u \otimes 1, v \otimes 1) = [f(u), v] \otimes 1$ for all $u, v \in L$.*

Proof. Define $\tilde{B}$ by

$$\tilde{B}\left(\sum x_i \otimes \lambda_i, \sum y_j \otimes \mu_j\right) = \sum_{i,j} B(x_i, y_j) \otimes \lambda_i \mu_j.$$

Using Lemma 4.2 it is easy to verify that $\tilde{B}$ has the desired properties.

We assume henceforth that R is GPI. Now R_C is a closed prime algebra over C and $\tilde{R}$ a closed prime algebra over F. Moreover, since R is GPI, so is $\tilde{R}$ so that $\tilde{R}$ is primitive and has nonzero socle. We recall [5] that in this case there are dual vector spaces V and W such that $\mathcal{F}_W(V) \subseteq \tilde{R} \subseteq \mathcal{L}_W(V)$ where $\mathcal{F}_W(V) \neq \{0\}$ is the socle of $\tilde{R}$, *i.e.*, the algebra of operators in $\tilde{R}$ of finite rank. Every element in $\mathcal{F}_W(V)$ can be written as

$$\sum_{i=1}^{n} v_i \otimes w_i$$

where $v_i \in V, w_i \in W$ and for $x, v \in V$, $w \in W$, $x(v \otimes w) = (x, w)v$. $\mathcal{L}_W(V)$ is the algebra of all F-linear operators on V having an *adjoint*, *i.e.*, $T \in \mathcal{L}_W(V)$ if there exists $T^* : W \to W$ such that $(vT, w) = (v, T^* w)$. We have for $a \in \tilde{R}$ that $(v\rho_a, w) = (v, \rho_a^* w)$ where $\rho_a^* w = aw$. By abuse of the notation we will write a instead of ρ_a and a^* instead of ρ_a^*. Thus $(va, w) = (v, a^* w)$.

We further note that, since F is a field, we can consider W as a *left* vector space isomorphic to W. The resulting algebra of operators, $\{\rho_a \mid a \in \tilde{R}\}$, will be isomorphic to $\tilde{R}$ as a vector space over F, but anti-isomorphic as an algebra. We will adopt these conventions and write both operators and scalars on the left. The reader will find that the anti-isomorphism causes no difficulty in the sequel.

We now return to consideration of $\tilde{L}$. Since $\tilde{L}$ is a Lie ideal of $\tilde{R}$, there is a nonzero ideal U of $\tilde{R}$ such that $[U, U] \subseteq \tilde{L}$ [4]. Since $\mathcal{F} = \mathcal{F}_W(V)$ is contained in any nonzero ideal of $\tilde{R}$ we have $[\mathcal{F}, \mathcal{F}] \subseteq \tilde{L}$.

Let $v_0 \in V$, $w_0 \in W$ be fixed nonzero vectors such that $(v_0, w_0) = 0$ and set $n = v_0 \otimes w_0$.

Lemma 4.5. *$n \in \tilde{L}$ and if $an = na$ for some $a \in \tilde{R}$ then $an = na = \mu n$ for some $\mu \in F$.*

Proof. Choose $x \in V$ such that $(x, w_0) = 1$ and note that

$$n = v_0 \otimes w_0 = (v_0 \otimes w_0)(x \otimes w_0) - (x \otimes w_0)(v_0 \otimes w_0) \in [\mathcal{F}, \mathcal{F}].$$

For the other part, if $x \in V$,

$$(an)(x) = a(v_0 \otimes w_0)(x) = a(x, w_0)v_0 = (x, w_0)av_0.$$

Also

$$(na)x = (v_0 \otimes w_0)ax = (ax, w_0)v_0.$$

Again choose x such that $(x, w_0) = 1$. Then $av_0 = \mu v_0$ and the result follows.

Now let $S = \{a \in \tilde{L} \mid a^* w_0 = 0\}$, $T = \{a \in \tilde{L} \mid av_0 = 0\}$. Of course, $Tn = nS = \{0\}$.

Recall that from Lemma 4.4(i) for every $x \in \tilde{L}$ there is $\tilde{x} \in \tilde{R}$ such that $\tilde{B}(x, y) = [\tilde{x}, y]$ for all $y \in \tilde{L}$.

Remark 4.6. It is easy to prove that $a\tilde{L}b = \{0\}$, $a, b \in \tilde{R}$, implies $a = 0$ or $b = 0$.

Lemma 4.7. *There is $\mu \in F$ such that $T(\tilde{n} - \mu) = (\tilde{n} - \mu)S = \{0\}$.*

Proof. By Lemma 4.4(ii) we have $\tilde{B}(n, n) = 0$ so that $[\tilde{n}, n] = 0$. By Lemma 4.5, $\tilde{n}n = n\tilde{n} = \mu n$ for some $\mu \in F$. For any $u \in \tilde{L}$ we have $[\tilde{n}, u] = \tilde{B}(n, u) = -\tilde{B}(u, n) = -[\tilde{u}, n]$. Since $Tn = nS = \{0\}$ it follows that $T[\tilde{n}, u]S = 0$, *i.e.*, $t\tilde{n}us = tu\tilde{n}s$ for all $s \in S$, $t \in T$, $u \in \tilde{L}$. Taking $s = n$ we see that $t\tilde{n}un = tu\tilde{n}n = \mu tun$. Hence $t(\tilde{n} - \mu)un = 0$ for all $t \in T$, $u \in \tilde{L}$, *i.e.*, $T(n - \mu)\tilde{L}n = \{0\}$. Since $n \neq 0$ this forces $T(\tilde{n} - \mu) = \{0\}$. Similarly $(\tilde{n} - \mu)S = \{0\}$.

Lemma 4.8. *There is $\lambda \in F$ such that $\tilde{n} - \mu = \lambda n$.*

Proof. Let $n' = \tilde{n} - \mu$. From Lemma 4.7 we have $Tn' = n'S = \{0\}$. Suppose there is v_1 in the range of n' such that v_0 and v_1 are independent.

Choose $x \in W$ such that $(v_0, x) = 0$ and $(v_1, x) \neq 0$ and set $t = v_0 \otimes x$. Note that $t \in T$ so $tn' = 0$. In particular, $0 = tv_1 = (x, v_1)v_0$ which is a contradiction. It follows that $n' = v_0 \otimes z$ for some $z \in W$. Suppose z and w_0 are independent. Then there is $y \in V$ such that $(y, w_0) = 0$, $(y, z) \neq 0$. Let $s = y \otimes w_0 \in S$ and note that $n's = 0$, a contradiction as before. Thus $z = \lambda w_0$, proving the lemma.

Now let $N = \{v \otimes w \mid (v, w) = 0\}$. We know that for any $n \in N$ there is $\lambda_n \in F$ such that $\tilde{B}(n, u) = [\tilde{n}, u] = \lambda_n[n, u]$ for every $u \in \tilde{L}$.

Lemma 4.9. *There is $\lambda \in F$ such that $\tilde{B}(n, u) = \lambda[n, u]$ for all $n \in N$, $u \in \tilde{L}$.* (That is, λ does not depend upon n.)

Proof. Take $n_1, n_2 \in N$, write $\lambda_i = \lambda_{n_i}$, and set $n_i = v_i \otimes w_i$ for $i = 1, 2$. We have $\lambda_1[n_1, n_2] = \tilde{B}(n_1, n_2) = -\tilde{B}(n_2, n_1) = -\lambda_2[n_2, n_1]$. Thus $\lambda_1 = \lambda_2$ unless $n_1 n_2 = n_2 n_1$, *i.e.*, $(v_2, w_1)(v_1 \otimes w_2) = (v_1, w_2)(v_2 \otimes w_1)$. Suppose $(v_2, w_1) \neq 0$. Then v_1 and v_2, as well as w_1 and w_2, are independent. Hence, $v_1 \otimes w_1$ and $v_2 \otimes w_2$ are independent, contradicting the relation resulting if n_1 and n_2 commute.

Thus we need only consider the case when $(v_1, w_2) = (v_2, w_1) = 0$ (we may assume v_1, v_2, w_1, $w_2 \neq 0$). Choose $v_3 \in V$ such that $(v_3, w_2) \neq 0$. As $(v_1, w_2) = 0$ we have v_1 and v_3 are independent. Therefore, there is $w_3 \in W$ such that $(v_1, w_3) \neq 0$ and $(v_3, w_3) = 0$. Then $v_3 \otimes w_3 \in N$. Since $(v_1, w_3) \neq 0$ we have $\lambda_1 = \lambda_3$. Similarly since $(v_3, w_2) \neq 0$ we have $\lambda_2 = \lambda_3$. Hence $\lambda_1 = \lambda_2$.

Lemma 4.10. *If $a \in \tilde{R}$ is such that $[a, N] = 0$, then $a \in F$.* (Here F is identified with $F \cdot 1$.)

Proof. Pick $0 \neq v \in V$. We can find $w \in W$ such that $(v, w) = 0$ and $w \neq 0$. We have $a(v \otimes w) = (v \otimes w)a$ which implies av and v are dependent. Since $v \in V$ is arbitrary it follows that $av = \alpha v$ for some $\alpha \in F$ and all

$v \in V$.

Lemma 4.11. *$\tilde{B}(x,y) = \lambda[x,y]$ for all $x, y \in \tilde{L}$.*

Proof. By Lemmas 4.4 and 4.5 we have $\tilde{B}(n,x) + \tilde{B}(x,n) = 0$ for all $n \in N$, $x \in \tilde{L}$. By Lemma 4.9, $\tilde{B}(n,x) = \lambda[n,x]$ so we have

$$0 = \tilde{B}(n,x) + \tilde{B}(x,n) = \lambda[n,x] + [\tilde{x},n] = [\tilde{x} - \lambda x, n].$$

By Lemma 4.10 this implies $\tilde{x} - \lambda x \in F$. Hence $\tilde{B}(x,y) = [\tilde{x},y] = \lambda[x,y]$ for all $x, y \in \tilde{L}$.

Lemma 4.12. *There is $\lambda \in C$ and an additive map $\xi : L \to C$ such that $f(u) = \lambda u + \xi(u)$ for all $u \in L$.*

Proof. Since $\tilde{B}(x,y) = \lambda[x,y]$ for some $\lambda \in F$ and all $x, y \in \tilde{L}$, we have $[f(u),v] \otimes 1 = \tilde{B}(u \otimes 1, v \otimes 1) = \lambda([u,v] \otimes 1) = [u,v] \otimes \lambda$ for all $u, v \in L$. But then $\lambda \in C$ and $[f(u),v] = \lambda[u,v]$ which yields $[f(u) - \lambda u, L] = 0$ for all $u \in L$. Hence $f(u) - \lambda u \in C$ for all $u \in L$.

Theorem 4.13. *Let R be a prime ring of characteristic $\neq 2$. If L is a noncommutative Lie ideal of R and if $f : L \to R$ is an additive commuting map, then there is $\lambda \in C$ and an additive map $\xi : L \to C$ such that $f(v) = \lambda v + \xi(v)$ for each $v \in L$.*

Proof. Lemmas 4.1 and 4.12.

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