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COMMUTING MONOTONE MAPPINGS OF DENDROIDS

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In this paper a continuum will be a compact connected Hausdorff space. A continuum X is hereditarily unicoherent if any two subcontinua of X meet in a continuum. In what follows X will denote a dendroid, i.e., a hereditarily unicoherent, arcwise connected metric continuum. A point e of X is an endpoint if e is an endpoint of any arc in which it lies.

In this paper we prove that if S is an abelian semigroup of continuous monotone surjections of X onto itself which leaves an endpoint of X fixed, then S has another fixed point. For a history of this problem see [1], where Professor L. E. Ward, Jr. proves the above result when S has not more than two generators, and conjectures that the result must be true in general.

If X is a dendroid and e is an endpoint of X we define a partial order $\leqq$, called the arc order on X with least element e, as follows: if x, y are two points of X let $A(x, y) = \{x\}$ if $x = y$; otherwise, there is a unique arc in X with x, y as endpoints, and this unique arc is denoted by $A(x, y)$. Define $x \leqq y$ if $x \in A(e, y)$. We define $[x, y] = \{z : x \leqq z \leqq y\}$. Then this order satisfies:

(1.1) e is the least element of X.

(1.2) If $x \in X$, then $[e, x] = A(e, x)$ is a closed chain, and the order topology and subspace topology coincide on $[e, x]$.

(1.3) Each non-empty subcontinuum of X has a least element, and each non-empty chain has a supremum.

(1.4) If $x, y \in X \setminus \{e\}$, there is a $z \in X \setminus \{e\}$ with $z \leqq x$, $z \leqq y$.

(1.5) If C is a subcontinuum of X and $x, y \in C$ then $A(x, y) \subset C$.

(1.6) If f is a continuous monotone surjection: $X \to X$ which leaves e fixed, then f preserves $\leqq$ and in fact $f(A(x, y)) = A(f(x), f(y))$, $x, y \in X$.

For (1.1)–(1.5), see Ward [2]; for (1.6), see [1].

THEOREM. *Let X be a non-trivial dendroid and let S be a finitely generated abelian semigroup of continuous monotone surjections of X onto X. If S leaves an endpoint e of X fixed, then S has a fixed point other than e.*

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PROOF. Let $\leqq$ be the arc order on X with e as least element. For each $t \in S$ and each $x \in X$, the set $t^{-1}(x)$ is a subcontinuum of X, hence has a least element, $t^*(x)$, under $\leqq$. This defines a map $t^*: X \to X$. We claim:

(1) $t\,t^* =$ identity on X.

(2) t^* is order preserving.

(3) If $s, t \in S$, then $s^*\,t^* = t^*\,s^*$.

(4) If $s, t \in S$ and $x \in X$ then $t^*\,s(x) \leqq s\,t^*(x)$.

In fact, (1) is immediate. To prove (2), let $x \leqq y$. Then if $z \in t^{-1}(y)$, we have $t([e, z]) = [t(e), t(z)] = [e, y]$; since $x \in [e, y]$, there is a $w \in [e, z]$ such that $t(w) = x$. Then $w \in t^{-1}(x)$, so $t^*(x) \leqq w \leqq z$. Since $z \in t^{-1}(y)$ was arbitrary, we have $t^*(x) \leqq t^*(y)$.

To prove (3), we need only show that $s^*t^* = (ts)^*$. Let $x \in X$. Because we have $(ts)\, s^*t^*\,(x) = x$, it must be that $(ts)^*\,(x) \leqq s^*\,t^*(x)$. On the other hand, if $w \in t^{-1}(x)$, by (2) we have $s^*(t^*(x)) \leqq s^*\,(x)$, and this implies that $s^*t^*\,(x) \leqq (ts)^*\,(x)$. Finally, (4) follows from the fact that $s\,t^{-1}(x) \subset t^{-1}s(x)$ for all $x \in X$.

Now let J be a finite set of generators for S. Choose a maximal element m of X. Since X is non-trivial we have $m \neq e$ and $e \notin t^{-1}(m)$ for all $t \in J$. Let $u = \mathrm{Sup}\{[e, m] \cap [e, t^*(m)] : t \in J\}$. Since J is finite, 1.4 implies that $e < u$. We have $u \leqq m$ and $t\,u \leqq t\,t^*(m) = m$, all $t \in J$, so by 1.2, u and $t(u)$ are comparable, all $t \in J$. Let $J_1 = \{t \in J : u \leqq t(u)\}$ and $J_2 = \{t \in J : t(u) < u\}$.

If $t \in J_2$, then $t([u, t^*(m)]) = [t(u), m]$, hence there is a point $p \in [u, t^*(m)]$ for which $t(p) = u$. Since $u \notin t^{-1}(u)$, we can show that $u < t^*(u)$. We now consider cases.

CASE I. $J_1 \neq \emptyset \neq J_2$. Let C_1 be a subset of X which is maximal with respect to the properties:

(5) $u \in C_1$.

(6) C_1 is a chain under $\leqq$.

(7) if $y \in C_1$ and $t \in J_1$, then $y \leqq t(y)$.

(8) if $y \in C_1$ and $t \in J_2$, then $y \leqq t^*(y)$.

Let $q = \mathrm{Sup}\, C_1$.

If $y \in C_1$, then $y \leqq t(y) \leqq t(q)$ for all $t \in J_1$. Thus for all $t \in J_1$, $t(q)$ is an upper bound for C_1, so

(9) $q \leqq t(q)$ for all $t \in J_1$.

Likewise

(10) $q \leqq t^*(q)$ for all $t \in J_2$.

Now if $s, t \in J_2$, from (10) we find

(11) $t^*(q) \leqq t^*\,s^*(q) = s^*(t^*(q))$.

If $s \in J_1$ and $t \in J_2$, we use (9) and (4) to find

(12) $t^*(q) \leqq t^*s(q) \leqq s(t^*(q))$.

Then by (9), (10), (11), (12), and the maximality of C_1, we obtain $t^*(q) \in C_1$, so

(13) $t^*(q) = q$, hence $t(q) = q$, for all $t \in J_2$, and $q \leqq s(q)$ for all $s \in J_1$.

To complete the proof, we choose a subset C_2 of X which is maximal with respect to the properties.

(14) $q \in C_2$.

(15) C_2 is a chain.

(16) If $t \in J$ and $y \in C_2$ then $y \leqq t(y)$.

Let $c = \text{Sup } C_2$. Arguing as before we find

(17) $c \leqq t(c)$ for all $t \in J$.

(18) $t(c) \leqq ts(c) = s(t(c))$ for all $s, t \in J$.

By (15), (16), and the maximality of C_2, we must have $t(c) \in C_2$, and so $t(c) = c$ for all $t \in J$. Thus S leaves c fixed, and the proof is complete in this case.

Case II ($J = J_1$) and Case III ($J = J_2$) are established by simplified versions of the argument of Case I, and are left to the reader.

We remark that the theorem is true under more general conditions: one may drop the condition of metrizability and add T_2. The arcwise connectedness requirement may be replaced by the condition that any two distinct points $x, y \in X$ lie in a subcontinuum of X which has x and y as its only non-cutpoints. The proof of the new theorem thus obtained is however essentially the same as the proof above.

References

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