

COMMUTING PROJECTIONS WITH ASSIGNED RANGES

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1. Let \mathfrak{H} be a Hilbert space. A projection is a bounded idempotent linear operator. Orthogonal projections form a proper subclass of the class of all projections. A problem is to get a condition for there to exist commuting projections E_1, \dots, E_n with the same ranges as given orthogonal projections P_1, \dots, P_n respectively. This will be settled in terms of properties of the sublattice, generated by P_1, \dots, P_n in the lattice of all orthogonal projections. In case $n=2$, commuting projections with minimum norms are constructed. Another problem is to find commuting projections E_1, \dots, E_n in a Hilbert space \mathfrak{K} , containing \mathfrak{H} as a subspace, such that $P_j x = P E_j x$ for $x \in \mathfrak{H}, j=1, 2, \dots, n$, where P is the orthogonal projection from \mathfrak{K} onto \mathfrak{H} . This will be proved to be always possible.

2. Let P and Q be orthogonal projections with ranges \mathfrak{M} and \mathfrak{N} respectively. Then $P \wedge Q$ and $P \vee Q$ will denote the orthogonal projections onto $\mathfrak{M} \cap \mathfrak{N}$ and the closure of $\mathfrak{M} + \mathfrak{N}$ respectively. $\mathfrak{M} \ominus \mathfrak{N}$ will stand for the subspace $\mathfrak{M} \cap (\mathfrak{M} \cap \mathfrak{N})^\perp$ and $P \ominus Q$ is the orthogonal projection onto it.

If there exist commuting projections E and F with ranges \mathfrak{M} and \mathfrak{N} respectively, $\mathfrak{M} + \mathfrak{N}$ becomes the range of the projection $E + F - EF$, therefore it is a closed subspace.

If, conversely, $\mathfrak{M} + \mathfrak{N}$ is closed, in view of the well-known theorem of Kober (see [3]) the operators E' and F' , which assign to each $x \in \mathfrak{M} + \mathfrak{N}$ the elements $u + v$ and $u + w$ respectively, are continuous, where u, v and w are uniquely determined by the relation $x = u + v + w$ with $u \in \mathfrak{M} \cap \mathfrak{N}, v \in \mathfrak{M} \ominus \mathfrak{N}$ and $w \in \mathfrak{N} \ominus \mathfrak{M}$. Then the operators $E = E'(P \vee Q)$ and $F = F'(P \vee Q)$ are commuting projections with ranges \mathfrak{M} and \mathfrak{N} respectively.

It was shown by Mackey [3, p. 166] that the closedness of $\mathfrak{M} + \mathfrak{N}$ is equivalent to the property that $(P \vee Q) \wedge Q' = (P \wedge Q') \vee Q$ for every orthogonal projection Q' with $Q \leq Q'$.

Suppose again that there exist commuting projections E and F with ranges \mathfrak{M} and \mathfrak{N} respectively. Since the commutativity implies that $E\mathfrak{N} \subseteq \mathfrak{M} \cap \mathfrak{N}$ for any $x, y \in \mathfrak{H}, (P \ominus Q)x$ and EQy are orthogonal, therefore

$$\|(P \ominus Q)x\|^2 \leq \|(P \ominus Q)x - EQy\|^2 \leq \|E\|^2 \cdot \|(P \ominus Q)x - Qy\|^2.$$

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Since, for fixed x , the minimum of $\|(P \ominus Q)x - Qy\|^2, y \in \mathfrak{S}$ is equal to $\|(P \ominus Q)x\|^2 - \|Q(P \ominus Q)x\|^2$, it follows that

$$\|Q(P \ominus Q)x\|^2 \leq \{1 - \|E\|^{-2}\} \cdot \|(P \ominus Q)x\|^2,$$

consequently

$$\|(Q \ominus P)(P \ominus Q)\|^2 \leq 1 - \|E\|^{-2} < 1,$$

because

$$(Q \ominus P)(P \ominus Q) = Q(P \ominus Q).$$

Suppose, conversely, that $\|(Q \ominus P)(P \ominus Q)\| < 1$. Then both $I - (Q \ominus P)(P \ominus Q)$ and $I - (P \ominus Q)(Q \ominus P)$ have bounded inverses and the operators

$$E_0 = P \wedge Q + (P \ominus Q)\{I - (Q \ominus P)(P \ominus Q)\}^{-1} \cdot (I - Q \ominus P)$$

and

$$F_0 = P \wedge Q + (Q \ominus P)\{I - (P \ominus Q)(Q \ominus P)\}^{-1} \cdot (I - P \ominus Q)$$

are commuting projections with ranges \mathfrak{M} and \mathfrak{N} respectively. This can be seen by using the expansion

$$\{I - (Q \ominus P)(P \ominus Q)\}^{-1} = \sum_{n=0}^{\infty} [(Q \ominus P)(P \ominus Q)]^n.$$

The definition shows that $E_0 F_0 = P \wedge Q$ and $E_0 + F_0 - E_0 F_0 = P \vee Q$ and that $E_0(I - F_0)$ and $F_0(I - E_0)$ have the same ranges as $P \ominus Q$ and $Q \ominus P$ respectively. Then it follows that

$$\begin{aligned} \|x\|^2 - \|E_0 F_0 x\|^2 &\geq \|(P \vee Q - P \wedge Q)x\|^2 \\ &= \|E_0(I - F_0)x + F_0(I - E_0)x\|^2 \\ &\geq \{1 - \|(Q \ominus P)(P \ominus Q)\|^2\} \cdot \|E_0(I - F_0)x\|^2, \end{aligned}$$

therefore

$$\|x\|^2 \geq \{1 - \|(Q \ominus P)(P \ominus Q)\|^2\} \|E_0 x\|^2,$$

which, together with the inequality already obtained, implies that

$$\|E_0\| = \{1 - \|(Q \ominus P)(P \ominus Q)\|^2\}^{-1/2}.$$

The norm of F_0 is shown to have the same value.

It is desired to give an expression for the quantity $\|(Q \ominus P)(P \ominus Q)\|$ in terms of $P \vee Q$ and $P \wedge Q$.

LEMMA. $\|(Q \ominus P)(P \ominus Q)\| = \|P \vee Q + P \wedge Q - P - Q\|.$

PROOF. Since

$$P \vee Q + P \wedge Q - P - Q = (P \ominus Q) \vee (Q \ominus P) - (P \ominus Q) - (Q \ominus P)$$

and $(P \ominus Q) \wedge (Q \ominus P) = 0$, it suffices to prove $\|I - P - Q\| = \|QP\|$ under the assumption that $P \wedge Q = 0$ and $P + Q = I$.

Since it is known [1, p. 70] that

$$\|I - P - Q\| = \text{Max}\{\|QP\|, \|(I - Q)(I - P)\|\},$$

the assertion is true in case $\|QP\| = 1$. If $\|QP\| < 1$, in view of the foregoing result there exist the unique commuting projections E and F with the same ranges as P and Q respectively and their norms are equal to $\{1 - \|QP\|^2\}^{-1}$. On the other hand, E^* and F^* are the unique commuting projections with the same ranges as $I - Q$ and $I - P$ respectively. The same discussion is applied to get

$$\|E\| = \|E^*\| = \{1 - \|(I - P)(I - Q)\|^2\}^{1/2},$$

therefore

$$\|(I - P)(I - Q)\| = \|QP\|.$$

Use, finally, the relation

$$\|(I - P)(I - Q)\| = \|(I - Q)(I - P)\|.$$

In view of this lemma, the preceding results can be summarized in the following theorem.

THEOREM 1. *For a pair of orthogonal projections P, Q the following conditions are mutually equivalent:*

(1) *there exist commuting projections E and F with the same ranges as P and Q respectively;*

(2) *$(P \vee Q) \wedge Q' = (P \wedge Q') \vee Q$ for every orthogonal projection Q' with $Q \leq Q'$;*

(3) *$\|P \vee Q + P \wedge Q - P - Q\| < 1$. When one (and all) of these conditions is fulfilled, there exist such commuting projections E_0 and F_0 with minimum norm:*

$$\|E_0\| = \|F_0\| = \{1 - \|P \vee Q + P \wedge Q - P - Q\|^2\}^{-1/2}.$$

A related problem is to ask when there exist doubly commuting projections E and F , i.e. $EF = FE$ and $E^*F = FE^*$, with the same ranges as P and Q respectively. The answer is, however, quite simple. It is the case (when and) only when $PQ = QP$. In fact, since the range of the projection $I - E^*$ is the orthogonal complement of the range of E and is invariant under F by the double commutativity, the range

of E reduces F , i.e. $FP = PF$. A similar argument yields $PQ = QP$.

3. Let P_1, \dots, P_n be orthogonal projections with ranges $\mathfrak{M}_1, \dots, \mathfrak{M}_n$ respectively. Consider the sublattice $L(P_1, \dots, P_n)$, generated by P_1, \dots, P_n in the lattice of all orthogonal projections, and the sublattice $L(\mathfrak{M}_1, \dots, \mathfrak{M}_n)$, generated by $\mathfrak{M}_1, \dots, \mathfrak{M}_n$ in the lattice of all (not necessarily closed) subspaces. If the lattice $L(\mathfrak{M}_1, \dots, \mathfrak{M}_n)$ consists of closed subspaces only, it is isomorphic to the lattice $L(P_1, \dots, P_n)$.

THEOREM 2. *For orthogonal projections P_1, \dots, P_n the following conditions are equivalent:*

(1) *there exist commuting projections E_1, \dots, E_n with the same ranges as P_1, \dots, P_n respectively;*

(2) *the sublattice $L(P_1, \dots, P_n)$, generated by P_1, \dots, P_n in the lattice of all orthogonal projections, is distributive and possesses one (and both) of the following equivalent properties:*

(a) $\|P \vee Q + P \wedge Q - P - Q\| < 1$ for $P, Q \in L(P_1, \dots, P_n)$,

(b) $(P \vee Q) \wedge Q' = (P \wedge Q') \vee Q$ for $P, Q \in L(P_1, \dots, P_n)$ and for every orthogonal projection Q' with $Q \leq Q'$.

PROOF. Suppose that the condition (1) is fulfilled, then the lattice $L(\mathfrak{M}_1, \dots, \mathfrak{M}_n)$ is distributive, where \mathfrak{M}_j is the range of $P_j, j = 1, 2, \dots, n$. In fact, for any three members $\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3$ in it there exist commuting projections F_1, F_2, F_3 , which are polynomials of E_1, \dots, E_n and have ranges $\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3$ respectively. Since $(\mathfrak{N}_1 + \mathfrak{N}_2) \cap \mathfrak{N}_3$ becomes the range of the projection $F_1(I - F_2)F_3 + F_2F_3$, it follows that

$$(\mathfrak{N}_1 + \mathfrak{N}_2) \cap \mathfrak{N}_3 = (\mathfrak{N}_1 \cap \mathfrak{N}_3) + (\mathfrak{N}_2 \cap \mathfrak{N}_3).$$

Further every member of $L(\mathfrak{M}_1, \dots, \mathfrak{M}_n)$ is a closed subspace, because it is a range of a projection. Then in view of Theorem 1 and the comment preceding Theorem 2, the lattice $L(P_1, \dots, P_n)$, which becomes isomorphic to the lattice $L(\mathfrak{M}_1, \dots, \mathfrak{M}_n)$, is distributive and possesses the properties (a) and (b).

Suppose, conversely, that the condition (2) is fulfilled. Then as in the above discussion, the lattice $L(\mathfrak{M}_1, \dots, \mathfrak{M}_n)$ is distributive and consists of closed subspaces only. Now let J_k (or L_k) denote generally a set, consisting of k integers in $\{1, 2, \dots, n\}$. Put for k and J_k

$$\mathfrak{M}(J_k) = \bigcap_{j \in J_k} \mathfrak{M}_j, \quad \mathfrak{M}^{(k)} = \sum_{\text{all } J_k} \mathfrak{M}(J_k)$$

and

$$\mathfrak{N}(J_k) = \mathfrak{M}(J_k) \ominus \mathfrak{M}^{(k+1)}, \quad (\mathfrak{M}^{(n+1)} = \{0\}).$$

Let us prove by induction that

$$\mathfrak{M}_j = \sum_{k=1}^n \sum_{j \in J_k} \mathfrak{N}(J_k), \quad j = 1, 2, \dots, n.$$

This is trivial for $n=1$. Assuming that the assertion is generally true for $n-1$, consider the subspaces $\mathfrak{M}_j^* = \mathfrak{M}_j \cap \mathfrak{M}_n, j=1, 2, \dots, n-1$ and define subspaces $\mathfrak{N}^*(J_k), \mathfrak{M}^{*(k)}$ and $\mathfrak{N}^*(J_k)$ (with $J_k \subset \{1, 2, \dots, n-1\}$) from $\mathfrak{M}_1^*, \dots, \mathfrak{M}_{n-1}^*$ just as $\mathfrak{M}(J_k), \mathfrak{M}^{(k)}$ and $\mathfrak{N}(J_k)$ were defined from $\mathfrak{M}_1, \dots, \mathfrak{M}_n$. Then the distributivity shows that

$$\mathfrak{M}^{*(k)} = \sum_{J_k} [\mathfrak{M}_n \cap \mathfrak{M}(J_k)] = \mathfrak{M}_n \cap \mathfrak{M}^{(k+1)};$$

hence $\mathfrak{N}^*(J_k) = \mathfrak{N}(J_k, n)$. Since the property, in question, of $L(\mathfrak{M}_1, \dots, \mathfrak{M}_n)$ implies that the lattice $L(\mathfrak{M}_1^*, \dots, \mathfrak{M}_{n-1}^*)$ is distributive and consists of closed subspaces only, it follows from the induction assumption that

$$\mathfrak{M}_j^* = \sum_{k=1}^n \sum_{j \in J_k} \mathfrak{N}^*(J_k), \quad j = 1, 2, \dots, n-1.$$

But this can be written in the form

$$\mathfrak{M}_j \cap \mathfrak{M}_n = \sum_{k=2}^n \sum_{j, n \in J_k} \mathfrak{N}(J_k).$$

A similar argument proves that for any distinct i, j

$$\mathfrak{M}_j \cap \mathfrak{M}_i = \sum_{k=2}^n \sum_{i, j \in J_k} \mathfrak{N}(J_k).$$

Since for any j the distributivity implies that

$$\mathfrak{M}_j = \mathfrak{N}(j) + \mathfrak{M}_j \cap \mathfrak{M}^{(2)} = \mathfrak{N}(j) + \sum_{i \neq j} (\mathfrak{M}_j \cap \mathfrak{M}_i),$$

the above result yields the required expression for \mathfrak{M}_j .

Next let us prove linear independence of the family of all nontrivial $\mathfrak{N}(J_k)$'s. To this end, it suffices to prove that for any J_k

$$\mathfrak{N}(J_k) \cap \left\{ \sum_{L_k \neq J_k} \mathfrak{N}(L_k) + \mathfrak{M}^{(k+1)} \right\} = \{0\},$$

because $\mathfrak{N}(J_i) \subseteq \mathfrak{M}^{(k+1)}$ whenever $k < i$. The assertion results from the following relations, based on the distributivity

$$\mathfrak{N}(J_k) \cap \left\{ \sum_{L_k \neq J_k} \mathfrak{N}(L_k) + \mathfrak{N}^{(k+1)} \right\} \subseteq \mathfrak{M}(J_k)$$

$$\cap \left\{ \sum_{L_k \neq J_k} \mathfrak{M}(L_k) + \mathfrak{M}^{(k+1)} \right\} = \mathfrak{M}(J_k) \cap \mathfrak{M}^{(k+1)}.$$

Now since $\mathfrak{N}(J_k)$'s have the closed sum $\mathfrak{M}^{(1)}$, in view of Kober's theorem there exist mutually annihilating projections $E(J_k)$ with ranges $\mathfrak{N}(J_k)$ respectively. Then the operators

$$E_j = \sum_{k=1}^n \sum_{j \in J_k} E(J_k), \quad j = 1, 2, \dots, n,$$

are commuting projections with ranges \mathfrak{M}_j . This completes the proof.

COROLLARY. *Let each of two families of orthogonal projections $\{P_1, \dots, P_n\}$ and $\{Q_1, \dots, Q_m\}$ satisfy the condition in Theorem 2. If each P_i commutes with all Q_j , then the combined family $\{P_1, \dots, P_n, Q_1, \dots, Q_m\}$ satisfies the same condition, too.*

PROOF. With the same notations as in the proof of Theorem 2, each subspace in $L(\mathfrak{M}_1, \dots, \mathfrak{M}_n)$ is invariant under all Q_j , hence so is each $\mathfrak{N}(J_k)$, which leads to the commutativity of each E_i with all Q_j . Similar argument shows that there exist commuting projections F_1, \dots, F_m which have the same ranges as Q_1, \dots, Q_m respectively and commute with all E_i .

4. When \mathfrak{H} is imbedded in a larger Hilbert space \mathfrak{R} , a linear operator S in \mathfrak{R} is called a dilation of a linear operator T in \mathfrak{H} , in case $Sx = PTx$ for $x \in \mathfrak{H}$, where P is the orthogonal projection from \mathfrak{R} onto \mathfrak{H} .

In a previous paper [2] we proved that if a pair of orthogonal projections admits dilations which are commuting orthogonal projections, then they necessarily commute with each other. In this connection the following theorem is of interest.

THEOREM 3. *Orthogonal projections P_1, \dots, P_n admit dilations E_1, \dots, E_n , which are commuting projections.*

PROOF. Consider the orthogonal sum

$$\mathfrak{R} = \mathfrak{H} \oplus \mathfrak{H}_1 \oplus \dots \oplus \mathfrak{H}_n$$

where each \mathfrak{H}_j is a copy of \mathfrak{H} . Imbed \mathfrak{H} and \mathfrak{H}_j canonically into \mathfrak{R} . The isomorphism from \mathfrak{H} to \mathfrak{H}_j will be denoted by I_j . Consider $n+1$ subspaces defined by

$$\mathfrak{N}_j = \{(P_j + I_j P_j)x; x \in \mathfrak{G}\}, \quad j = 1, 2, \dots, n$$

and

$$\mathfrak{N} = \left\{ \left(I - \sum_{j=1}^n P_j - \sum_{j=1}^n I_j P_j \right) x; x \in \mathfrak{G} \right\}.$$

Then it is easy to see that these subspaces are linearly independent with a closed sum; therefore in view of Kober's theorem there exist mutually annihilating projections E_1, \dots, E_n with ranges $\mathfrak{N}_1, \dots, \mathfrak{N}_n$ respectively. Since each $x \in \mathfrak{G}$ is written in the form

$$x = \sum_{j=1}^n (P_j + I_j P_j)x + \left(I - \sum_{j=1}^n P_j - \sum_{j=1}^n I_j P_j \right) x,$$

it follows that $E_j x = P_j x + I_j P_j x$, which means that E_j is a dilation of P_j .

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REFERENCES

1. N. I. Akhiezer and I. M. Glazman, *Theory of linear operators in Hilbert space*, Ungar, New York, 1961.
2. T. Ando and M. Takaguchi, *Commuting dilations of self-adjoint operators*, Proc. Japan Acad. **41** (1965), 696–698.
3. G. Mackey, *On infinite-dimensional linear spaces*, Trans. Amer. Math. Soc. **57** (1945), 155–207.

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