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COMPACT 3-MANIFOLDS WITH A FLAT CARNOT–CARATHÉODORY METRIC

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§ 0. Introduction. A nonintegrable subbundle Δ of a tangent bundle TM yields an interesting geometry on a manifold M. A Riemannian metric on Δ enables one to measure the length of differentiable curves tangent to Δ. The metric obtained by minimizing the length of such curves is called the *Carnot–Carathéodory metric*. The standard example of such a metric arises if we take for Δ the invariant 2-dimensional subbundle in the tangent bundle of the Heisenberg group generated by two noncentral vectors of its Lie algebra, and then equip Δ with the invariant Riemannian metric. This metric space, described more precisely in §1, will be denoted by H_c.

The aim of this paper is to classify, up to isometry, all 3-dimensional compact manifolds with a Carnot–Carathéodory metric satisfying the property of being locally differentiably isometric to H_c. This property of a Carnot–Carathéodory metric will be called *flatness*. The above problem is in many aspects similar to the classification problem for compact flat Riemannian manifolds. All classified manifolds have the same universal covering space, isometric to H_c, thus they are quotients of it by a free discrete action of some group of isometries (cf. §2). The Heisenberg group translations of H_c behave like ordinary translations of $\mathbb{R}^n$, in particular, an analogue of the Bieberbach theorem holds: the holonomy group of a compact manifold with flat Carnot–Carathéodory metric is finite, thus the manifold is finitely covered by a nilmanifold (Theorem 5.3). The Heisenberg group nilmanifolds (an analogue of tori in the Riemannian case) are dealt with in §4, where they are classified up to isometry.

In §5 all possible holonomy groups (in the above sense) are determined, and the classification is completed in §6 by considering each holonomy group separately.

§ 1. Preliminaries. Let Δ denote a *distribution* on a manifold M, i.e. a subbundle of the tangent bundle TM.

1.1. Definition. The distribution Δ is said to satisfy the *Hörmander* *condition* if any vector fields locally generating $\varDelta$, together with their commutators, generate the tangent space T_pM at each point $p \in M$.

1.2. A Riemannian metric on the distribution $\varDelta$ enables us to measure the length of *horizontal curves*, i.e. differentiable curves tangent to $\varDelta$ at each point. According to a theorem of Chow (cf. [M], p. 35), if $\varDelta$ satisfies the Hörmander condition, then any two points of M can be joined by a horizontal curve. This enables us to define the function

$$d_c(a,b) = \inf\{\mathrm{length}\,\gamma : \gamma \text{ horizontal joining } a \text{ and } b\},$$

which is obviously a metric, called a *Carnot–Carathéodory* (C–C for short) *metric* (cf. [M], [G], [P]).

1.3. Example. The *Heisenberg group* H is $\mathbb{R}^3$ with the following multiplication:

$$(x_1,y_1,z_1)(x_2,y_2,z_2) = (x_1+x_2, y_1+y_2, z_1+z_2+\tfrac{1}{2}(x_1y_2 - x_2y_1)).$$

The left-invariant vector fields

$$X = \partial_x - \tfrac{1}{2}y\partial_z, \qquad Y = \partial_y + \tfrac{1}{2}x\partial_z, \qquad Z = \partial_z \tag{1.3.1}$$

form a basis of the Lie algebra $\mathbf{h}$ of H with the commutation relations

$$[X,Y] = Z, \qquad [X,Z] = [Y,Z] = 0. \tag{1.3.2}$$

The distribution $\varDelta$ spanned at each point by X and Y satisfies, of course, the Hörmander condition, and the Riemannian metric on $\varDelta$ for which X and Y are orthonormal yields a left invariant C–C metric on H. Denote the Heisenberg group with this metric by H_c.

1.4. Definition. A C–C metric on a 3-dimensional manifold is called *flat* if it is locally isometric to H_c.

1.5. Remark. Every C–C metric is an example of a metric for which the *tangent cone* exists (cf. [G] or [M], Definition 2.2). For a Riemannian manifold M the tangent cone at a point p is the tangent space T_pM with the euclidean metric given by the Riemannian scalar product on T_pM. In particular, the tangent cone of the euclidean space is that space itself. This observation leads to the following *general definition of flatness*: a metric space is called *flat* if it has a tangent cone at each point and if the tangent cone is locally isometric to the space itself. H is flat in this sense because it is, like the euclidean space, its own tangent cone. Since H_c is also the tangent cone of any 3-dimensional C–C manifold, Definition 1.4 is compatible with the general definition of flatness.

§ 2. Universal covering space. All results of this section seem to be known, but we do not know any references. A proof of the crucial fact 2.3 is given in the Appendix. The main result is Corollary 2.6, which is an analogue of the result for flat Riemannian manifolds (cf. [W], Section 3.2). We begin with a description of the group of all differentiable isometries of H_c.

2.1. Left group translations are, of course, isometries of H_c. We will identify elements of H with the corresponding left translations, and the whole group of such translations will also be denoted by H.

Another family of isometries of H_c is the group of isomorphisms of H obtained from the Lie algebra isomorphisms

$$X \mapsto \cos tX - \sin tY, \quad Y \mapsto \sin tX + \cos tY, \quad Z \mapsto Z\,. \tag{2.1.1}$$

This group of differentiable isometries will be denoted by SO(2) and its elements will be called *rotations*.

The isomorphism of H given by

$$X \mapsto -X, \quad Y \mapsto Y, \quad Z \mapsto -Z\,, \tag{2.1.2}$$

is also an isometry of H_c, and the group of isometries generated by it and SO(2), being isomorphic to the orthogonal group of $\mathbb{R}^2$, will be denoted by O(2).

2.2. Denote by G_c the group generated by O(2) and H. G_c is a 4-dimensional Lie group, a semidirect product of H and O(2).

2.3. Theorem. *G_c is the group of all differentiable isometries of H_c.*

2.4. Theorem. *If $\phi : U_1 \to U_2$ is a differentiable isometry of open subsets of H_c then there exists a unique differentiable isometry $\widetilde{\phi} \in G_c$ which extends ϕ.*

2.5. Theorem. *Let M be a compact flat C–C* 3*-manifold. Then its universal covering space with the lifted C–C metric is differentiably isometric to H_c.*

A proof of 2.3 is sketched in the Appendix; 2.4 can be proved by the same methods; 2.5 is a consequence of 2.4. The results of the next sections are based on the following corollary to Theorem 2.5:

2.6. Corollary. *Any compact flat C–C* 3*-manifold is the quotient of H_c by a group Γ of differentiable isometries satisfying the following conditions*:

(2.6.1)
- (i) *Γ acts freely*, *i.e.* $\gamma \in \Gamma$ *has no fixed points unless* $\gamma = \mathrm{id}$;
- (ii) *Γ acts totally discontinuously*, *i.e. there exists an open subset $U \subseteq H_c$ such that $\{\gamma(U) : \gamma \in \Gamma\}$ consists of disjoint sets*;
- (iii) *the action of Γ is cocompact*, *i.e. the quotient space H_c/Γ is compact.*

Furthermore, *two groups satisfying* (2.6.1) *give isometric quotients if and only if they are conjugate in* G_c.

§ 3. Computations in G_c.

This section contains technical results and formulas.

3.1. The elements of the identity component of G_c will be called *even isometries*; the remaining ones will be called *odd*.

3.2. Denote by Z the center of the Heisenberg group H; it consists of elements of the form $(0,\ 0,\ z)$. Since it is isomorphic to $\mathbb{R}$, we will apply the additive notation to it. The identification of Z with $\mathbb{R}$ enables us to multiply a central element by a real scalar.

The same remarks on notation concern the group of central translations (also denoted by Z). We will use the letters z and h to denote both central elements and central translations.

3.3. Central translations, by definition, commute with elements of H. They also commute with elements of $\mathrm{O}(2)$ as each $\alpha \in \mathrm{SO}(2)$ is the identity on Z, and so, recalling that α is an isomorphism of H, we have

$$\alpha \circ z(x) = \alpha(z(x)) = \alpha(zx) = \alpha(z)\alpha(x) = z\alpha(x) = z \circ \alpha(x) \tag{3.3.1}$$

where $z \in Z$ and x is any element of H. Using the fact that $m(z) = -z$ for $m \in \mathrm{O}(2)\backslash\mathrm{SO}(2)$ and $z \in Z$ one can prove similarly that $m \circ z = (-z) \circ m$.

3.4. Any orbit of Z in H_c will be called a *vertical line*. We give the vertical lines the orientations induced by a fixed orientation of Z. All isometries in G_c transform vertical lines onto vertical lines; even isometries preserve the orientation and odd ones reverse it.

3.5. It is useful to consider the action of G_c on the quotient space H_c/Z, the space of vertical lines, which is diffeomorphic to $\mathbb{R}^2$. Since central translations act on H_c/Z identically, the group acting on this space is G_c/Z, which is isomorphic to the group $\mathrm{E}(2)$ of isometries of the euclidean plane $H/Z \cong \mathbb{R}^2$. Let Π denote both quotient maps $H_c \to H_c/Z$ and $G_c \to G_c/Z$.

3.6. If $p \in H_c/Z$ then the projection of the scalar product from Δ_x to $T_p(H_c/Z)$ does not depend on the choice of a point $x \in H_c$ such that $\Pi(x) = p$. The Riemannian metric on H_c/Z defined in this way makes this space isometric to the euclidean plane E^2.

The transformations in G_c/Z acting on H_c/Z are then isometries and so the identification of G_c/Z with $\mathrm{E}(2)$ makes sense. In particular, $\Pi(H) = H/Z$ corresponds to the group of translations of E^2. The even isometries of H_c are transformed by Π onto the group $\mathrm{SE}(2)$ of orientation preserving isometries of the euclidean plane, and the odd ones onto the second component of $\mathrm{E}(2)$ consisting of orientation reversing isometries.

3.7. Let α_p denote the rotation through angle α about p in H_c/Z. Then $\Pi^{-1}(\alpha_p)$ consists of transformations in G_c such that any two of them differ by a central translation. *$\Pi^{-1}(\alpha_p)$ includes a unique transformation that preserves each point of the vertical line $\Pi^{-1}(p)$.* It will be denoted by α_x, and called the *rotation about the vertical line $\Pi^{-1}(p)$* (here x is any point of this line).

P r o o f. Transformations in $\Pi^{-1}(\alpha_p)$ preserve the vertical line $\Pi^{-1}(p)$ and, since by 3.6 they are even, they preserve its orientation (cf. 3.4). When restricted to this vertical line, they are translations, and only one of them is the identity. ■

From the above definition we see that for any $x \in H_c$ and $z \in Z$

(3.7.1) $$\alpha_x = \alpha_{z(x)}\,.$$

3.8. Let k_p denote the reflection of H_c/Z with axis parallel to a direction k and containing a point $p \in H_c/Z$. *For each $x \in \Pi^{-1}(p)$ there exists a unique transformation in $\Pi^{-1}(k_p)$ preserving x.* Denote it by k_x.

P r o o f. Transformations in $\Pi^{-1}(k_p)$ preserve the vertical line $\Pi^{-1}(p)$ but, as they are odd, they reverse the orientation. When restricted to that line, they are reflections, and since they differ by central translations, any reflection may be obtained uniquely by composing a fixed one with the elements of Z. ■

From the above considerations one obtains the formula

(3.8.1) $$(2z) \circ k_x = k_x \circ (-2z) = k_{z(x)}\,.$$

We will use the letters k, m to denote the directions on the plane H_c/Z and call isometries of the form k_x *reflections in H_c*.

3.9. Denote by δ the subspace of the Lie algebra **h** of the Heisenberg group generated by the vectors X and Y (cf. 1.3). If $\Pi(x) = p$, then Π yields an isomorphism between $T_p(H_c/Z)$ and Δ_x. We may treat this isomorphism as an identification of vectors of the plane H_c/Z with vectors of δ (any vector of Δ_x can be uniquely extended to a left-invariant vector field on H which is an element of δ).

For u being a translation of H_c/Z (which we identify with an element of δ in the above sense) and for any $x \in H$

(3.9.1) $$f(t) = x \circ \exp(tu) \circ x^{-1}$$

is the only one-parameter subgroup of H such that $\Pi(f(1)) = u$ and the orbit $f(t)(x)$ is tangent to Δ at x. $f(1)$ is then an element of G_c which is uniquely determined by u and x; we denote it by u_x. The orbit of $f(t)$ containing x will be denoted by $L_{u,x}$ and called the *horizontal line parallel to u and passing through x*.

In the following we will use the letters u, w and v to denote translations (or equivalently vectors) of H_c/Z.

3.10. Note that $u_y = u_x$ for $y \in L_{u,x}$. Indeed,

$$y = x \circ \exp(su) \circ x^{-1}(x) = x \exp(su) \tag{3.10.1}$$

for some $s \in \mathbb{R}$. Applying (3.9.1) with $t = 1$ we obtain

$$\begin{aligned} u_y &= y \circ \exp u \circ y^{-1} = x \circ \exp(su) \circ \exp u \circ \exp(-su) \circ x^{-1} \\ &= x \circ \exp u \circ x^{-1} = u_x\,. \end{aligned} \tag{3.10.2}$$

We will denote points of $L_{u,x}$ of the form $x \circ \exp(su) = (su)_x(x)$ by $su(x)$.

3.11. The isometry defined by (2.1.2) is the reflection k_e, where e is the unit of H and k is the direction on H_c/Z parallel to Y. It follows from (2.1.2) that k_e *preserves the horizontal line* $L_{Y,e}$. We will sometimes denote this line by $L_{k,e}$, as it depends only on the direction of Y. Similarly *for any direction* m *and any* $y \in H_c$ *the reflection* m_y *preserves the horizontal line* $L_{m,y}$. An easy consequence is that if $a \in L_{m,y}$ then

$$m_y = m_a\,. \tag{3.11.1}$$

3.12. Observe that *if* $\gamma_1, \gamma_2 \in G_c$ *then* $\gamma_1 = \gamma_2$ *if and only if* $\Pi(\gamma_1) = \Pi(\gamma_2)$ *and there exists* $x \in H_c$ *such that* $\gamma_1(x) = \gamma_2(x)$; if $\Pi(\gamma_1) = \Pi(\gamma_2)$ then γ_1 and γ_2 may differ at most by a central translation (cf. 3.5).

3.13. *If* u *is parallel to a direction* k*, then the isometries* u_x *and* k_x *commute.* That follows from 3.12 and the fact that the plane isometries $u = \Pi(u_x)$ and $k_p = \Pi(k_x)$ commute, and both u_x and k_x preserve the horizontal line $L_{u,x}$ and commute on it (cf. 3.9 and 3.11). In particular,

$$u_x \circ k_x \circ u_x \circ k_x = u_x \circ u_x \circ k_x \circ k_x = (2u)_x\,. \tag{3.13.1}$$

3.14. An argument very similar to that in 3.11 shows that *if* k *is orthogonal to* u *then*

$$k_x \circ u_x \circ k_x = (-u)_x\,. \tag{3.14.1}$$

From this it follows that *the reflection* k_x *preserves the horizontal line* $L_{u,x}$ (*reflecting it with respect to* x). But this in turn implies that

$$k_{u(x)} \circ k_x = (2u)_x\,, \tag{3.14.2}$$

and combining this with (3.8.1) we see that *each group translation* $g \in H$ *can be expressed as a composition of two reflections:*

$$g = k_y \circ k_x \tag{3.14.3}$$

with k orthogonal to $\Pi(g)$ and $y = \gamma(x)$, where $\gamma \circ \gamma = g$ (we will denote that uniquely determined γ by $\frac{1}{2}g$).

3.15. If the composition of two reflections on the plane is a rotation, i.e. $m_p \circ k_p = \alpha_p$ for $p \in H_c/Z$, we have $m_x \circ k_x = \alpha_x \circ z$ for some $z \in Z$. But $m_x \circ k_x(x) = x$, hence $z = 0$ and we have

$$m_x \circ k_x = \alpha_x \,. \tag{3.15.1}$$

3.16. Let us calculate $u_x \circ \pi_x$ where π_x is a rotation through π in H_c. Denote by k the direction parallel to u and by m the orthogonal one. Then, writing $y = \frac{1}{2}u(x)$, we have $u_x = m_y \circ m_x$ and $\pi_x = m_x \circ k_x$, and we obtain

$$u_x \circ \pi_x = m_y \circ m_x \circ m_x \circ k_x = m_y \circ k_x \,. \tag{3.16.1}$$

But since $y = \frac{1}{2}u(x) \in L_{x,k}$ (u is parallel to k), $k_y = k_x$ and so

$$u_x \circ \pi_x = m_y \circ k_y = \pi_y \,. \tag{3.16.2}$$

3.17. Since [**h**,[**h**,**h**]]=0 for the Lie algebra **h** of the Heisenberg group, the Campbell–Hausdorff formulas simplify considerably, for example

$$\exp V \cdot \exp W \cdot \exp(-V) = \exp(W + [V,W]) = \exp W \cdot \exp([V,W]) \,, \tag{3.17.1}$$

$$[\exp V, \exp W] = \exp([V,W]) \,, \tag{3.17.2}$$

$$\exp V \cdot \exp W = \exp(V+W) \cdot \exp(\tfrac{1}{2}[V,W]) \tag{3.17.3}$$

for any $V, W \in h$.

3.18. From (3.17.2) it is clear that the center Z is also the commutant $[H,H]$ of the Heisenberg group. Let $[a,b] = aba^{-1}b^{-1}$ for $a, b \in H$. Since $[a,b] = [a \cdot z_1, b \cdot z_2]$ for $z_1, z_2 \in Z$, the value of $[a,b]$ depends only on $\Pi(a)$ and $\Pi(b)$ and so, if u, w are translations of H_c/Z then $[u,w] = [u_x, w_y]$ is well defined. [,] may be viewed as a bilinear antisymmetric form on the vector space H_c/Z with values in Z. Indeed,

$$\begin{aligned}[tu, sw] &= [(tu)_x, (sw)_x] = [\exp(tU), \exp(sW)] = \exp([tU, sW]) \\ &= \exp(ts[U,W]) = ts \cdot \exp([U,W]) = ts[\exp U, \exp W] \\ &= ts[u_x, w_x] = ts[u,w] \,.\end{aligned}$$

3.19. If $V, W \in \mathbf{h}$ and $\exp W = u_x$, $\exp V = g$, we obtain from (3.17.1)

$$g \circ u_x \circ g^{-1} = u_x \circ [g, u_x] \,. \tag{3.19.1}$$

Notice that $g \circ u_x \circ g^{-1} = u_{g(x)}$ because, recalling 3.9, if $\exp(tW)x$ is tangent to Δ at x then $g \circ \exp(tW) \circ g^{-1}(g(x))$ is tangent to Δ at $g(x)$. Then

$$u_{g(x)} = u_x \circ [g, u_x] \,. \tag{3.19.2}$$

The particular case with g replaced by $z \in Z$ gives

$$u_{z(x)} = u_x \,. \tag{3.19.3}$$

3.20. If the vectors V and W in (3.17.3) are taken from δ then $\exp V = V_e$, $\exp W = W_e$ and $\exp(V+W) = (V+W)_e$, where e is the unit of H.

Then we have

$$V_e \circ W_e = (V+W)_e \circ \tfrac{1}{2}[V,W]\,. \tag{3.20.1}$$

Using (3.19.2) we obtain a similar result for any $x \in H_c$ and any translations u, w of H_c/Z:

$$u_x \circ w_x = (u \circ w)_x \circ \tfrac{1}{2}[u,w]\,. \tag{3.20.2}$$

3.21. We wish to calculate the composition of a nonzero rotation α_x with a translation u_x. Choose directions m and k so that $\alpha_x = k_x \circ m_x$ and $u_x = k_{\frac{1}{2}u(x)} \circ k_x$. If $p = \Pi(x)$ we have

$$\Pi(u_x \circ \alpha_x) = u \circ \alpha_p = k_{\frac{1}{2}u(p)} \circ k_p \circ k_p \circ m_p = k_{\frac{1}{2}u(p)} \circ m_p = \alpha_q \tag{3.21.1}$$

where q is the intersection point of the axes of the reflections $k_{\frac{1}{2}u(p)}$ and m_p. From (3.21.1) we obtain

$$u_x \circ \alpha_x = k_{\frac{1}{2}u(x)} \circ m_x = \alpha_y \circ z \tag{3.21.2}$$

for any $y \in \Pi^{-1}(q)$ and some $z \in Z$. Our aim is to calculate z.

We are interested in the horizontal lines $L_{m,x}$ and $L_{k,\frac{1}{2}u(x)}$ which correspond to the sets of fixed points of the two reflections in the middle term of (3.21.2). Both lines intersect the vertical line $\Pi^{-1}(q)$; denote the points of intersection by a and b respectively. Then, by (3.10.2) and (3.11.1), $m_x = m_a$ and $k_{\frac{1}{2}u(x)} = k_b$, and consequently $u_x \circ \alpha_x = m_a \circ k_b$. If $h(a) = b$ for $h \in Z$, then from (3.8.1), $u_x \circ \alpha_x = \alpha_a \circ 2h$. We will calculate h.

Denote by w the vector on H_c/Z perpendicular to u and such that $(w + \frac{1}{2}u)(p) = q$. Since the transformation $(\frac{1}{2}u)_x$ preserves the distribution Δ, it transforms the line $L_{k,x}$ onto the line $L_{k,\frac{1}{2}u(x)}$, and therefore transforms the point $w_x(x)$ into $b \in L_{k,\frac{1}{2}u(x)}$. Hence $b = (\frac{1}{2}u)_x \circ w_x(x)$. Now, since $a \in L_{m,x}$ and $\frac{1}{2}u + w$ is parallel to m, we have by (3.20.2)

$$a = (\tfrac{1}{2}u + w)_x(x) = (\tfrac{1}{2}u)_x \circ w_x \circ \tfrac{1}{2}[w, \tfrac{1}{2}u](x) = \tfrac{1}{4}[w,u](b)\,, \tag{3.21.4}$$

and therefore $h = \frac{1}{4}[u,w]$. So we obtain the following formula:

$$u_x \circ \alpha_x = \alpha_a \circ \tfrac{1}{2}[u,w]\,. \tag{3.21.5}$$

§ 4. Nilmanifolds

4.1. Definition. Any subgroup of G_c satisfying conditions (2.6.1) will be called a *discrete uniform subgroup*.

Note that usually "discrete uniform" only means that the group is discrete and its orbit space is compact. Here we assume additionally that it acts totally discontinuously.

4.2. Remark. If a discrete uniform subgroup Γ of G_c is contained in H then H_c/Γ is a Heisenberg group nilmanifold carrying the induced C–C metric. Such a Γ is then, of course, a discrete uniform subgroup of H.

By Theorem 2.21 of [R] we have the following

4.3. Theorem. *Any discrete uniform subgroup of H has the form*

$$\Gamma = \mathrm{gp}\{a, b, c\} \tag{4.3.1}$$

where $[a, b] \neq 0$ and $c = \frac{1}{n}[a, b]$ for some natural n, and where $\mathrm{gp}\, A$ denotes the group generated by the set A.

The number n is determined by Γ and will be called the *index* of Γ.

4.4. Proposition. *If Γ is a discrete uniform subgroup of H then $\Pi(\Gamma)$ is a rank two lattice of translations of the euclidean plane H_c/Z.*

Proof. If Γ is as in (4.3.1) then $\Pi(a)$ and $\Pi(b)$ generate $\Pi(\Gamma)$. They are linearly independent since otherwise, according to 3.18, $[a, b] = 0$. ∎

4.5. Definition. Any orbit of the action of $\Pi(\Gamma)$ on the plane H_c/Z will be called a *fundamental lattice* of Γ.

The importance of the introduced notions is revealed by

4.6. Theorem. *Two discrete uniform subgroups of H are conjugate in G_c if and only if they have the same indexes and isometric fundamental lattices.*

Proof. Let $\Gamma_j = \mathrm{gp}\{\underline{a}_j, \underline{b}_j, c_j\}$ for $j = 1, 2$, $\Pi(\underline{a}_j) = a_j, \Pi(\underline{b}_j) = b_j$, and $\underline{a}_1 = (a_1)_x \circ z_a$, $\underline{b}_1 = (b_1)_x \circ z_b$. Then, by 3.18 and (3.19.2), for any real numbers t and s, setting $p = (ta_1 + sb_1)_x(x)$, we have

$$\underline{a}_1 = (a_1)_x \circ z_a = (a_1)_p \circ ([a_1, ta_1 + sb_1] + z_a) = (a_1)_p \circ (t[a_1, b_1] + z_a)\,,$$

and similarly $\underline{b}_1 = (b_1)_p \circ (s[a_1, b_1] + z_b)$. Now, since $[a_1, b_1] \neq 0$, we can choose t and s such that $\underline{a}_1 = (a_1)_p$ and $\underline{b}_1 = (b_1)_p$. In an analogous manner $\underline{a}_2 = (a_2)_q$ and $\underline{b}_2 = (b_2)_q$ for some q, and we have

$$\Gamma_1 = \mathrm{gp}\{(a_1)_p, (b_1)_p, c_1\}, \qquad \Gamma_2 = \mathrm{gp}\{(a_2)_q, (b_2)_q, c_2\}\,.$$

Suppose that the isometry of the fundamental lattices is given by $i \circ a_1 \circ i^{-1} = a_2$ and $i \circ b_1 \circ i^{-1} = b_2$, where i is an isometry of H_c/Z, and take any $\gamma \in G_c$ with $\Pi(\gamma) = i$. If we compose this γ with a translation in H obtaining ϕ such that $\phi(p) = q$ then $\Pi(\phi) \circ a_1 \circ \Pi(\phi)^{-1} = a_2$. Let us calculate $\phi \circ (a_1)_p \circ \phi^{-1}$. It must belong to $\Pi^{-1}(a_2)$, but since the orbit $\{\phi\circ(ta_1)_p\circ\phi^{-1}(q) : t \in \mathbb{R}\}$ is tangent to Δ at q (because $(ta_1)_p(p)$ is tangent at p), we simply have

$$\phi \circ (a_1)_p \circ \phi^{-1} = (a_2)_q \quad \text{and similarly} \quad \phi \circ (b_1)_p \circ \phi^{-1} = (b_2)_q\,,$$

thus ϕ conjugates Γ_1 with Γ_2. The inverse implication is obvious. ∎

§ 5. Holonomy. In this section we introduce an invariant of a flat C–C manifold which is an analogue of the holonomy group of a Riemannian manifold. We prove that this group is finite and that only a finite number of groups appear as a holonomy group. The main result of this section is a classification of the holonomy groups. The classification in §6 is its refinement.

Recall that G_c is a semidirect product of H and $\mathrm{O}(2)$. Denote by $\Psi : G_c \to \mathrm{O}(2)$ the canonical quotient homomorphism onto the second factor.

5.1. DEFINITION. The *holonomy group* of a compact flat C–C 3-manifold is the group $\Psi(\Gamma)$ where Γ is such that H_c/Γ is isometric to M.

Since $\Psi(\gamma\Gamma\gamma^{-1}) = \Psi(\gamma)\Psi(\Gamma)\Psi(\gamma)^{-1}$, $\Psi(\Gamma)$ depends up to isomorphism only on the conjugacy class of Γ in G_c, so it is a well defined invariant of M.

5.2. Remark. It is possible to define a *parallel translation* on M by taking the group translation in the universal covering H_c of M. The holonomy group obtained with the use of this parallel translation can be easily identified with the group defined above.

The following result is a special case of Theorem 1 of [A]:

5.3. THEOREM. *Let Γ be a discrete uniform subgroup of G_c. Then $\Gamma^* = \Gamma \cap H$ is a normal subgroup of finite index in Γ and so it is also a discrete uniform subgroup of G_c.*

COROLLARY. *The holonomy group $\Psi(\Gamma)$ is a finite subgroup of* $\mathrm{O}(2)$.

Proof. Γ^* is the kernel of the homomorphism $\Psi : \Gamma \to \mathrm{O}(2)$. ∎

Remark. Any manifold H_c/Γ is finitely covered by a nilmanifold.

Recall that $\mathrm{E}(2)$, the group of isometries of the euclidean plane, is a semidirect product of $\mathbb{R}^2$ and $\mathrm{O}(2)$, hence there exists a natural homomorphism $\Phi : \mathrm{E}(2) \to \mathrm{O}(2)$. We mention without proof the following easy

5.4. PROPOSITION. *The transformation $\Psi(\gamma) \mapsto \Phi \circ \Pi(\gamma)$ for $\gamma \in \Gamma$ is well defined and is an isomorphism of* $\mathrm{O}(2)$.

Take $\gamma \in \Gamma$ such that $\Psi(\gamma) \in \mathrm{SO}(2)\backslash\{\mathrm{id}\}$. Then γ is even, hence $\Pi(\gamma)$ is an orientation preserving isometry of H_c/Z. As it cannot be a translation, it must be a rotation about some $x \in H_c/Z$.

5.5. LEMMA. *Let γ and x be as above and let $\Lambda(x)$ be the orbit of x under the action of $\Pi(\Gamma^*)$. Then the rotation $\Pi(\gamma)$ preserves the lattice $\Lambda(x)$.*

Proof. Let $y \in \Lambda(x)$. By definition, there exists $g \in \Gamma^*$ such that $y = \Pi(g)(x)$. Then $\Pi(\gamma)(y) = \Pi(\gamma)\circ\Pi(g)(x) = \Pi(\gamma)\circ\Pi(g)\circ\Pi(\gamma)^{-1}(x) =$

$\Pi(\gamma g\gamma^{-1})(x)$. Since Γ^* is a normal subgroup of Γ, $\gamma g\gamma^{-1} \in \Gamma^*$, hence $\Pi(\gamma)(y) \in \Lambda(x)$. ∎

By the argument of Lemma 3.5.2 in [W] the following corollary can be obtained from Lemma 5.5:

5.6. Corollary. *If $\Psi(\gamma) \in \mathrm{SO}(2)\backslash\{\mathrm{id}\}$ then $\Pi(\gamma)$ is a rotation through an angle of $\pi/3$, $\pi/2$, $2\pi/3$ or π. In view of Proposition 5.4 the same holds for $\Psi(\gamma)$.*

The corollary shows that *if the holonomy group is contained in* SO(2) *then it can only be the cyclic rotation group of order* 2, 3, 4 *or* 6. Now we shall study other possibilities.

5.7. Lemma. *If the reflection k_e belongs to $\Psi(\Gamma)$ then there exists in Γ an element of the form $k_x \circ u_x$ where u is parallel to k.*

Proof. Let $\gamma \in \Gamma$ and $\Psi(\gamma) = k_e$. Then $\Pi(\gamma)$ is the composition of a reflection with a nonzero translation u parallel to k, $\Pi(\gamma) = u \circ k_p$ (if $u = 0$ then $\Pi(\gamma)$ has a fixed point in $\Pi^{-1}(p)$). Then $\gamma = u_x \circ k_y \circ z$ for some x, $y \in \Pi^{-1}(p)$ and $z \in Z$. The lemma now follows from (3.8.1) and (3.19.3). ∎

5.8. Lemma. *If $k_e \in \Psi(\Gamma)$ then one can choose $x \in H_c$ and u, a vector on H_c/Z, in such a way that u is parallel to k, $k_x \circ u_x \in \Gamma$, and there is $v \in \Pi(\Gamma^*)$ orthogonal to u such that $2u$ and v generate the fundamental lattice of Γ (which is rectangular).*

Proof. u can be chosen to be the shortest possible vector parallel to k with $k_x \circ u_x \in \Gamma$ for some x. Then, since by (3.13.1), $(2u)_x = (k_x \circ u_x)^2 \in \Gamma^*$, there exists an element $w_x \circ z \in \Gamma^*$ such that $2u$ and w generate the fundamental lattice of Γ^*. By (3.19.1) we have

$$\begin{aligned}(k_x \circ u_x) \circ (w_x \circ z) \circ (k_x \circ u_x)^{-1} &= k_x \circ u_x \circ w_x \circ z \circ u_x^{-1} \circ k_x \\ &= k_x \circ ([u, w] + z) \circ w_x \circ k_x = k_x \circ w_x \circ k_x \circ ([w, u] - z)\,.\end{aligned}$$

Denote $\Pi(k_x \circ w_x \circ k_x)$ by w^k. Since w^k and w are symmetric with respect to k and both belong to $\Pi(\Gamma^*)$, $w + w^k = 2nu$ for some integer n (recall that u is parallel to k). The number n cannot be odd since in this case the transformation $(w_x \circ z) \circ (k_x \circ u_x)^{-n}$ would have a fixed point, which is impossible as it belongs to a discrete uniform Γ. So $\Pi((w_x \circ z) \circ (2u)_x^{-n/2})$ can be taken for v. ∎

5.9. Corollary. *If $k_e \in \Psi(\Gamma)$ then the index of Γ^* is even.*

Proof. By Lemma 5.8, let u, v generate $\Pi(\Gamma^*)$ with $k_x \circ u_x \in \Gamma$, $v_x \circ z \in \Gamma$ and v orthogonal to u and k. Then using (3.14.1) and (3.19.1) we get

$$(k_x \circ u_x)^{-1} \circ (v_x \circ z) \circ k_x \circ u_x = u_x^{-1} \circ k_x \circ v_x \circ z \circ k_x \circ u_x$$

$$= u_x^{-1} \circ (v_x \circ z)^{-1} \circ u_x = [u,v] \circ (v_x \circ z)^{-1} ,$$

which proves that $[u,v] \in \Gamma^*$. But $[u,v] = \frac{1}{2}[2u,v]$, so the index is even. ∎

As a consequence of Lemma 5.8 the following restriction on the holonomy group can be formulated:

5.10. Corollary. *If $\Psi(\Gamma)$ contains a reflection then it contains at most one more reflection with axis orthogonal to the first one*, *and also a rotation through* π.

P r o o f. This follows from the fact that a lattice can have at most one orthogonal basis. ∎

§ 6. Classification. The classification of compact flat C–C 3-manifolds given in this section is done separately for different holonomy groups. In fact, it is a classification of discrete uniform subgroups of G_c up to conjugacy (compare Corollary 2.6). In the whole section the results and notation of §3 are used. The symbol $|u|$ will denote the length of a vector u in the canonical metric described in 3.6.

6.1. $\Psi(\Gamma) = \{\mathrm{id}\}$. This case has been considered in §4. According to Theorems 4.3 and 4.6, the general form of Γ is

$$\Gamma = \mathrm{gp}\left\{u_x, v_x, \frac{1}{n}[u,v]\right\}$$

where u, v are linearly independent vectors in H_c/Z.

The conjugacy class for groups of this type is completely determined by the isometry class of the fundamental lattice and the index number n.

6.2. $\Psi(\Gamma) = \{\mathrm{id}, k_e\}$. By Lemma 5.8 together with Corollary 5.9 we have

$$\Gamma = \mathrm{gp}\left\{u_x \circ z, k_x \circ v_x, \frac{1}{n}[u,v]\right\}$$

where u, v are orthogonal and k is parallel to v. According to 3.10, (3.11.1) and (3.19.2), we can choose $y \in L_{x,k}$ such that $u_x \circ z = u_y$, $k_x = k_y$ and $v_x = v_y$. This gives the more convenient form

$$\Gamma = \mathrm{gp}\left\{u_y, k_y \circ v_y, \frac{1}{n}[u,v]\right\} . \tag{6.2.1}$$

For any orthogonal u, v the group of the above form is discrete uniform. In order to prove that, it suffices to show that no $\phi \in \Gamma, \phi \neq \mathrm{id}$, has fixed points. That is obvious for $\phi \in \Gamma^* \backslash \{\mathrm{id}\}$. For $\phi \in \Psi^{-1}(k_e) = k_y \circ v_y \circ \Gamma^*$ that follows from the fact that $\Pi(\phi)$ has no fixed points on H_c/Z.

Two groups of the form (6.2.1) are conjugate in G_c if and only if $n_1 = n_2$, $|u_1| = |u_2|$ and $|v_1| = |v_2|$. To prove this observe that Γ_1^* and Γ_2^*

are conjugate under these conditions, and the conjugating map $\phi \in G_c$ transforms L_{y_1,k_1} onto L_{y_2,k_2}, so it conjugates the reflections too.

6.3. $\Psi(\Gamma) = \{\mathrm{id}, k_e, m_e, \pi_e\}$; k is orthogonal to m. According to Lemma 4.9 and Corollary 4.11

$$\Gamma = \mathrm{gp}\{k_x \circ u_x, m_y \circ w_y, z_0\}\,, \tag{6.3.1}$$

where the vectors $2u$ and $2w$, parallel to k and m respectively, generate the fundamental lattice of Γ^*.

Since the lines $\Pi(L_{k,x})$ and $\Pi(L_{m,y})$ are orthogonal, they intersect at a point of H_c/Z. Hence, by 3.10, we may assume that, in (6.3.1), $y = z(x)$ for some $z \in Z$. But then $m_y \circ w_y = m_y \circ w_x = m_x \circ w_x \circ 2z$ (cf. (3.19.1) and (3.8.1)), and denoting $2z$ by h we obtain

$$\Gamma = \mathrm{gp}\{k_x \circ u_x, m_x \circ w_x \circ h, z_0\}\,. \tag{6.3.2}$$

Moreover, $[u,w] \in \Gamma^*$. To prove this, we first combine (3.14.1), (3.20.2) and (3.16.2) to obtain

$$\begin{aligned}(k_x \circ u_x) \circ (m_x \circ w_x \circ h) &= k_x \circ m_x \circ (-u)_x \circ w_x \circ h \\ &= \pi_x \circ (w-u)_x \circ (\tfrac{1}{2}[-u,w] + h) = \pi_a \circ (h - \tfrac{1}{2}[u,w]) = \pi_a \circ h_0\,,\end{aligned} \tag{6.3.3}$$

where $a = (\frac{1}{2}(w-u))_x(x)$. Next we use the following relations: $(2u)_x = (2u)_a \circ [u,w]$ by (3.19.2), $\pi_a = k_a \circ m_a$ by 3.15, $k_a \circ (2u)_a = (2u)_a \circ k_a$ by 3.13, $m_a \circ (2u)_a = m_{-u(a)}$ by (3.14.2), $k_a = k_{-u(a)}$ by 3.13, to calculate

$$\begin{aligned}(\pi_a \circ h_0) \circ (2u)_x &= m_a \circ k_a \circ (2u)_a \circ (h_0 + [u,w]) \\ &= m_b \circ k_b \circ (h_0 + [u,w]) = \pi_b \circ (h_0 + [u,w])\,,\end{aligned} \tag{6.3.4}$$

where $b = (-u)_a(a)$. Now, observe that $2h_0 \in \Gamma^*$ $(2h_0 = (\pi_a \circ h_0)^2)$ and $2[u,w] \in \Gamma^*$ (Corollary 5.9), but $h_0 \notin \Gamma^*$ (otherwise $\pi_a \in \Gamma$ and the action of Γ is not free). If $[u,w] \notin \Gamma^*$ then $(h_0 + [u,w]) \in \Gamma^*$ and consequently, since the left-hand side of (6.3.4) is a composition of elements of Γ, $\pi_b \in \Gamma$. But π_b has fixed points, which gives a contradiction. Thus $[u,w] \in \Gamma^*$ and the index of Γ^* is a multiple of 4.

Returning to (6.3.3) we have $2h_0 = 2h - [u,w]$ and so $2h \in \Gamma^*$. Moreover, if $\frac{1}{2}[u,w] \in \Gamma^*$ we may assume $h = \frac{1}{2}z_0$ and if $\frac{1}{2}[u,w] \notin \Gamma^*$ we may assume $h = 0$ (we know that $h \notin \Gamma^*$ and $2h \in \Gamma^*$). We then obtain two possible forms of Γ:

$$\begin{aligned}\Gamma &= \mathrm{gp}\left\{k_x \circ u_x, m_x \circ w_x, \frac{1}{2n+1}[u,w]\right\}, \\ \Gamma &= \mathrm{gp}\left\{k_x \circ u_x, m_x \circ w_x \circ h, \frac{1}{2n}[u,w]\right\}, \qquad h = \frac{1}{4n}[u,w]\,,\end{aligned} \tag{6.3.5}$$

where $k \perp m \perp u \perp w$.

By the argument of 6.2 we see that in groups of the form (6.3.5) no maps in $\Psi^{-1}(\{k_e, m_e\})$ have fixed points. Further, arguments similar to the above discussion show that the same holds for $\Psi^{-1}(\pi_e)$. This implies that if we choose orthogonal u, w of any length and any natural n then the group of the form (6.3.5) will be discrete uniform. It is obvious that Γ_1, Γ_2 are conjugate if and only if Γ_1^*, Γ_2^* are. The conjugacy class is then determined by the fundamental lattice and the index number.

6.4. $\Psi(\Gamma) = \{\mathrm{id}, \pi_e\}$. By 4.3 we may assume that $\Gamma^* = \mathrm{gp}\{f, g, z_0\}$, $z_0 = \frac{1}{n}[f, g]$. Then

$$\Gamma = \mathrm{gp}\{f, g, \pi_x \circ z, z_0\} = \mathrm{gp}\{f, g, \pi_x \circ z\} \quad \text{for some } x \in H_c\,. \tag{6.4.1}$$

Note that $2z = (\pi_x \circ z)^2 \in \Gamma^*$, hence $z = \frac{1}{2}z_0$ may be assumed, and then $z_0 = (\pi_x \circ z)^2$.

Define $u = \Pi(f)$, $w = \Pi(g)$ and let $f = u_x \circ z_u$, $g = w_x \circ z_w$ for some $z_u, z_w \in Z$. Then, by 3.16, $\pi_x \circ z \circ u_x \circ z_u = \pi_{\frac{1}{2}u(x)} \circ (z + z_u)$, hence $2z_u \in \Gamma^*$. Thus, by composing g with a multiple of z_0, z_u may be assumed to equal 0 or $\frac{1}{2}z_0$. The second possibility implies, however, that $(z + z_u) \in \Gamma^*$, hence $\pi_{\frac{1}{2}u(x)} \in \Gamma$, a contradiction. Similarly one may prove that $z_w = 0$.

To check that no ϕ in $\Psi^{-1}(\pi_e)$ has fixed points we examine (by use of (3.20.2) and 3.16) the composition

$$\begin{aligned}(u_x)^k \circ (w_x)^m \circ \pi_x \circ z &= (ku)_x \circ (mw)_x \circ \pi_x \circ z \\ &= (ku + mw)_x \circ \tfrac{1}{2}[mw, ku] \circ \pi_x \circ z \\ &= \pi_{\frac{1}{2}(ku+mw)(x)} \circ (z + \tfrac{1}{2}km[w, u])\,.\end{aligned}$$

If $\frac{1}{2}[w, u] \notin \Gamma^*$ then putting $k = m = 1$ in the above formula we find that $z + \frac{1}{2}[w, u] \in \Gamma^*$ and consequently $\pi_{\frac{1}{2}(u+w)(x)} \in \Gamma$, a contradiction. Thus $\frac{1}{2}[w, u] \in \Gamma^*$, hence n in (6.4.1) is even. We then obtain

$$\Gamma = \mathrm{gp}\{u_x, w_x, \pi_x \circ z\}, \qquad z = \frac{1}{4n}[u, w]\,, \tag{6.4.2}$$

and notice that by the above argument, for any u, w linearly independent, a group of the form (6.4.2) is discrete uniform.

We omit the easy proof of the fact that the conjugacy class of any group of the type (6.4.2) is determined by the fundamental lattice and the index number.

6.5. $\Psi(\Gamma) = \mathrm{gp}\{(2\pi/3)_e\}$. Denote $2\pi/3$ by θ. By Lemma 5.5 the fundamental lattice $\Pi(\Gamma^*)$ is invariant under the rotation through θ. The only possibility then is that it is generated by two vectors u, w of the same length making angle θ. Thus we start with

$$\Gamma = \mathrm{gp}\{\theta_x \circ z, u_x \circ z_u, w_x \circ z_w, z_0\}$$

for some z_u, $z_w \in Z$ and $z = \frac{1}{3}z_0$, which can be assumed because $(\theta_x \circ z)^3 = 3z \in \Gamma^*$. Moreover, since $\theta_x \circ u_x \circ \theta_x^{-1} = w_x$, we obtain $(\theta_x \circ z) \circ (u_x \circ z_u) \circ (\theta_x \circ z)^{-1} = \theta_x \circ u_x \circ \theta_x^{-1} \circ z_u = w_x \circ z_u$, hence we may assume $z_u = z_w$.

Furthermore, by 3.20 we calculate

$$(u_x \circ z_u)^n \circ (\theta_x \circ z) = \theta_{y_n} \circ \left(z + nz_u + \frac{n^2}{6}[u,w]\right) , \tag{6.5.1}$$

where $y_n = (\frac{2n}{3}u + \frac{n}{3}w)_x(x)$, and by 3.18

$$u_x \circ z_u = u_{y_n} \circ \left(z_u + \frac{n}{3}[u,w]\right) . \tag{6.5.2}$$

Putting $n = 2$ in (6.5.1) and calculating the third power of the right-hand side we see that $6z_u \in \Gamma^*$. The possible values of z_u depend on the index number n such that $nz_0 = [u,w]$. We should investigate the cases where $n = 6k + r$ for $r = 0, 1, 2, 3, 4, 5$ in order to make sure that Γ is discrete uniform (use (6.5.1) and exclude the fixed points) and to find the more convenient form of Γ (use (6.5.2) and replace x by y_n to simplify the form or reduce it to the case previously considered). These elementary considerations will be omitted. We summarize the possible forms of Γ:

$$\Gamma = \mathrm{gp}\{\theta_x \circ z, u_x, w_x\}, \qquad z = \frac{1}{3n}[u,w],\ n \neq 6k + 4\,; \tag{6.5.3}$$

$$\Gamma = \mathrm{gp}\{\theta_x \circ z, u_x \circ h, w_x \circ h\}, \qquad z = \frac{1}{3n}[u,w]\,, \tag{6.5.4}$$

where $n = 6k + 3$ or $n = 6k + 5$, $h = \frac{1}{2}z_0 = \frac{1}{2n}[u,w]$. In both cases the conjugacy class is determined by the length of u and by n.

6.6. $\Psi(\Gamma) = \mathrm{gp}\{(\pi/3)_e\}$. As in 6.5 the fundamental lattice of Γ^* is generated by vectors u, w of the same length making angle $\pi/3$. Since Γ with the holonomy considered contains subgroups of the form (6.5.3) or (6.5.4) and (6.4.2), one may easily prove that

$$\Gamma = \mathrm{gp}\{u_x, w_x, (\pi/3)_x \circ z\}\,,$$

with $z = \frac{1}{12n}$, is the general form of a discrete uniform subgroup of G_c having such holonomy. The conjugacy class is determined by the length of u and the index number $2n$.

6.7. $\Gamma = \mathrm{gp}\{(\pi/2)_e\}$. The group with such holonomy contains a subgroup of the form (6.4.2), and following the methods of 6.4 and 6.5 one easily proves that

$$\Gamma = \mathrm{gp}\{(\pi/2)_x \circ z, u_x, w_x\}$$

where u, w generate the square fundamental lattice of Γ^* and $z = \frac{1}{8n}[u,w]$. Again, the conjugacy class is determined by the length of u and the index number $2n$.

Since, according to 5.10 and the remark after 5.6, all the holonomy groups have been taken into account, the classification is complete.

Appendix. We begin with listing the geometric properties of the C–C metric d_c on H_c.

A.1. For each C–C manifold there exist geodesics, i.e. differentiable curves locally realizing the distance d_c. For every distributional direction at each point of H_c there exists a family of geodesics tangent to it, naturally parametrized by the set $\mathbb{R}$ of real numbers (cf. [H], [V-G], p. 334).

A.2. A direct computation shows the following formula for geodesics γ_1, γ_2 tangent to each other at a point of H_c:

$$d_c(\gamma_1(t), \gamma_2(t)) = C(\gamma_1, \gamma_2)t^{3/2} + O(t^2)\,.$$

The function $\rho : (\gamma_1, \gamma_2) \to C(\gamma_1, \gamma_2)^2$ is a metric on the family of tangent geodesics, making it a space isometric to the euclidean line.

A.3. In each family of geodesics in H_c tangent to a fixed vector there is exactly one having no conjugate points. For each $r > 0$ there are exactly two geodesics having their first conjugate points at a distance r from the initial point (cf. [H] or [V-G], pp. 333–337).

A.4. It is clear that a differentiable isometry of a C–C manifold preserves the distribution defining the metric and the scalar product on it. Such an isometry transforms geodesics to geodesics and preserves the structure of conjugate points. In the case of H_c a differentiable isometry also preserves the metric structure (described in A.2) on the family of geodesics tangent to a fixed vector.

Now we prove a lemma which is a crucial step in the proof of Theorem 2.3:

LEMMA. *Let $f : \Delta_p \to \Delta_q$ be a linear transformation of fibres of the distribution Δ preserving the invariant scalar product* (*defining the metric d_c of H_c*)*. Then there exists at most one differentiable isometry F such that the tangent map dF restricted to Δ_p is equal to f.*

Proof. It is enough to show that for any geodesic starting from p there is only one geodesic starting from q that can be its image. Choose any geodesic γ starting at p and let $v \in \Delta_p$ be a vector tangent to γ at p. Then the geodesic $F(\gamma)$ must be tangent to $f(v) \in \Delta_q$. If γ has no conjugate points then the only candidate for $F(\gamma)$ is the unique geodesic tangent to $f(v)$ having no conjugate points (cf. A.3). If γ has its first point conjugate to p at a distance r from p then, again by A.3, among the geodesics tangent to $f(v)$ there are two having this property with respect to q, thus being the only candidates for $F(\gamma)$. Each choice determines F for all other geodesics tangent to the same vector, since the metric structure on the family described in A.2 must be preserved. But a direct computation of distances between points of different geodesics shows that only one of the choices leads to an isometry, which concludes the proof.

Proof of Theorem 2.3. The theorem follows from the easy observation that for any $f : \Delta_p \to \Delta_q$ as in the Lemma there exists an isometry $\phi \in G_c$ such that $d\phi|_{\Delta_p} = f$. ∎

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