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COMPACT EINSTEIN WARPED PRODUCT SPACES WITH NONPOSITIVE SCALAR CURVATURE

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Dedicated to Professor Bang-yen Chen on the occasion of his sixtieth birthday

ABSTRACT. We study Einstein warped product spaces. As a result, we prove the following: if M is an Einstein warped product space with nonpositive scalar curvature and compact base, then M is simply a Riemannian product space.

0. INTRODUCTION

Let $B = (B^m, g_B)$ and $F = (F^k, g_F)$ be two Riemannian manifolds. We denote by π and σ the projections of $B \times F$ onto B and F, respectively. For a positive smooth function f on B the warped product $M = B \times_f F$ is the product $M = B \times F$ furnished with the metric tensor g defined by $g = \pi^* g_B + f^2 \sigma^* g_F$, where * denotes the pull back. The function f is referred to as the warping function. The notion of warped product $B \times_f F$ generalizes that of a surface of revolution. It was introduced in [3] for studying manifolds of negative curvature.

A Riemannian manifold M is called Einstein if its Ricci tensor Ric is proportional to the metric g, that is, $\text{Ric} = \lambda g$, where λ is a constant on M. Obviously the Riemannian product $M = B \times F$ is Einstein if B and F are Einstein with the same scalar curvature. A warped product $B \times_f F$ with a constant warping function f can be considered as a Riemannian product.

In search of a new compact Einstein space in [2] (p. 265), A. L. Besse asked the following:

"Does there exist a compact Einstein warped product with nonconstant warping function?"

In this article, we give a negative partial answer as follows (cf. [1]):

Theorem 1. *Let $M = B \times_f F$ be an Einstein warped product space with base B a compact space. If M has nonpositive scalar curvature, then the warped product is simply a Riemannian product.*

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1. Proofs

We denote by $\mathrm{Ric}^B, \mathrm{Ric}^F$ the lifts to M of the Ricci curvatures of B and F, respectively. Then we have the following ([6]):

Proposition 2. *The Ricci curvature Ric of the warped product $M = B \times_f F$ with $k = \dim F$ satisfies*
(1) $\mathrm{Ric}(X,Y) = \mathrm{Ric}^B(X,Y) - \frac{k}{f} H^f(X,Y)$,
(2) $\mathrm{Ric}(X,V) = 0$,
(3) $\mathrm{Ric}(V,W) = \mathrm{Ric}^F(V,W) - g(V,W)f^{\#}$, $f^{\#} = \frac{-\Delta f}{f} + \frac{k-1}{f^2} g_B(\nabla f, \nabla f)$ *for any horizontal vectors X, Y and any vertical vectors V, W, where H^f and Δf denote the Hessian of f and the Laplacian of f given by $-tr(H^f)$, respectively.*

Hence the Einstein equations become

Corollary 3. *The warped product $M = B \times_f F$ is Einstein with $\mathrm{Ric} = \lambda g$ if and only if*
(1.1) $\mathrm{Ric}_B = \lambda g_B + \frac{k}{f} H^f$,
(1.2) (F, g_F) *is Einstein with* $\mathrm{Ric}_F = \mu g_F$,
(1.3) $-f\Delta f + (k-1)|\nabla f|^2 + \lambda f^2 = \mu$.

Now we prove a lemma.

Lemma 4. *Let f be a smooth function on a Riemannian manifold B. Then for any vector X, the divergence of the Hessian tensor H^f satisfies*

$$\mathrm{div}(H^f)(X) = \mathrm{Ric}(\nabla f, X) - \Delta(df)(X), \tag{1.4}$$

where $\Delta = d\delta + \delta d$ denotes the Laplacian on B acting on differential forms.

Proof. The well-known Ricci identity implies (cf. [5], p. 159)

$$D^2 df(X,Y,Z) - D^2 df(Y,X,Z) = df(R_{XY}Z) \tag{1.5}$$

for all vector fields X, Y, and Z where $D^2_{XY} = D_X D_Y - D_{D_X Y}$ denotes the second order covariant differential operator and $R_{XY} = -D_X D_Y + D_Y D_X + D_{[X,Y]}$ is the curvature tensor acting on tensors as a derivation. Since df is closed, it is easily proved that

$$D^2 df(X,Y,Z) = D^2 df(X,Z,Y) \tag{1.6}$$

for any vector fields X, Y and Z.

For a fixed $p \in B$ we may choose a local orthonormal frame $E_1, E_2, \cdots, E_m$ of the space B such that $D_{E_i} E_j(p) = 0$ for all i, j. Also, we may assume $D_{E_i} Y(p) = 0$ for a vector field Y. Taking the trace with respect to X and Z in (1.5) and using (1.6), we have

$$\sum_i (D^2 df)(E_i, E_i, Y) = -d\Delta f(Y) + \mathrm{Ric}(Y, \nabla f)$$

at p. Since $\mathrm{div} H^f(Y) = \sum_i (D^2 df)(E_i, E_i, Y)$ is straightforward, (1.4) is proved. □

Proposition 5. *Let (B^m, g_B) be a compact Riemannian manifold of dimension $m \geqq 2$. Suppose that f is a nonconstant smooth function on B satisfying* (1.1) *for a constant $\lambda \in R$ and a natural number $k \in N$. Then f satisfies* (1.3) *for a constant $\mu \in R$. Hence for a compact Einstein space (F, g_F) of dimension k with* $\mathrm{Ric}_F =$

μg_F, we can make a compact Einstein warped product space $M = B \times_f F$ with $\mathrm{Ric} = \lambda g$.

Proof. By taking the trace of both sides of (1.1), we have

$$S = m\lambda - \frac{k}{f}\Delta f, \tag{1.7}$$

where S denotes scalar curvature of B given by tr(Ric). Note that the second Bianchi identity implies ([6], p. 88)

$$dS = 2\mathrm{div}(\mathrm{Ric}). \tag{1.8}$$

From (1.7) and (1.8), we obtain

$$\mathrm{div}\,\mathrm{Ric}(X) = \frac{k}{2f^2}\{\Delta f df - f d(\Delta f)\}(X). \tag{1.9}$$

On the other hand, by definition we have

$$\mathrm{div}(\frac{1}{f}H^f)(X) = \sum_i (D_{E_i}(\frac{1}{f}H^f))(E_i, X) = -\frac{1}{f^2}H^f(\nabla f, X) + \frac{1}{f}\mathrm{div}H^f(X)$$

for any vector field X and an orthonormal frame $E_1, E_2, \cdots, E_m$ of B. Since $H^f(X, \nabla f) = (D_X df)(\nabla f) = \frac{1}{2}d(|\nabla f|^2)(X)$, the last equation becomes

$$\mathrm{div}(\frac{1}{f}H^f)(X) = -\frac{1}{2f^2}d(|\nabla f|^2)(X) + \frac{1}{f}\mathrm{div}H^f(X)$$

for a vector field X on B. Hence, from (1.1) and (1.4) it follows that

$$\mathrm{div}(\frac{1}{f}H^f) = \frac{1}{2f^2}\{(k-1)d(|\nabla f|^2) - 2fd(\Delta f) + 2\lambda f df\}. \tag{1.10}$$

But, (1.1) gives divRic $= \mathrm{div}(\frac{k}{f}H^f)$. Therefore, (1.9) and (1.10) imply that $d(-f\Delta f + (k-1)|\nabla f|^2 + \lambda f^2) = 0$, that is, $-f\Delta f + (k-1)|\nabla f|^2 + \lambda f^2 = \mu$ for some constant μ. Thus the first part of the proposition is proved. For a compact Einstein manifold (F, g_F) of dimension k with $\mathrm{Ric}_F = \mu g_F$, we can construct a compact Einstein warped product $M = B \times_f F$ by the sufficiencies of Corollary 3. □

Now we give the proof of Theorem 1. Note that (1.3) becomes

$$\mathrm{div}(f\nabla f) + (k-2)|\nabla f|^2 + \lambda f^2 = \mu. \tag{1.11}$$

By integrating (1.11) over B we have

$$\mu = \frac{k-2}{V(B)}\int_B |\nabla f|^2 + \frac{\lambda}{V(B)}\int_B f^2, \tag{1.12}$$

where $V(B)$ denotes the volume of B.

1) Suppose $k \geq 3$. Let p be a maximum point of f on B. Then, we have $f(p) > 0, \nabla f(p) = 0$ and $\Delta f(p) \geq 0$. Hence from (1.3) and (1.12) we obtain the following:

$$\begin{aligned} 0 &\leqq f(p)\Delta f(p) \\ &= \lambda f(p)^2 - \mu \\ &= \frac{2-k}{V(B)}\int_B |\nabla f|^2 + \frac{\lambda}{V(B)}\int_B (f(p)^2 - f^2) \\ &\leqq 0. \end{aligned}$$

The last inequality follows from the hypothesis on λ. Thus, f is constant.

2) When $k = 1, 2$, we choose q as a minimum point of f on B. Then, we have $f(q) > 0, \nabla f(q) = 0$ and $\Delta f(q) \leq 0$. Hence we obtain from (1.3) and (1.12)

$$\begin{aligned} 0 &\geqq f(q)\Delta f(q) \\ &= \lambda f(q)^2 - \mu \\ &= \frac{2-k}{V(B)} \int_B |\nabla f|^2 + \frac{\lambda}{V(B)} \int_B (f(q)^2 - f^2) \\ &\geqq 0. \end{aligned} \tag{1.13}$$

As in case 1), the last inequality follows from the hypothesis on λ. If $k = 1$ or $\lambda < 0$, then (1.13) shows that f is constant. If $k = 2$ and $\lambda = 0$, (1.11) and (1.12) imply that f^2 is harmonic on B, and hence f is constant. This completes the proof of the theorem.

In a similar manner, we may prove the following (cf. [4]):

Remark 6. Let (M, g) be a compact Riemannian manifold. If the Ricci tensor satisfies $\mathrm{Ric} = \lambda g + H^f$ for a nonpositive constant $\lambda \in R$ and a smooth function f on M, then f is constant.

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