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Compact ergodic groups of automorphisms — Source link

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Compact ergodic groups of automorphisms

by

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Abstract. It is shown that if G is a compact ergodic group of $*$-automorphisms on a unital C^*-algebra A then the unique G-invariant state is a trace. Hence if A is a von Neumann algebra then it is finite.

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1. Introduction. Let A be a unital C^*-algebra, G a compact group and α a strongly continuous representation of G as an ergodic group of $*$-automorphisms of A , i.e. $\alpha_g(x) = x$ for all $g \in G$ implies x is a scalar operator. It was shown in [9] that if G is abelian and A a von Neumann algebra then A is automatically finite and the (necessarily unique) G-invariant state is a trace. Since then it has been an open problem whether the same is true without the assumption that G be abelian, see the introduction to [6]. In the present paper we solve this problem to the affirmative by showing that if G acts ergodically on the unital C^*-algebra A , then the G-invariant state is a trace. In the course of the proof of the theorem it will be shown that if D is an irreducible representation of G and A(D) the corresponding spectral subspace in A , see below, then the multiplicity of D in A(D) is not greater than the dimension of D . A consequence of this is that if G is second countable acting on a C^*-algebra then the action is cyclic if and only if it is ergodic.

The problem solved in this paper immediately raises the problem of classification of compact ergodic actions on C^*- or von Neumann algebras. If G is abelian this has been done completely in [1] and [6], and we can from those examples find nonabelian finite extensions of abelian ergodic actions on the hyperfinite II_1-factor. Another construction is to let for each positive integer i , G_i be an ergodic compact group of automorphisms on the complex $n_i \times n_i$ matrices, and then let the product group $G = \prod_{i=1}^{\infty} G_i$ act on the infinite tensor product of the matrix algebras in the obvious way. Then the GNS-representation due to the trace gives rise to an ergodic action of G on the hyperfinite factor. This is as far as we can go at present and we leave two basic problems open: (1) If a compact group acts ergodically on a II_1-factor M, is M hyperfinite ? (2) Find an example of a simple compact group acting ergodically on a II_1-factor.

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2. Compact ergodic groups. Let A be a unital C^*-algebra, G a compact group, and suppose α is a strongly continuous representation of G as $*$-automorphisms of A, so $g \to \alpha_g(x)$ is norm continuous for all $x \in A$. We assume the action is ergodic on A, i.e. $\alpha_g(x) = x$ for all $g \in G$ only if x is a scalar operator. Then for each $x \in A$, $\int \alpha_g(x)dg$ is a scalar operator $\omega(x)1$, where dg is the normalized Haar measure on G. ω so defined is the unique G-invariant state on A.

If $f \in L^1(G)$ we denote by $\alpha(f)$ the operator on A defined by

$$\alpha(f)(x) = \int f(g)\alpha_g(x)dg .$$

Let D be an irreducible unitary representation of G and χ_D its normalized character $\chi_D(g) = \dim D \ \mathrm{Tr}(D_g^{-1})$, where Tr is the usual trace on the Hilbert space of dimension $\dim D$. Then $\alpha(\chi_D)$ is a projection of A onto a norm closed subspace A(D) of A called the <u>spectral subspace of D in</u> A, see [3]. By [11, § 4.4.2] A(D) is the set of $x \in A$ such that the linear span of $\alpha_g(x)$, $g \in G$, is finite dimensional and splits into a direct sum of irreducible components all unitarily equivalent to D.

<u>Proposition 2.1</u>. Let A be a unital C^*-algebra, G a compact group and α a strongly continuous representation of G as an ergodic group of *-automorphisms of A. Let D be an irreducible unitary representation of G, A(D) the spectral subspace of D in A and m(D) the multiplicity of D in A(D). Then we have

(i) $m(D) \leq d$.

(ii) $\dim A(D) \leq d^2$.

<u>Proof</u>. If E is an irreducible unitary representation of G either α has no subrepresentation equivalent to E or there is an irreducible subspace V_E of A such that $\alpha|V_E$ is equivalent to E. Then $V_E \subset A(E)$, as follows from the characterization of A(E) given above. Let D be as in the proposition. We may assume $V_D \neq 0$.

Consider A as imbedded in the Hilbert space obtained in the GNS-representation due to the invariant state ω. Thus $(a,b) = \omega(b^*a)$ is the inner product on A. Let $d = \dim D$. Then we can choose $a_1, \ldots, a_d$ in V_D so they form an orthonormal basis for V_D. Then the map P_D defined by

$$P_D(a) = \sum_{i=1}^{d} (a, a_i) a_i$$

is a projection of A onto V_D, and since ω is G-invariant $\alpha_g(P_D(a)) = P_D(\alpha_g(a))$ for all $a \in A$. Thus the subspace

$(\iota-P_D)(A(D))$ of A, ι denoting the identity map, is a closed G-invariant subspace of A orthogonal to V_D. If $(\iota-P_D)(A(D)) \neq 0$ it contains an irreducible subspace V_E [7], and E is unitarily equivalent to D. Considering $P_D + P_E$ we have found a norm continuous projection onto $V_D + V_E$, and we can do this for any finite set of irreducible representations D_i equivalent to D, such that the spaces V_{D_i} are pairwise mutually orthogonal.

We fix now a finite set J of unitarily equivalent irreducible representations $D_1,\dots,D_N$ such that their irreducible subspaces V_{D_k} of $A(D)$ are nonzero and pairwise mutually orthogonal. We shall show $N \leq d$, which will prove the proposition.

Choose $a_{ik} \in V_{D_k}$, $i = 1,\dots,d$, so that they form an orthonormal basis for V_{D_k}, and such that they have the same action under G, i.e. there is an irreducible unitary representation $g \to (u_{rs}(g))$ of G into the complex $d \times d$ matrices M_d satisfying

$$(2.1) \qquad \alpha_g(a_{ik}) = \sum_{j=1}^{d} u_{ij}(g) a_{jk}, \qquad k \in J.$$

For each pair $j,k \in J$ we have

$$\begin{aligned}\alpha_g\Big(\sum_{i=1}^{d} a_{ij}^* a_{ik}\Big) &= \sum_i \alpha_g(a_{ij})^* \alpha_g(a_{ik}) \\ &= \sum_{i,r,s} \overline{u_{ir}(g)}\, a_{rj}^* u_{is}(g) a_{sk} \\ &= \sum_r a_{rj}^* a_{rk}.\end{aligned}$$

Since G is ergodic $\sum_i a_{ij}^* a_{ik}$ is a scalar operator, the scalar being found by the computation

$$\omega\Big(\sum_i a_{ij}^* a_{ik}\Big) = \sum_i (a_{ik}, a_{ij}) = \sum_i \delta_{jk} = \delta_{jk} d.$$

Thus we have shown

$$(2.2) \qquad \sum_{i=1}^{d} a_{ij}^* a_{ik} = \delta_{jk} d 1, \qquad j,k \in J.$$

Similarly we can find complex numbers c_{jk} such that

$$\sum_{i=1}^{d} a_{ij} a_{ik}^* = c_{jk}\, d\, 1\ , \qquad j,k \in J\ . \tag{2.3}$$

The $N \times N$ matrix (c_{jk}) is clearly self-adjoint, so we can find a unitary $N \times N$ matrix (α_{rs}) such that

$$\sum_{l,m=1}^{N} \alpha_{kl}\, c_{lm}\, \overline{\alpha_{jm}} = \delta_{jk} \lambda_j\ , \qquad j,k \in J$$

with $\lambda_j \in \mathbb{R}$. Let $a'_{ij} = \sum_{k=1}^{N} \alpha_{jk} a_{ik}$. Then $a'_{ij} \in \sum_{k=1}^{N} V_{D_k}$, and they form an orthonormal basis for $\sum_{k=1}^{N} V_{D_k}$. Note that

$$\alpha_g(a'_{ij}) = \sum_{r=1}^{d} u_{ir}(g) a'_{rj}\ ,$$

as is easily computed, hence we may replace a_{ij} by a'_{ij}, $i = 1,\dots,d$, $j \in J$, and still have that (2.1) is satisfied. We shall therefore do this and thus assume (2.1), (2.2), and the diagonal form of (2.3)

$$\sum_{i=1}^{d} a_{ij}\, a_{ik}^* = \delta_{jk} \lambda_j\, d\, 1\ , \qquad j,k \in J\ , \tag{2.4}$$

where $\lambda_j \in \mathbb{R}$. From (2.4) it is clear that $\lambda_j > 0$.

Denote by e the $d \times d$ matrix operator

$$e = \{ \sum_{k=1}^{N} a_{ik}\, a_{jk}^* \} \in A \otimes M_d\ , \quad i,j \in \{1,\dots,d\}\ .$$

Clearly e is self-adjoint, and by (2.2) it satisfies

$$e^2 = \{ \sum_{k,l=1}^{N} \sum_{s=1}^{d} a_{ik}\, a_{sk}^*\, a_{sl}\, a_{jl}^* \} = \{ \sum_{k=1}^{N} a_{ik}\, a_{jk}^*\, d \} = de\ .$$

Hence $e = dp$ with p a projection, in particular $0 \le e \le d\,1$. Let τ denote the normalized trace on M_d. Then $\omega \otimes \tau$ is a state on $A \otimes M_d$, so by (2.4) we have

$$d \ge \omega \otimes \tau(e) = d^{-1} \sum_{k=1}^{N} \omega\Big(\sum_{i=1}^{d} a_{ik}\, a_{ik}^* \Big) = \sum_{k=1}^{N} \lambda_k\ . \tag{2.5}$$

We next assert that

(2.6) $$\omega(a_{ik}\, a_{jl}^{*}) = \delta_{ij}\,\delta_{kl}\,\lambda_k\,, \qquad i,j \in \{1,\ldots,d\},\ k,l \in J\,.$$

Indeed, fix $k,l \in J$, and let $\beta_{ij} = \omega(a_{ik}\, a_{jl}^{*})$. Then (β_{ij}) is a $d \times d$ matrix which by (2.1) satisfies

$$\begin{aligned}\sum_{s=1}^{d} u_{is}(g)\beta_{sj} &= \omega(\sum_s u_{is}(g) a_{sk}\, a_{jl}^{*})\\ &= \omega(\alpha_g(a_{ik}) a_{jl}^{*})\\ &= \omega(a_{ik}\, \alpha_{g^{-1}}(a_{jl})^{*})\\ &= \sum_s \beta_{is}\, u_{sj}(g)\,.\end{aligned}$$

Therefore the matrix (β_{ij}) commutes with $(u_{is}(g))$ for all $g \in G$. Since the representation $g \to (u_{is}(g))$ is irreducible (β_{ij}) is a scalar operator, so (2.6) follows from (2.4).

Now consider the conjugate representation $\bar{D}$ to D. Since $a \in A(E)$ if and only if $\alpha(\chi_E)(a) = a$ for E an irreducible representation, it is immediate from the definition of $\alpha(\chi_E)$ that $a \in A(D)$ if and only if $a^* \in A(\bar{D})$. Thus by (2.6) if $b_{ij} = \lambda_j^{-\frac{1}{2}} a_{ij}^{*}$ then $\{b_{ij} : i = 1,\ldots,d\,,\ j \in J\}$ form an orthonormal set in $A(\bar{D})$ for which (2.1) is replaced by

$$\alpha_g(b_{ik}) = \sum_{j=1}^{d} \overline{u_{ij}(g)} b_{jk}\,.$$

Since $g \to (\overline{u_{ij}(g)})$ is irreducible the space spanned by $\{b_{ik} : i = 1,\ldots,d\}$ is irreducible in $A(\bar{D})$ for each $k \in J$. Thus our previous discussion for D and the a_{ij} is valid for $\bar{D}$ and the b_{ij}. We have in particular by the equations (2.2) - (2.5)

(2.7) $$\sum_{i=1}^{d} b_{ij}\, b_{ik}^{*} = \delta_{jk}\, \mu_j\, d\, 1\,, \qquad j,k \in J\,,$$

where $\mu_j > 0$ and $\sum_{j=1}^{N} \mu_j \leq d$. Computing we find by (2.2)

$$\omega(\sum_{i=1}^{d} b_{ij} b_{ij}^*) = \lambda_j^{-1}\omega(\sum_i a_{ij}^* a_{ij}) = \lambda_j^{-1} d ,$$

so that $\mu_j = \lambda_j^{-1}$ and therefore

$$\sum_{j=1}^{N} \lambda_j^{-1} \leq d . \tag{2.8}$$

Since $x+x^{-1} \geq 2$ whenever $x > 0$ we have by (2.5) and (2.8) that $2N \leq \sum_{j=1}^{N} (\lambda_j+\lambda_j^{-1}) \leq 2d$, so that $N \leq d$, as we wanted to show. Q.E.D.

Let A, G, and α be as in Proposition 2.1. Representing A in the GNS-representation defined by the G-invariant state ω we may assume $\omega(a) = (a\xi_0, \xi_0)$ for some cyclic vector ξ_0 for A in the Hilbert space. Furthermore there is a continuous unitary representation $g \to u_g$ of G on H such that $\alpha_g(a) = u_g a u_g^{-1}$ and $u_g \xi_0 = \xi_0$ for all $g \in G$, $a \in A$. Since ω is the unique G-invariant state on A, ω is the unique normal G-invariant state on the weak closure A^- of A, hence by [5], G is ergodic on A^- as well as A. Since the support projection for ω is a G-invariant projection in A^-, it is 1, hence ω is faithful on A^-, and ξ_0 is a separating vector for A^-. Let Δ denote the modular operator for ξ_0 with respect to A^-, and J the corresponding conjugation, so $a^*\xi_0 = J\Delta^{\frac{1}{2}} a\xi_0$ for all $a \in A^-$, see [10]. By [8] $u_g\Delta = \Delta u_g$ and $Ju_g = u_g J$, $g \in G$, hence in particular the finite dimensional subspace $A(D)\xi_0$ is invariant under the action of $\Delta^{\frac{1}{2}}$, so under Δ, recall $A(D)\xi_0 = \{\int \chi_D(g) u_g\, a\, dg\, \xi_0 : a \in A\}$. By equation (2.6) we have with $N = m(D)$, so $\sum_{k=1}^{N} V_{D_k} = A(D)$,

$$(\Delta a_{ij}\xi_0, a_{kl}\xi_0) = (a_{kl}^*\xi_0, a_{ij}^*\xi_0) = \delta_{ik}\delta_{jl}\lambda_j = \lambda_j (a_{ij}\xi_0, a_{kl}\xi_0) .$$

Hence $a_{ij}\xi_0$ is an eigenvector for Δ with eigenvalue λ_j. Hence we have from (2.5) and (2.8)

Corollary 2.2. Let A, G, α, D be as in Proposition 2.1. Let ξ_0 be the cyclic vector in the GNS-representation defined by the G-invariant state ω. Then ξ_0 is separating for A^-, and if Δ is its modular operator then Δ leaves the finite dimensional vector space $A(D)\xi_0$ invariant. If λ is an eigenvalue for $\Delta|A(D)\xi_0$ then both $\lambda \leq \dim D$ and $\lambda^{-1} \leq \dim D$.

We shall also need the probably well known observation

Lemma 2.3. Let M be a von Neumann algebra and G an ergodic group of *-automorphisms of M. Suppose V is a nonzero globally G-invariant linear subspace of M. If $x \in M$, denote by $r(x)$ and $s(x)$ respectively the range and support projections of x. Then we have

$$\bigvee_{x\in V} r(x) = \bigvee_{x\in V} s(x) = 1 .$$

Proof. If α is a *-automorphism of M then α is ultraweakly continuous, so by the construction of $r(x)$ by spectral theory on the positive operator xx^*, we see that $\alpha(r(x)) = r(\alpha(x))$ for $x \in M$. Thus $\bigvee_{x\in V} r(x)$ and $\bigvee_{x\in V} s(x)$ are nonzero G-invariant projections in M, hence are equal to 1 by ergodcity.

Q.E.D.

3. Tensor representations. In this section we shall apply Herman Weyl's classical theory for representations of groups, to obtain estimates for the dimensions of irreducible subspaces of powers of G-invariant subspaces of an ergodic group G .

If V is a finite dimensional complex vector space we denote by $V^{(m)}$ the tensor product $V\otimes\cdots\otimes V$ (m times). If π is a representation of a group G on V , π has a corresponding representation π^m of G on $V^{(m)}$ defined by $\pi^m(g) = \pi(g)\otimes\cdots\otimes\pi(g)$.

Lemma 3.1. Let V be a finite dimensional complex vector space with $\dim V = n$. Consider $Gl(n,\mathbb{C})$ as acting on V and consider the corresponding representation of $Gl(n,\mathbb{C})$ on $V^{(m)} = V\otimes\cdots\otimes V$. Then any irreducible subspace U of $V^{(m)}$ satisfies

$$\dim U \leq (1+m)^{\frac{n(n-1)}{2}} .$$

Proof. Let π denote the representation of $Gl(n,\mathbb{C})$ on V . By [2, p. 192] we can decompose the representation π^m of $Gl(n,\mathbb{C})$ on $V^{(m)}$ into irreducible components as follows:

$$V^{(m)} = \sum_{\lambda_1+\cdots+\lambda_n = m} l_\lambda D_\lambda$$

where $\lambda = (\lambda_1,\ldots,\lambda_n)$, λ_i is a nonnegative integer for each $i \in \{1,\ldots,n\}$,

$$l_\lambda = \frac{m!}{\prod_{i,j} h_{ij}} , \qquad h_{ij} = 1+\lambda_i+\bar{\lambda}_j-(i+j) ,$$

and $\bar{\lambda}_j$ is the number of boxes in the j^{th} column in the Young tableau corresponding to λ [2, p. 192, eq. (23)]. $l_\lambda D_\lambda$ means that the irreducible representation D_λ is repeated l_λ times, and D_λ is the irreducible representation of $Gl(n,\mathbb{C})$ with highest weight $\lambda = (\lambda_1,\ldots,\lambda_n)$.

Set now $l_j = \lambda_j + n-j$ and $l_j^o = n-j$. Then the Weyl formula, see [2, p. 283, eq. (32)] gives that

$$\dim D_\lambda = \frac{\prod_{i<j}(l_i - l_j)}{\prod_{i<j}(l_i^o - l_j^o)} .$$

Hence

$$\dim D_\lambda = \prod_{1 \le i<j \le n} \left(1 + \frac{\lambda_i - \lambda_j}{i-j}\right) \le (1+m)^{\frac{n(n-1)}{2}}$$

Q.E.D.

Proposition 3.2. Let G be a group of $*$-automorphisms on a C^*-algebra A, and suppose V is a finite dimensional linear subspace of A which is globally invariant under G. Let $\dim V = n$, and let for $m \in \mathbb{N}$, V^m denote the linear subspace of A generated by products of m elements in V. Then V^m is again globally invariand under G, and for each subspace $U \subset V^m$ globally invariant and irreducible under the action of G we have

$$\dim U \le (1+m)^{\frac{n(n-1)}{2}}$$

Proof. Let π be the representation of G on V and π^m the corresponding representation on $V^{(m)}$. Let j_m be the m-linear map of $V^{(m)}$ onto V^m given by

$$j_m(x_1 \otimes \cdots \otimes x_m) = x_1 \cdots x_m .$$

Then j_m intertwines the representation π^m and the action of G on V^m, i.e.

$$j_m \circ \pi^m(g) = \pi(g) \circ j_m , \qquad g \in G .$$

Therefore j_m takes invariant subspaces of $V^{(m)}$ onto invariant subspaces of V^m. Since the dimension of the image of a subspace is not greater than the dimension of the subspace, it suffices to show that for any invariant subspace U of $V^{(m)}$ irreducible under the action of $\pi^m(G)$ we have $\dim U \leq (1+m)^{\frac{n(n-1)}{2}}$.

Denote by ι the representation of $Gl(n,\mathbb{C})$ on V, and ι^m the corresponding representation on $V^{(m)}$. Then $\pi^m(G) \subset \iota^m(Gl(n,\mathbb{C}))$. By Lemma 3.1 any irreducible invariant subspace for $\iota^m(Gl(n,\mathbb{C}))$ has dimension at most $(1+m)^{\frac{n(n-1)}{2}}$. Hence any subgroup and especially $\pi^m(G)$ also has the property that any irreducible invariant subspace has dimension at most $(1+m)^{\frac{n(n-1)}{2}}$. Thus $\dim U \leq (1+m)^{\frac{n(n-1)}{2}}$.

Q.E.D.

4. The main results.

<u>Theorem 4.1</u>. Let A be a unital C^*-algebra, G a compact group, and α a strongly continuous representation of G as an ergodic group of $*$-automorphisms of A. Then the unique G-invariant state on A is a trace.

<u>Proof</u>. Since G is compact A is generated by the spectral subspaces $A(D)$, as D runs through the irreducible unitary representations of G [7]. Thus it suffices to show that each $A(D)$ is

contained in the centralizer of the invariant state, or equivalently by Corollary 2.2 and [10], to show that all the eigenvalues of Δ restricted to $A(D)\xi_0$ are equal to 1, ξ_0 being the G-invariant separating and cyclic vector in the GNS-representation due to the invariant state. Suppose λ is one of them. By Corollary 2.2 we may assume $\lambda \geq 1$. Let V be a G-invariant subspace of $A(D)$ such that $\Delta a\xi_0 = \lambda a\xi_0$ for all $a \in V$ and such that V is irreducible under the action of G. This is possible since $\Delta u_g = u_g\Delta$ for all $g \in G$. For each $m \in \mathbb{N}$, if V^m is the space generated by products of m elements in V, for each $a \in V^m$, $a\xi_0$ is an eigenvector for Δ with eigenvalue λ^m, as is easily seen since $y \to \Delta^{it} y \Delta^{-it}$ is an automorphism of the weak closure of A. Since G is ergodic an easy induction argument based on Lemma 2.3 shows that $V^m \neq 0$, and by Proposition 3.2 each subspace U of V^m which is globally invariant and irreducible under the action of G has dimension $\dim U \leq (1+m)^{\frac{n(n-1)}{2}}$, where $n = \dim V$. By Corollary 2.2 $\lambda^m \leq \dim U$, hence $\lambda^m \leq (1+m)^{\frac{n(n-1)}{2}}$. Thus

$$0 \leq \log\lambda \leq \frac{n(n-1)}{2m}\log(1+m),$$

which is arbitrarily small for large m, so that $\log\lambda = 0$, and $\lambda = 1$. Since λ was an arbitrary eigenvalue for Δ restricted to an arbitrary subspace $A(D)\xi_0$ with D an irreducible representation of G, $\Delta = 1$, and ξ_0 is a trace vector for A.

Q.E.D.

If M is a von Neumann algebra, G a topological group and α a representation if G as *-automorphisms of M, we say α is continuous if $g \to \rho(\alpha_g(x))$ is continuous on G for each $\rho \in M_*$, $x \in M$.

<u>Corollary 4.2</u>. Let M be a von Neumann algebra and G a compact group. If there is a continuous representation of G as an ergodic group of *-automorphisms on M then M is finite.

<u>Proof</u>. It is well known that the set A of $x \in M$ such that the function $g \to \alpha_g(x)$ is norm continuous on G is a C^*-algebra globally invariant under G and weakly dense in M. Let ω be a normal G-invariant state on M. Then $\omega|A$ is G-invariant, hence is a trace by Theorem 4.1. By density of A in M, ω is a trace on M. Since by ergodicity ω is faithful, M is finite.

Q.E.D.

The next result is a generalization of Corollary 4.2 and shows that compact automorphism groups in general have very large fixed point algebras.

<u>Corollary 4.3</u>. Let M be a von Neumann algebra of type III, G a compact group, and α a continuous representation of G as *-automorphisms of M. Then the fixed point algebra M^G of G in M contains no minimal projections.

<u>Proof</u>. $M^G = \{x \in M : \alpha_g(x) = x,\ g \in G\}$. Suppose to the contrary that e is a nonzero minimal projection in M^G. Then G acts ergodically on the reduced algebra M_e by $\alpha_g(exe) = e\alpha_g(x)e$. By Corollary 4.2 M_e is finite contradicting the fact that it is of type III since M is.

Q.E.D.

Let A be a C^*-algebra, G a group, and α a representation of G as *-automorphisms of A. Suppose ω is a G-invariant state. We say α is <u>cyclic with respect to</u> ω if there is $x \in A$

such that $\omega(y\alpha_g(x)) = 0$ for all $g \in G$ implies $y = 0$. We shall see below that if G is compact and α is a continuous representation of G as an ergodic group, then cyclicity of G means that the orbit of $x\xi_0$ under G in the GNS-representation due to the unique G-invariant trace, is dense in the Hilbert space.

Lemma 4.4. Let A be a unital C^*-algebra, G a compact group and α a strongly continuous representation of G as $*$-automorphisms of A. Suppose ω is a G-invariant state such that α is cyclic with respect to ω. Then α is an ergodic representation, and ω is the unique G-invariant state.

Proof. Let A^G denote the fixed point algebra of G in A. Since G is compact the adjoint of the map

$$y \to \int_G \alpha_g(y)dg$$

of A onto A^G defines an affine isomorphism between the G-invariant states of A and the state space of A^G. Suppose there is $x \in A$ such that $\omega(y\alpha_g(x)) = 0$ for all $g \in G$ implies $y = 0$. Then if $y \in A^G$ we have $\omega(y\alpha_g(x)) = \omega(\alpha_g^{-1}(y)x) = \omega(yx)$, so the functional $y \to \omega(yx)$ is injective on A^G. But this is only possible if A^G is the scalars. Q.E.D.

The next theorem is a direct analogue for representations of compact groups as $*$-automorphisms on C^*-algebras, of a result of Greenleaf and Moskowitz on unitary representations on Hilbert space [4].

Theorem 4.5. Let A be a unital C^*-algebra and G a second countable compact group. Suppose α is a strongly continuous representation of G as $*$-automorphisms of A. Then α is an ergodic representation if and only if α is cyclic with respect to some G-invariant state.

Proof. By Lemma 4.4 we only have to show that if α is ergodic and ω is the unique G-invariant state, then α is cyclic with respect to ω. By Proposition 2.1 if D is an irreducible representation of G then its multiplicity in the spectral subspace $A(D)$ of A is not greater than $\dim D$. Thus there is $x_D \in A(D)$ of norm one such that the linear span of $\alpha_g(x_D)$, $g \in G$, equals $A(D)$. Indeed, in the notation of the proof of Proposition 2.1 we may choose $x_D = c\sum_{i=1}^{m(D)} a_{ii}$ for a suitable scalar $c > 0$. Since G is second countable and compact its dual $\hat{G}$ is countable, hence there is a countable number of spectral subspaces $A(D)$. Number them by $A(D_k)$, $k \in \mathbb{N}$. For each k choose $x_{D_k} \in A(D_k)$ of norm one as above, and let $x = \sum_{k=1}^{\infty} 2^{-k} x_{D_k}$ (if $\hat{G}$ is finite let the sum be finite). Then $\|x\| \leq 1$ and $x \in A$. We show that the linear span of the orbit of $x\xi_0$, ξ_0 being the G-invariant separating and cyclic vector in the GNS-representation due to ω, is dense in the underlying Hilbert space H, hence in particular that α is cyclic with respect to ω.

Let $\xi \in H$ satisfy $(\xi, \alpha_g(x)\xi_0) = 0$ for all $g \in G$. Let u denote the unitary representation of G on H such that $u_g a u_g^{-1} = \alpha_g(a)$ and $u_g \xi_0 = \xi_0$ for all $g \in G$, $a \in A$. Let D be an irreducible representation of G and χ_D the corresponding normalized character. Then $u(\chi_D) = \int \chi_D(g) u_g \, dg$ is the orthogonal projection of H onto the subspace $A(D)\xi_0$. Let $D = D_k$ be one of the irreducible representations described above. Then $\alpha(\chi_D)(x) = 2^{-k}x_D$. Let $h \in G$, then $\alpha(\chi_D)(\alpha_h(x))\xi_0 \in A(D)\xi_0$, hence we have

$$\begin{aligned}(u(\chi_D)\xi, u_h x_D \xi_0) &= (\xi, u(\chi_D)u_h x_D \xi_0) \\ &= 2^k(\xi, \alpha(\chi_D)\alpha_h(x)\xi_0) \\ &= 2^k\int \chi_D(g)(\xi, \alpha_{gh}(x)\xi_0)dg \\ &= 0\end{aligned}$$

by assumption on ξ. Since $\text{span}\{u_h x_D \xi_0 : h \in G\} = A(D)\xi_0$, $u(\chi_D)\xi = 0$ for each $D = D_k$. Since the subspaces $A(D_k)\xi_0$ are mutually orthogonal and span H, $\xi = \sum_{k=1}^{\infty} u(\chi_{D_k})\xi = 0$.

Q.E.D.

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