COMPACT HERMITIAN SURFACES OF POINTWISE CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

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1. Introduction. Let $M = (M, J, g)$ be an almost Hermitian manifold and $U(M)$ the unit tangent bundle of M. Then the holomorphic sectional curvature $H = H(x)$ can be regarded as a differentiable function on $U(M)$. If the function H is constant along each fibre, then M is called a space of pointwise constant holomorphic sectional curvature. Especially, if H is constant on the whole $U(M)$, then M is called a space of constant holomorphic sectional curvature. An almost Hermitian manifold with an integrable almost complex structure is called a Hermitian manifold. A real 4-dimensional Hermitian manifold is called a Hermitian surface. Hermitian surfaces of pointwise constant holomorphic sectional curvature have been studied by several authors (cf. **[2]**, **[3]**, **[5]**, **[6]** and so on).

In this paper, we shall prove the following.

THEOREM A. *Let $M = (M, J, g)$ be a compact Hermitian surface of pointwise constant holomorphic sectional curvature. If the scalar curvature of M is nonpositive constant, then M is an Einstein Kähler surface.*

THEOREM B. *Let $M = (M, J, g)$ be a Hermitian surface of pointwise constant holomorphic sectional curvature satisfying the condition*

$$R(X, Y) \cdot R = 0 \textit{ for any differentiable vector fields } X \textit{ and } Y. \tag{1.1}$$

If the curvature operator is non-singular at each point of M, then M is a weakly $*$-Einstein manifold.

Taking account of the solution of Yamabe's problem, the classification problem of compact self-dual (resp. anti-self-dual) Hermitian surfaces can be reduced to the one of compact self-dual (resp. anti-self-dual) Hermitian surfaces with constant scalar curvature. We may easily show that a 4-dimensional almost Hermitian manifold of pointwise constant holomorphic sectional curvature is self-dual (cf. **[2]**). Therefore Theorem A gives a partial solution to the classification problem of compact self-dual Hermitian surfaces and also a partial improvement to the previous result of the present authors (**[3]**, Theorem A). In the course of the proof, we have used the following fact (**[3]**, Proposition 2.1).

PROPOSITION. **[3]** *Let $M = (M, J, g)$ be a compact Einstein Hermitian surface of pointwise constant holomorphic sectional curvature. Then M is a locally conformal Kähler surface and the tensor field S defined by*

$$S(X, Y) = (\nabla_X \omega)Y - (\nabla_{JX}\omega)JY + \tfrac{1}{2}(\omega(X)\omega(Y) - \omega(JX)\omega(JY))$$

vanishes on M.

However the proof of the proposition is not right (more precisely, the equality (2.19)

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in [3] is false in sign), which has been pointed out by T. Sato. We give a correct proof of the proposition after proving Theorem A and B.

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2. Preliminaries. Let $M = (M, J, g)$ be a $2n$-dimensional almost Hermitian manifold with the almost Hermitian structure (J, g), and Ω the Kähler form of M defined by $\Omega(X, Y) = g(X, JY), X, Y \in \mathscr{X}(M)$. We assume that M is oriented by the volume form $dM = \frac{(-1)^n}{n!}\Omega^n$. We denote by $\nabla, R, \rho, \tau, \rho^*$ and τ^* the Riemannian connection, the Riemannian curvature tensor, the Ricci tensor, the scalar curvature, the $*$-Ricci tensor and the $*$-scalar curvature of M respectively:

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$
$$\rho(x, y) = \text{trace of } (z \to R(z, x)y),$$
$$\tau = \text{trace of } \rho,$$
$$\rho^*(x, y) = \tfrac{1}{2}\text{ trace of } (z \to R(x, Jy)Jz),$$
$$\tau^* = \text{trace of } \rho^*,$$

where $X, Y \in \mathscr{X}(M), x, y, z \in T_p(M), p \in M$.

An almost Hermitian manifold $M = (M, J, g)$ is called a weakly $*$-Einstein manifold if it satisfies $\rho^* = \lambda^* g$ for some function λ^* on M.

Now we assume that M is a Hermitian surface. Then we have

$$d\Omega = \omega \wedge \Omega,$$

where $\omega = \delta\Omega \circ J$. The 1-form ω is called the Lee form of M. The Lee form ω satisfies the following (see [7], [8]):

$$\begin{aligned} &J^{ij}\nabla_i\omega_j = 0,\\ &2\nabla_i J_{jk} = \omega_a J_j{}^a g_{ik} - \omega_a J_k{}^a g_{ij} + \omega_j J_{ki} - \omega_k J_{ji}, \\ &\tau - \tau^* = 2\delta\omega + \|\omega\|^2. \end{aligned} \tag{2.1}$$

Let M be a Hermitian surface of pointwise constant holomorphic sectional curvature $c = c(p) (p \in M)$. Then we have (see [5])

$$\begin{aligned} R_{ijkl} = {} & \frac{1}{4}\|\omega\|^2 C_{ijkl} + \left(\frac{c}{4} - \frac{1}{16}\|\omega\|^2\right)H_{ijkl} \\ & + \frac{1}{96}\{g_{ik}A_{jl} - g_{il}A_{jk} + g_{jl}A_{ik} - g_{jk}A_{il} \\ & + J_{ik}B_{jl} - J_{il}B_{jk} + J_{jl}B_{ik} - J_{jk}B_{il} \\ & + 2J_{ij}B_{kl} + 2J_{kl}B_{ij}\}, \end{aligned}$$

where

$$\begin{aligned} C_{ijkl} &= g_{il}g_{jk} - g_{ik}g_{jl}, \\ H_{ijkl} &= g_{il}g_{jk} - g_{ik}g_{jl} + J_{il}J_{jk} - J_{ik}J_{jl} - 2J_{ij}J_{kl}, \\ A_{ij} &= 21(\nabla_i\omega_j + \nabla_j\omega_i + \omega_i\omega_j) - 3J_i{}^aJ_j{}^b(\nabla_a\omega_b + \nabla_b\omega_a + \omega_a\omega_b), \\ B_{ij} &= 7(J_j{}^a\nabla_i\omega_a - J_i{}^a\nabla_j\omega_a) - (J_j{}^a\nabla_a\omega_i - J_i{}^a\nabla_a\omega_j) + 3(J_j{}^a\omega_i\omega_a - J_i{}^a\omega_j\omega_a). \end{aligned}$$

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We put

$$T_{ij} = \nabla_i\omega_j + \nabla_j\omega_i + \omega_i\omega_j - J_i^a J_j^b(\nabla_a\omega_b + \nabla_b\omega_a + \omega_a\omega_b),$$
$$T^*_{ij} = \nabla_i\omega_j - \nabla_j\omega_i - J_i^a J_j^b(\nabla_a\omega_b - \nabla_b\omega_a). \tag{2.2}$$

Then we have

$$\rho = \frac{\tau}{4}g - \frac{1}{4}T,$$
$$\rho^* = \frac{\tau^*}{4}g + \frac{1}{4}T^*. \tag{2.3}$$

We may easily get (see [5])

$$c = \frac{\tau + 3\tau^*}{24}. \tag{2.4}$$

We have the following integral formula (see [5]).

$$\int_M \|T\|^2\, dM = \int_M (4\,\|d\omega\|^2 + 2(\tau - \tau^*)^2 - 4\tau^*\,\|\omega\|^2)\, dM. \tag{2.5}$$

PROPOSITION 2.1. [5] *Let M be a compact Hermitian surface of pointwise constant holomorphic sectional curvature c. Then the Euler class of M is given by*

$$\chi(M) = \frac{1}{32\pi^2}\int_M \{12c^2 - \tfrac{1}{16}(\tau - \tau^*)^2 + \tfrac{1}{2}\tau^*\,\|\omega\|^2\}\, dM. \tag{2.6}$$

PROPOSITION 2.2. [5] *Let M be a compact Hermitian surface of pointwise constant holomorphic sectional curvature. Then the square of the first Chern class of M is given by*

$$c_1(M)^2 = \frac{1}{32\pi^2}\int_M \{(\tau^*)^2 + \tau^*\,\|\omega\|^2 + \|d\omega\|^2\}\, dM. \tag{2.7}$$

THEOREM 2.3. [4] *Let $M = (M, J)$ be a compact connected complex surface. Then we have*

$$c_1(M)^2 \leqq \max\{2c_2(M), 3c_2(M)\}. \tag{2.8}$$

3. Proof of Theorem A. In this section, we shall prove Theorem A. Before proceeding to the proof, we recall the following fact.

THEOREM 3.1. [5] *Let $M = (M, J, g)$ be a compact Hermitian surface of constant nonpositive holomorphic sectional curvature. Then M is a Kähler surface.*

We assume that $M = (M, J, g)$ is a compact Hermitian surface of pointwise constant holomorphic sectional curvature $c = c(p)$, $p \in M$.

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First we assume that $c_2(M)(=\chi(M))<0$. Then Miyaoka's inequality (2.8) implies $c_1(M)^2 \leqq 2c_2(M)$. Then by (2.4), (2.6) and (2.7), we have

$$0 \leqq \int_M \left\{\frac{1}{24}(\tau+3\tau^*)^2 - \frac{1}{8}(\tau-\tau^*)^2 + \tau^* \|\omega\|^2 - (\tau^*)^2 - \tau^* \|\omega\|^2 - \|d\omega\|^2\right\} dM$$
$$= \int_M \left\{\frac{1}{24}(-2\tau^2 + 12\tau\tau^* - 18(\tau^*)^2) - \|d\omega\|^2\right\} dM$$
$$= \int_M \left\{-\frac{1}{12}(\tau - 3\tau^*)^2 - \|d\omega\|^2\right\} dM \leqq 0.$$

Thus we have

$$\tau = 3\tau^* \quad \text{and} \quad d\omega = 0. \tag{3.1}$$

In this case, by the assumption that M has nonpositive constant scalar curvature τ, c is nonpositive constant on M. By Theorem 3.1, M is a Kähler surface. And then we have $\tau = \tau^* = c = 0$. This contradicts $\chi(M) < 0$.

Hence it follows that $c_2(M)(=\chi(M)) \geqq 0$. Then Miyaoka's inequality implies

$$c_1(M)^2 \leqq 3c_2(M).$$

Then from (2.4), (2.6) and (2.7), we have

$$0 \leqq \int_M \left\{\frac{1}{16}(\tau+3\tau^*)^2 - \frac{3}{16}(\tau-\tau^*)^2 + \frac{3}{2}\tau^* \|\omega\|^2 - (\tau^*)^2 - \tau^* \|\omega\|^2 \right.$$
$$\left. - \frac{1}{4}\|T\|^2 + \frac{1}{2}(\tau-\tau^*)^2 - \tau^* \|\omega\|^2\right\} dM. \tag{3.2}$$

From (3.2) and (2.5) we have

$$\int_M \left\{\frac{1}{16}(\tau+3\tau^*)^2 - (\tau^*)^2 + \frac{1}{16}(\tau-\tau^*)^2\right\} dM$$
$$\geqq \int_M \left\{\frac{1}{2}\tau^* \|\omega\|^2 - \frac{1}{4}(\tau-\tau^*)^2 + \frac{1}{8}\|T\|^2\right\} dM + \frac{1}{8}\int_M \|T\|^2 \, dM$$
$$= \frac{1}{2}\int_M \|d\omega\|^2 \, dM + \frac{1}{8}\int_M \|T\|^2 \, dM \geqq 0. \tag{3.3}$$

The left hand side of the above inequality reduces to

$$\int_M \left(\left(\frac{1}{4}(\tau+3\tau^*) - \tau^*\right)\left(\frac{1}{4}(\tau+3\tau^*) + \tau^*\right) + \frac{1}{16}(\tau-\tau^*)^2\right) dM$$
$$= \frac{1}{16}\int_M ((\tau-\tau^*)(\tau+7\tau^*) + (\tau-\tau^*)^2) \, dM$$
$$= -\frac{3}{8}\int_M (\tau-\tau^*)^2 \, dM + \frac{\tau}{2}\int_M (\tau-\tau^*) \, dM$$
$$= -\frac{3}{8}\int_M (\tau-\tau^*)^2 \, dM + \frac{\tau}{2}\int_M \|\omega\|^2 \, dM \leqq 0. \tag{3.4}$$

Thus by (3.3) and (3.4), we have finally $d\omega = 0$, $T = 0$ and hence $S = 0$, where S is the tensor field defined by

$$S(X, Y) = (\nabla_X \omega)Y - (\nabla_{JX}\omega)JY + \tfrac{1}{2}(\omega(X)\omega(Y) - \omega(JX)\omega(JY)). \tag{3.5}$$

Thus, from (2.3), we see that M is an Einstein locally conformal Kähler surface and the tensor field S vanishes on M. In particular, Proposition 1.2 of [**3**] is valid in the case where the Einstein constant is nonpositive. Thus by the argument after Proposition 2.1 of [**3**], we may conclude that M is Kähler surface.

This completes the proof of Theorem A.

4. Proof of Theorem B. In this section, we shall prove Theorem B. The condition (1.1) implies

$$R_{ija}{}^{t}R_{tbcd} + R_{ijb}{}^{t}R_{atcd} + R_{ijc}{}^{t}R_{abtd} + R_{ijd}{}^{t}R_{abct} = 0. \tag{4.1}$$

Now by (2.3) we have

$$\begin{aligned} J^{ia}J^{jc}R_{ija}{}^{t}R_{tbcd} &= \frac{1}{2}J^{ia}J^{jc}(R_{ija}{}^{t} - R_{aji}{}^{t})R_{tbcd} \\ &= -\frac{1}{2}J^{ia}J^{jc}R_{aij}{}^{t}R_{tbcd} \\ &= -\rho^{*tc}R_{tbcd} \\ &= \frac{\tau^*}{4}\rho_{bd} - \frac{1}{4}T^{*tc}R_{tbcd} \\ &= \frac{\tau^*}{4}\rho_{bd} - \frac{1}{8}T^{*tc}(R_{tbcd} - R_{cbtd}) \\ &= \frac{\tau^*}{4}\rho_{bd} - \frac{1}{8}T^{*tc}R_{tcbd}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} J^{ia}J^{jc}R_{ijc}{}^{t}R_{abtd} &= \frac{1}{2}J^{ia}J^{jc}(R_{ijc}{}^{t} - R_{icj}{}^{t})R_{abtd} \\ &= -\frac{1}{2}J^{ia}J^{jc}R_{jci}{}^{t}R_{abtd} \\ &= \rho^{*ta}R_{abtd} \\ &= -\frac{\tau^*}{4}\rho_{bd} - \frac{1}{8}T^{*ta}R_{tabd}, \end{aligned} \tag{4.3}$$

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$$J^{ia}J^{jc}R_{ijb}{}^{t}R_{atcd} = \tfrac{1}{2}J^{ia}J^{jc}R_{ijb}{}^{t}(R_{atcd} - R_{ctad})$$
$$= -\tfrac{1}{2}J^{ia}J^{jc}R_{ijb}{}^{t}R_{catd}$$
$$= -\tfrac{1}{2}J^{ia}J^{jc}R_{ijb}{}^{t}R_{acdt}, \tag{4.4}$$

$$J^{ia}J^{jc}R_{ijd}{}^{t}R_{abct} = \tfrac{1}{2}J^{ia}J^{jc}R_{ijd}{}^{t}(R_{abct} - R_{cbat})$$
$$= -\tfrac{1}{2}J^{ia}J^{jc}R_{ijd}{}^{t}R_{cabt}$$
$$= \tfrac{1}{2}J^{ai}J^{cj}R_{ijd}{}^{t}R_{acbt}$$
$$= \tfrac{1}{2}J^{ia}J^{jc}R_{acd}{}^{t}R_{ijbt}. \tag{4.5}$$

Thus, transvecting (4.1) with $J^{ia}J^{jc}$ and taking account of (4.2)–(4.5), we have

$$R_{abcd}T^{*ab} = 0. \tag{4.6}$$

Since the curvature operator is non-singular at each point of M, (4.6) implies $T^* = 0$ on M. Hence by (2.3) we see that M is a weakly $*$-Einstein manifold.

This completes the proof of Theorem B.

Finally we shall prove Proposition 2.1 of [**3**]. We assume that M is a compact Einstein Hermitian surface of pointwise constant holomorphic sectional curvature $c = c(p)(p \in M)$. Taking account of the proof of Theorem A in Section 3, it suffices to consider the case where $\tau > 0$. N. Hitchin proved the following.

THEOREM 4.1. [**1**] *Let $M = (M, g)$ be a 4-dimensional half-conformally flat Einstein manifold of positive scalar curvature. Then M is isometric to a 4-dimensional sphere or a complex projective space with the respective standard metric.*

Since a 4-dimensional almost Hermitian manifold of pointwise constant holomorphic sectional curvature is self-dual, then by Theorem 4.1, the manifold $M = (M, J, g)$ under consideration satisfies the conditions of Theorem B. Then from Theorem B we get $T^* = 0$. On the other hand, we have (see (3.13) of [**5**])

$$\int_M J^{ia}J^{jb}\nabla_a\omega_b\nabla_i\omega_j\,dM = \int_M J^{ia}J^{jb}\nabla_a\omega_b\nabla_j\omega_i\,dM. \tag{4.7}$$

By (2.2) and (4.7) we obtain

$$\int_M \|T^*\|^2\,dM = 4\int_M \|d\omega\|^2\,dM. \tag{4.8}$$

Hence we have $d\omega = 0$, that is M is a locally conformal Kähler surface. Furthermore by (2.2) and (2.3) we have $S = 0$, since M is assumed to be Einstein.

This completes the proof of Proposition 2.1 of [**3**].

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