Compact Hyperbolic Extra Dimensions: Branes, Kaluza-Klein Modes and Cosmology

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We reconsider theories with low gravitational (or string) scale M_* where Newton's constant is generated via new large-volume spatial dimensions, while Standard Model states are localized to a 3-brane. Utilizing compact hyperbolic manifolds (CHM's) we show that the spectrum of Kaluza-Klein (KK) modes is radically altered. This allows an early universe cosmology with normal evolution up to substantial temperatures , and completely negates the constraints on M_* arising from astrophysics. Furthermore, an exponential hierarchy between the usual Planck scale and the true fundamental scale of physics can emerge with only $\mathcal{O}(1)$ coefficients. The linear size of the internal space remains small. The proposal has striking testable signatures.

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Recent work [1–4] has heralded a renewed interest in higher-dimensional space-times, a key new concept being the localization of matter, and even gravity, to branes embedded in the extra dimensions [5]. In the canonical example of [2], space-time is a direct product of ordinary 4D space-time and a (flat) spatial d-torus of common linear size R and volume $V_{\rm new} = R^d$, while Standard Model particles are localized on a 3-brane of thickness $\sim M_*^{-1}$, where M_* is the new *fundamental* higher-dimensional gravitational (or string) scale. The low energy effective 4D Planck scale M_P is then given by the Gauss's Law relation, $M_P^2 = M_*^{2+d} R^d$. The hierarchy between M_P and M_* can be very large if $RM_* \gg 1$. For example, if $d = 2$ and $R \sim$ mm, then $M_* \sim$ TeV. The hierarchy $M_P/$TeV thus becomes a problem of understanding the size of the extra dimensions in such a model [6].

Remarkably, models with R approaching the sub-millimeter range are not excluded [7], but astrophysics and cosmology do place significant bounds. In particular, the evolution of the early universe at temperatures just above those at the epoch of Big Bang Nucleosynthesis (BBN) is inevitably, and dramatically altered. This narrow range of normal evolution prior to BBN makes it difficult to implement baryogenesis, moduli dilution etc.

The most important model-independent constraints on such models arise from the production of light KK modes of the graviton. These KK modes are the eigenmodes of the appropriate Laplace operator Δ on the internal space, and it is of central importance in the following that all the constraints depend on the form of the spectral density of this operator, which in turn depends completely on the topology and geometry of the internal space.

In this letter we argue that attractive alternate choices of compactification imply significantly weaker constraints, admitting in particular a standard 4D Friedmann-Robertson-Walker (FRW) evolution up to high temperatures. These compactifications employ a topologically non-trivial internal space— a d-dimensional compact hyperbolic manifold (CHM). They also throw into a new light the problem of explaining the large hierarchy $M_P/$TeV, since even though the volume of these manifolds is large, their linear size L is only slightly larger than the new fundamental length scale ($L \sim 30 M_*^{-1}$ for example), thus only requiring numbers of $\mathcal{O}(10)$.

CHM's are obtained from H^d, the universal covering space of hyperbolic manifolds (those admitting constant negative curvature), by modding out by an appropriate freely acting discrete subgroup Γ of the isometry group of H^d [8]. (If Γ is not freely-acting, then the resulting quotient is a non-flat non-smooth orbifold. We will not discuss this interesting case here.) These manifolds have been much discussed recently as the possible structure of ordinary 3-space [9], and play an important role in the theory of classical and quantum "chaotic" systems, where the spectra of Laplacian operators are also vital [10]. Here we will consider space-times of the form $M^4 \times (H^d/\Gamma|_{\rm free})$ (M^4 is a FRW 4-manifold) with metric

$$G_{IJ} dz^I dz^J = g^{(4)}_{\mu\nu}(x) dx^\mu dx^\nu + R_c^2 g^{(d)}_{ij}(y) dy^i dy^j. \quad (1)$$

Here R_c is the physical curvature radius of the CHM, so that $g_{ij}(y)$ is the metric on the CHM normalized so that its Ricci scalar is $\mathcal{R} = -1$, and $\mu = 0, \ldots, 3$, $i = 1, \ldots, d$.

Because they are locally negatively curved, CHM's exist only for $d \geq 2$. Their properties are well understood only for $d \leq 3$; however, it is known that CHM's in dimensions $d \geq 3$ possess the important property of *rigidity* [11]. As a result, these manifolds have *no massless shape moduli.* Moreover, the volume of the manifold, in units of the curvature radius R_c, cannot be changed while maintaining the homogeneity of the geometry. Hence, *the stabilization of such internal spaces reduces to the problem of stabilizing a single modulus,* the curvature length or the "radion". Of course, in a complete high-energy theory, (e.g. string theory), there will be massive $\mathcal{O}(M_*)$ excitations of the would-be shape moduli, and more important for the constraints, the massive KK modes.

To uncover the physics of these models one must consider the spectrum of small fluctuations h in the metric around the background eq. (1), $G_{IJ} \rightarrow G_{IJ} + \mathrm{e}^{ip.x} h_{IJ}(y)$. There are 3 different types of KK fluctuations that so arise: $h_{\mu\nu}$, the spin-2 piece; h_{ij}, with indices only in the internal directions, giving spin-0 fields for the 4D observer; and the mixed case $h_{i\mu}$, giving spin-1 4D fields. The 4D KK masses of these states are the eigenvalues of the appropriate internal-space Laplacians acting on $h(y)$, the correct Laplacian differing between these 3 cases. In the most important spin-2 case the operator is the Laplace-Beltrami operator Δ_{LB} (the Laplacian on scalar functions in the internal space), defined by

$$\Delta_{LB}\phi(y) = |g(y)|^{-1/2}\partial_i \left(|g(y)|^{1/2} g^{ij}\partial_j \phi(y)\right) . \qquad (2)$$

There are no known analytic expressions for the individual eigenvalues of Δ_{LB} on a CHM of any dimension. However, despite the extremely complicated topology and geometry of CHM's with arbitrarily large volume, a number of simple facts are generally true. First, by a variational argument, the spectrum of Δ_{LB} is bounded from below, and the lowest eigenmode is just the constant function on the CHM. This zero mode is the internal space wave-function of the massless spin-2 4D graviton. As it is a constant, the effective 4D Planck mass depends only on the *volume* of the (highly curved) internal space.

For example, suppose that the internal space was a 3-sphere of radius r, cut out of an H^3 of curvature radius R_c. Its volume $\mathrm{Vol}(r)$ grows exponentially for $r \gg R_c$,

$$\mathrm{Vol}(r) = \pi R_c^3 [\sinh(2r/R_c) - 2r/R_c] . \qquad (3)$$

In general, the total volume of a smooth compact hyperbolic space in any number of dimensions is

$$\mathrm{Vol}_{\mathrm{new}} = R_c^d \; e^{\alpha} , \qquad (4)$$

where α is a constant, *determined by topology.* (For $d = 3$ it is known that there is a countable infinity of orientable CHM's, with dimensionless volumes, e^{α}, bounded from below, but unbounded from above. Moreover, the e^{α} do not become sparsely distributed with large volume.) In addition, because the topological invariant e^{α} characterizes the volume of the CHM, it is also a measure of the largest distance L around the manifold. CHM's are globally anisotropic; however, since the largest linear dimension gives the most significant contribution to the volume, one can employ eq. (3), or its generalizations to $d \neq 3$, to find an approximate relationship between L and $\mathrm{Vol}_{\mathrm{new}}$. For $L \gg R_c/2$ the appropriate asymptotic relation, dropping irrelevant angular factors, is

$$e^{\alpha} \simeq \exp\bigl((d-1)L/R_c\bigr) . \qquad (5)$$

Thus, in strong contrast to the flat case, the expression for M_P depends *exponentially on the linear size*,

$$M_P^2 = M_*^{2+d} R_c^d e^{\alpha} = M_*^{2+d} R_c^d \exp\bigl((d-1)L/R_c\bigr). \qquad (6)$$

The most interesting case (and as we will see later, most reasonable) is the smallest possible curvature radius, $R_c \sim M_*^{-1}$. Taking $M_* \sim$ TeV then yields

$$L \simeq 35 M_*^{-1} = 10^{-15}\mathrm{mm} . \qquad (7)$$

Therefore, one of the most attractive features of a CHM internal space is that to generate an exponential hierarchy between $M_* \sim$ TeV, and M_P requires only that the linear size L be very mildly tuned.

We now return to the important topic of the non-zero eigenmodes of Δ_{LB} on CHM's, and to the astrophysical and cosmological implications of these KK modes. Recall that in flat models, the KK modes are extremely light, $m_{KK} \geq R^{-1} \geq 10^{-4}$eV, and very numerous, $N_{KK} \simeq M_P^2/M_*^2 \leq 10^{32}$ [2]. As a result, even though these modes are individually only weakly coupled, with strength $1/M_P$, they can be copiously produced by energetic processes on our brane, and observational limits then constrain the fundamental scale. The tightest astrophysical constraint comes from supernova physics, leading to a lower bound of $M_* \geq 50$TeV if $d = 2$, and of $M_* \geq 3$TeV for $d = 3$ [7,12]. There are also severe limits on the maximum temperature (the "normalcy temperature" T_*) above which the evolution of the universe must be non-standard [7]. This temperature is found by equating the rates for cooling by the usual process of adiabatic expansion, and by the new process of evaporation of KK gravitons into the bulk. This gives $T_* \leq 10$ MeV for $d = 2$, up to $T_* \leq 10$ GeV when $d = 6$. As we will now see, for us the situation is much improved.

First, by the compactness of the internal space, the spectrum of Δ_{LB} on a CHM is discrete and has a gap between the zero mode and the first excited KK state. The size of this gap is all important. Second, most of the eigenmodes of Δ_{LB} on a CHM have wavelengths less than R_c, and the number density of these modes is well approximated by the usual Weyl asymptotic formula

$$n(k) = (2\pi)^{-d}\Omega_{(d-1)} V_d k^{d-1} , \qquad (8)$$

where $\Omega_{(d-1)} = \mathrm{Area}(S^{d-1})$. There can also be a few lighter *supercurvature modes*, with wavelengths as large as the longest linear distance in the manifold, and masses thus bounded below by L^{-1}. There is no simple expression for the spectral density of supercurvature modes, although the Selberg trace formula provides some information on the full spectrum of Δ_{LB}. Nevertheless bounds on the first non-zero eigenvalue are known. In the best-studied CHM case of $d = 2$ we have the following theorem [13]: Consider a compact (oriented) Riemann surface S_g of arbitrary genus $g \geq 2$, with metric of constant negative curvature -1. Then for every ϵ, there exists $N \in Z^+$ such that for $g > N$ there exists an S_g with first eigenvalue

$$\lambda_1(S_g) \geq (C - \epsilon) , \qquad (9)$$

where $C \geq 171/784$ by earlier work [13]. Restoring units, a large enough volume (and thus genus) $d = 2$ CHM will have first eigenvalue $\geq 171/(784R_c^2)$. Moreover, Brooks has conjectured that for $d = 2$ a typical CHM chosen at random will have first eigenvalue $\geq 1/4R_c^2$ with positive probability P, perhaps even with $P \to 1$ as the genus $g \to \infty$ [14]. The analogous conjecture in $d = 3$ is more problematic, but has also been made [14]. Numerical studies of the spectra of even small volume $d = 3$ CHM's show that they have very few modes with $\lambda < R_c$ [15].

The crucial result is that the first KK modes are exponentially more massive than the very light $m_{KK} \geq 1/V^{1/d}$ found in the flat case. This drastically raises the threshold for their production. Even making the pessimistic assumption that the spectral density of the supercurvature modes satisfies eq. (8) for $k > 1/L$, *the astrophysical bounds of [7] and [12] completely disappear* since the lightest KK mode has a mass (at least 30 GeV), much greater than the temperature of even the hottest astrophysical object. Similarly the large KK masses imply a much higher normalcy temperature T_*, up to which the evolution of our brane-localized 4D universe can be normal radiation-dominated FRW. Approximate numerical evaluation shows that T_* is understandably sensitive to the gap to the first non-zero KK mass, ranging from 2 GeV to 10 GeV (for $d = 2$ to $d = 6$) if $m_{KK,1} \simeq 1/L \simeq$ TeV/35, and from 20 GeV to 40 GeV if $m_{KK,1} \simeq$ TeV/2 as suggested by the Brooks conjecture. (In all cases taking $M_* = 1$ TeV. Raising M_* raises T_*.)

So far we have concentrated on the spectrum of Δ_{LB} appropriate for the spin-2 KK excitations. What about the spin-0(1) excitations? In both cases the detailed form of the Laplacian is modified. For example, in the spin-0 case the correct operator is the Liechnerowicz Laplacian,

$$(\Delta_{\rm LL} h)_{ij} = -(D^k D_k h_{ij} + R_{ikjl} h^{kl}), \qquad (10)$$

where D_i is the covariant derivative. The Mostow-Prasad rigidity theorem for CHM's of dimension $d \geq 3$ tells us that $\Delta_{\rm LL}$ has no zero modes. Although we know of no rigorous bounds for the first eigenvalue of this operator, an inspection of the generalized Selberg trace formulae supports the conjecture that the gap is of similar size to the Laplace-Beltrami case, a result that is physically reasonable. Finally for the spin-1 fluctuations $h_{i\mu}$ recall that these zero modes would correspond to KK gauge-bosons (the original motivation of Kaluza and Klein!), and are directly related to the continuous isometries of the compact space. But, as a result of the quotient by Γ, CHM's have no such isometries, and thus there are no massless KK gauge bosons. The non-zero KK modes once again have a mass gap that is at least as large as $1/L$ and is more likely close to $\sim 1/R_c$, as in the previous cases. Thus these additional types of fluctuation do not disturb our estimates.

We have not yet addressed why it is almost automatic that there exist solutions of the form of eq. (1). Since CHM's are just quotients of H^d by a *discrete identification* under $\Gamma \subset {\rm Isom}(H^d)$, it is possible to find solutions of our form whenever there exists a uniform negative bulk cosmological constant (CC), given one constraint: $R_c \sim M_*^{-1}$ and $e^\alpha \simeq \exp((d-1)L/R_c) \gg 1$ must be realized consistently with our ansatz of a factorizable geometry with a static internal space, *together with the vanishing of the 4D long-distance* ($\gg L$) CC. To ensure a static internal space, the small curvature radius of the internal space must be balanced in the field equations by the bulk CC, $\Lambda_{4+d} \sim M_*^{4+d}$. Both these quantities contribute to the effective long-distance 4D CC, Λ_4, on our brane, and typically do not cancel. Furthermore, one cannot just set Λ_4 to zero by adjusting the tension or energy density f^4 of our 3-brane, because this requires $f^4 \gg M_*^4$, violating our basic assumption that a low-energy effective theory is valid on the brane (and perturbing the geometry, possibly destroying our assumption that it is factorizable). To address this problem we must examine the form of the total 4D potential energy density V, which in the effective theory depends only on R_c (e^α is an invariant), and which arises from the dimensional reduction of the full bulk and brane actions [6].

For a 3-brane embedded in $(4+d)$ dimensions, the bulk and brane actions are respectively:

$$S_{\rm bulk} = \int d^{4+d}x \sqrt{-|g_{(4+d)}|} \left(M_*^{d+2} \mathcal{R} + \Lambda - \mathcal{L}_m \right) \qquad (11)$$

$$S_{\rm brane} = \int d^4x \sqrt{-|g_{(4)}^{\rm induced}|} \left(f^4 + \dots \right), \qquad (12)$$

where $\mathcal{L}_m$ is the bulk matter field Lagrangian. Reduction of these actions gives a 4D potential energy density of the form

$$V(R_c) = \Lambda R_c^d e^\alpha - M_*^4 e^\alpha (M_* R_c)^{d-2} + W(R_c) \ , \qquad (13)$$

to which we must add the brane tension f^4. The first two terms arise from the $(4+d)$ bulk CC term, and the curvature of the internal space. Now consider expanding $W(R_c)$, which comes from $\mathcal{L}_m$, as a Laurent series in R_c

$$W(R_c) = \sum_p a_p \frac{M_*^4}{(R_c M_*)^p} \ , \qquad (14)$$

with dimensionless coefficients a_p. (Gauge or scalar field kinetic energies can give such terms with $p > 0$ [6].) If the determination of the minimum is dominated by a competition between any *two* terms in V, then at this minimum $V \equiv V_{\rm min} \neq 0$. Moreover, $V_{\rm min}$ is enhanced by e^α over the "natural" value M_*^4. However, the vanishing of the 4D CC demands $V_{\rm min}|_{\rm tot} = 0$. This cannot be achieved by adjusting the brane tension such that $|f^4| \leq M_*^4$.

Fortunately there is an attractive alternative. If *three* or more R_c-dependent terms in $V(R_c)$ are all important at the minimum (for example the CC and curvature terms, and one of the matter terms from W) then we can tune the coefficients a_p such that $V_{\rm min} = 0$, without needing $f^4 \gg M_*^4$. Thus, our basic assumptions remain consistent. Moreover, this tuning is particularly natural in our case precisely because we want the minimum to occur for a curvature radius close to the fundamental scale $R_c \sim M_*^{-1}$, at which we expect the high-scale theory to produce many different terms that contribute roughly in an equal way. (This is exactly the opposite situation from the large flat extra dimension case where the minimum has to occur at a length scale much greater than M_*^{-1}.) This one fine-tuning is just the usual 4d CC problem, about which we have nothing to add.

Having shown that there do exist solutions of our form, another significant result follows from this analysis. The most severe problem bedeviling the usual large extra dimension scenario is the radion moduli problem in the early universe [16]. In our case this problem is much weakened. The radion, which is the light mode corresponding to dilations of the internal space, gets its mass from the stabilizing potential $V(R_c)$. Generally, in the flat extra dimension scenario, the radion mass m_r is of size $M_*^2/M_P \simeq 10^{-3}$ eV, so that it is very easily excited during the exit from inflation. Furthermore, since its couplings are $1/M_P$ suppressed, its life-time is longer than the age of the universe, so that it would unacceptably dominate our current expansion. In our case, however, the radion mass is greatly increased because the second derivative of the potential at its minimum is enhanced by a factor of e^α, $V''_{\rm min} = \mathcal{O}(e^\alpha M_*^6)$. Thus

$$m_r^2 = \frac{1}{2}\frac{R_c^2 V''(R_c)}{e^\alpha M_*^{d+2} R_c^d} \simeq \frac{1}{R_c^2} \ , \tag{15}$$

which is close to $M_*^2 \sim \mathrm{TeV}^2$. Therefore, the radion lifetime is $T \sim M_P^2/M_*^3$, much shorter than in the case of flat extra dimensions, and only slightly longer than the age of the universe at nucleosynthesis, even if $M_* \sim$ TeV. Moreover, it is (comparatively) easy to dilute away any unwanted radion oscillations by a period of late inflation.

While cosmologically and astrophysically much safer, models with internal compact hyperbolic spaces do have testable signatures accessible to collider experiments. Since KK modes abound close to the fundamental scale, Standard Model particle collisions with center-of-mass energies near this scale will result in the production of KK particles, detectable by a distinctive missing energy signature [17]. Although this is qualitatively similar to the scenario of [3], the spectrum of KK modes one will see is quite distinctive. While the scale of KK masses is set by R_c^{-1}, their ratios and multiplicities are in almost one-to-one correspondence with the topology of the internal manifold [18]. A full exploration of these experimental signatures will require a more complete investigation of the spectrum of large CHM's, in particular the issues of isospectrality and homophonicity of such manifolds. It is quite likely that such CHM's have other implications for higher-dimensional physics. Besides a more detailed study of the question of radion stabilization, effects such as wavefunction scarring [10] and brane-manifold dynamics are currently under investigation.

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