

COMPACT KAEHLER MANIFOLDS WITH CONSTANT GENERALIZED SCALAR CURVATURE

BANG-YEN CHEN & KOICHI OGIUE

1. Introduction

Let M be an n -dimensional Kaehler manifold with fundamental 2-form Φ and Ricci 2-form γ . Following [2], we call M a *cohomological Einstein manifold* if $[\gamma] = a \cdot [\Phi]$ for some a , where $[*]$ denotes the cohomology class represented by $*$. The first Chern class of M is represented by γ .

It is well-known that a compact cohomological Einstein Kaehler manifold is Einsteinian if the scalar curvature is constant. The purpose of this paper is to generalize this result.

In [2], the second author introduced the notion of *generalized scalar curvatures*: Let $\omega^1, \dots, \omega^n$ be a local field of unitary coframes, so that the Kaehler metric of M is given by $g = \frac{1}{2} \sum (\omega^\alpha \otimes \bar{\omega}^\alpha + \bar{\omega}^\alpha \otimes \omega^\alpha)$. Let $S = \frac{1}{2} \sum (R_{\alpha\beta} \omega^\alpha \otimes \bar{\omega}^\beta + \bar{R}_{\alpha\beta} \bar{\omega}^\alpha \otimes \omega^\beta)$ be the Ricci tensor of M . We define n scalars ρ_1, \dots, ρ_n by

$$\det(\delta_{\alpha\beta} + tR_{\alpha\beta}) = 1 + \sum_{k=1}^n \rho_k t^k.$$

If we denote the scalar curvature of M by ρ , then it is easily seen that $\rho = 2\rho_1$. It is also clear that $\rho_n = \det(R_{\alpha\beta})$.

Our main theorem is the following.

Theorem. *Let M be an n -dimensional compact Kaehler manifold ($n \geq 2$).*

If

- (i) ρ_k is constant,
- (ii) $[\gamma^k] = a \cdot [\Phi^k]$ for some a ,
- (iii) $\text{rank}(R_{\alpha\beta}) \geq k$ (or equivalently $\rho_k \neq 0$) for some $k < n$,

then M is Einsteinian.

Assumption (iii) is redundant if $k = 1$, but it is essential if $k > 1$. Immediately from the above theorem we have

Corollary. *Let M be an n -dimensional compact Kaehler manifold ($n \geq 2$).*

If

Received August 9, 1974. The first author was partially supported by NSF Grant GP-36684, and the second author by NSF Grant GP-36684 and the Matsunaga Science Foundation.

- (i) ρ_k is constant,
 - (ii) M is cohomologically Einsteinian,
 - (iii) $\text{rank}(R_{\alpha\beta}) \geq k$ (or equivalently $\rho_k \neq 0$) for some $k < n$,
- then M is Einsteinian.

2. Proof of the theorem

Let Φ be the fundamental 2-form of M , that is, a closed 2-form defined by

$$\Phi = \frac{1}{2}\sqrt{-1} \sum \omega^\alpha \wedge \bar{\omega}^\alpha .$$

Let γ be the Ricci 2-form of M , that is, a closed 2-form defined by

$$\gamma = \frac{\sqrt{-1}}{4\pi} \sum R_{\alpha\beta} \omega^\alpha \wedge \bar{\omega}^\beta .$$

Let A be the operator of interior product by Φ . Then we have (cf. [2])

$$A^k \gamma^k = \frac{k!k!}{(2\pi)^k} \rho_k ,$$

which, together with assumption (i), implies that

$$(1) \quad dA^k \gamma^k = 0 .$$

Let δ be the codifferential operator, and C the operator defined by $C\alpha = (\sqrt{-1})^{r-s}\alpha$, where α is a form of bidegree (r, s) . Then they satisfy $dA^k - A^k d = kC^{-1}\delta CA^{k-1}$. Therefore from (1) and the fact that γ is closed we obtain

$$(2) \quad \delta A^{k-1} \gamma^k = 0 .$$

We prove the following general lemma.

Lemma. *Let η be a form of bidegree (p, q) with $p > 1$ and $q > 1$. If $A\delta\eta = 0$, then $\delta\eta = 0$.*

Proof. If we denote the star isomorphism by $*$, then $A\delta\eta = 0$ is equivalent to $*A*d*\eta = 0$. If we denote the dual operator of A by L , then $*A*d*\eta = 0$ is equivalent to $Ld*\eta = 0$. Since L is an isomorphism (cf. for example [1]), $Ld*\eta = 0$ is equivalent to $d*\eta = 0$. Applying $*$ we obtain $*d*\eta = 0$ which is equivalent to $\delta\eta = 0$. q.e.d.

Since $A\delta = \delta A$, it follows from (2) that $A\delta A^{k-2} \gamma^k = 0$. Therefore by Lemma we have

$$\delta A^{k-2} \gamma^k = 0 .$$

Repeatedly applying this process we finally obtain

$$\delta\gamma^k = 0 ,$$

which, together with the fact that $d\gamma^k = 0$, implies that γ^k is harmonic. Therefore assumption (ii) yields that

$$(3) \quad \gamma^k = a\bar{\Phi}^k .$$

At each point of M , we can choose a unitary coframe $\omega^1, \dots, \omega^n$ with respect to which $(R_{\alpha\bar{\beta}})$ is of the form

$$\begin{pmatrix} \lambda_1 & & & 0 \\ & \ddots & & \\ 0 & & & \lambda_n \end{pmatrix}$$

so that $\gamma = \frac{\sqrt{-1}}{4\pi} \sum \lambda_\alpha \omega^\alpha \wedge \bar{\omega}^\alpha$. Therefore

$$\gamma^k = \left(\frac{\sqrt{-1}}{4\pi}\right)^k k! \sum_{\alpha_1 < \dots < \alpha_k} \lambda_{\alpha_1} \dots \lambda_{\alpha_k} \omega^{\alpha_1} \wedge \bar{\omega}^{\alpha_1} \wedge \dots \wedge \omega^{\alpha_k} \wedge \bar{\omega}^{\alpha_k} .$$

Hence from (3) we obtain the following system of simultaneous equations

$$(4) \quad \begin{aligned} \lambda_1 \dots \lambda_{k-1}(\lambda_k - \lambda_{k+1}) &= 0 , \\ \lambda_1 \dots \lambda_{k-1}(\lambda_k - \lambda_{k+2}) &= 0 , \\ \dots \dots \dots \dots \dots \dots \dots \end{aligned}$$

which consists of $\binom{n}{k-1} \binom{n-k+1}{2}$ equations. It is easily seen that, under assumption (iii), (4) implies that $\lambda_1 = \dots = \lambda_n$. Therefore M is Einsteinian.

Remark. Assumption (iii) is essential if $k > 1$. In fact, let $M = P_{k-1}(C) \times T^{n-k+1}$, where $P_{k-1}(C)$ denotes a $(k-1)$ -dimensional complex projective space with the Fubini-Study metric, and T^{n-k+1} denotes an $(n-k+1)$ -dimensional complex torus with the flat metric. Then M satisfies assumptions (i) and (ii), but M is not Einsteinian.

References

[1] S. I. Goldberg, *Curvature and homology*, Academic Press, New York, 1962.
 [2] K. Ogiue, *Generalized scalar curvatures of cohomological Einstein Kaehler manifolds*, *J. Differential Geometry* **10** (1975) 201-205.

MICHIGAN STATE UNIVERSITY
 TOKYO METROPOLITAN UNIVERSITY

