Compact Kähler manifolds with hermitian semipositive anticanonical bundle

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1. Main results

This short note is a continuation of our previous work [DPS93] on compact Kähler manifolds X with semipositive Ricci curvature. Our purpose is to state a splitting theorem describing the structure of such manifolds, and to raise some related questions. The foundational background will be found in papers by Lichnerowicz [Li67], [Li71], and Cheeger-Gromoll [CG71], [CG72]. Recall that a *Calabi-Yau manifold* X is a compact Kähler manifold with $c_1(X) = 0$ and finite fundamental group $\pi_1(X)$, such that the universal covering $\widetilde{X}$ satisfies $H^0(\widetilde{X}, \Omega^p_{\widetilde{X}}) = 0$ for all $1 \le p \le \dim X - 1$. A *symplectic manifold* X is a compact Kähler manifold admitting a holomorphic symplectic 2-form ω (of maximal rank everywhere); in particular $K_X = \mathcal{O}_X$. We denote here as usual

$$\Omega_X = T^\star_X, \qquad \Omega^p_X = \Lambda^p T^\star_X, \qquad K_X = \det(T^\star_X).$$

The following structure theorem generalizes the Bogomolov-Kobayashi-Beauville structure theorem for Ricci-flat manifolds ([Bo74a], [Bo74b], [Ko81], [Be83]) to the Ricci semipositive case.

Structure Theorem. *Let X be a compact Kähler manifold with $-K_X$ hermitian semipositive. Then*

i) *The universal covering $\widetilde{X}$ admits a holomorphic and isometric splitting*

$$\widetilde{X} \simeq \mathbb{C}^q \times \prod X_i$$

with X_i being either a Calabi-Yau manifold or a symplectic manifold or a manifold with $-K_{X_i}$ semipositive and $H^0(X_i, \Omega^{\otimes m}_{X_i}) = 0$ for all $m > 0$.

ii) *There exists a finite étale Galois covering $\widehat{X} \to X$ such that the Albanese variety $\mathrm{Alb}(\widehat{X})$ is a q-dimensional torus and the Albanese map $\alpha : \widehat{X} \to \mathrm{Alb}(\widehat{X})$ is a locally trivial holomorphic fibre bundle whose fibres are products $\prod X_i$ of the type described in* i)*, all X_i being simply connected.*

iii) *We have $\pi_1(\widehat{X}) \simeq \mathbb{Z}^{2q}$ and $\pi_1(X)$ is an extension of a finite group Γ by the normal subgroup $\pi_1(\widehat{X})$. In particular there is an exact sequence*

$$0 \to \mathbb{Z}^{2q} \to \pi_1(X) \to \Gamma \to 0,$$

and the fundamental group $\pi_1(X)$ is almost abelian.

Recall that a line bundle L is said to be hermitian semipositive if it can be equipped with a smooth hermitian metric of semipositive curvature form. A sufficient condition for hermitian semipositivity is that some multiple of L is spanned by global sections; on the other hand, the hermitian semipositivity condition implies that L is numerically effective (nef) in the sense of [DPS94], which, for X projective algebraic, is equivalent to saying that $L \cdot C \geq 0$ for every curve C in X. Examples contained in [DPS94] show that all three conditions are different (even for X projective algebraic). By Yau's solution of the Calabi conjecture (see [Au76], [Yau78]), a compact Kähler manifold X has a hermitian semipositive anticanonical bundle $-K_X$ if and only if X admits a Kähler metric g with $\mathrm{Ricci}(g) \geq 0$. The isometric decomposition described in the theorem refers to such Kähler metrics.

In view of "standard conjectures" in minimal model theory it is expected that projective manifolds X with no nonzero global sections in $H^0(X, \Omega_X^{\otimes m})$, $m > 0$, are rationally connected. We hope that most of the above results will continue to hold under the weaker assumption that $-K_X$ is nef instead of hermitian semipositive. However, the technical tools needed to treat this case are still missing.

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2. Bochner formula and holomorphic differential forms

Our starting point is the following well-known consequence of the Bochner formula.

Lemma. *Let X be a compact n-dimensional Kähler manifold with $-K_X$ hermitian semipositive. Then every section of $\Omega_X^{\otimes m}$, $m \geq 1$ is parallel with respect to the given Kähler metric.*

Proof. The Lemma is an easy consequence of the Bochner formula

$$\Delta(\| u \|^2) = \| \nabla u \|^2 + Q(u),$$

where $u \in H^0(X, \Omega_X^{\otimes m})$ and $Q(u) \geq m\lambda_0 \| u \|^2$. Here λ_0 is the smallest eigenvalue of the Ricci curvature tensor. For details see for instance [Ko83]. Q.E.D.

The following definition of a modified Kodaira dimension $\kappa_+(X)$ is taken from Campana [Ca93]. As the usual Kodaira dimension $\kappa(X)$, this is a birational invariant of X. Other similar invariants have also been considered in [BR90] and [Ma93].

Definition. *Let Y be a compact complex manifold. We define*

i) $\kappa_+(Y) = \max\{\kappa(\det \mathcal{F}) : \mathcal{F}$ *is a subsheaf of* Ω_Y^p *for some* $p > 0\}$,

ii) $\kappa_{++}(Y) = \max\{\kappa(\det \mathcal{F}) : \mathcal{F}$ *is a subsheaf of* $\Omega_Y^{\otimes m}$ *for some* $m > 0\}$.

Here we let as usual $\det \mathcal{F} = (\Lambda^r \mathcal{F})^{**}$, *where* $r = \mathrm{rank}\mathcal{F}$ *and* κ *is the usual Iitaka dimension of a line bundle.*

Clearly, we have $-\infty \le \kappa(Y) \le \kappa_+(Y) \le \kappa_{++}(Y)$ where $\kappa(Y) = \kappa(K_Y)$ is the usual Kodaira dimension. It would be interesting to know whether there are precise relations between $\kappa_+(Y)$ and $\kappa_{++}(Y)$, as well as with the weighted Kodaira dimensions defined by Manivel [Ma93]. The above lemma implies:

Proposition. *Let X be a compact Kähler manifold with $-K_X$ hermitian semipositive. Then $\kappa_{++}(X) \le 0$.*

Proof. Assume $\kappa_{++}(X) > 0$. Then we can find $m > 0$ and a subsheaf $\mathcal{F} \subset \Omega_X^{\otimes m}$ with $\kappa(\det \mathcal{F}) > 0$. Hence there is some $\mu \in \mathbb{N}$ and $s \in H^0(X, (\det \mathcal{F})^\mu)$ with $s \ne 0$. Since $\kappa(\det \mathcal{F}) > 0$, s must have zeroes. Hence the induced section $\tilde{s} \in H^0(X, \Omega_X^{\otimes \mu r m})$ has zeroes too, r being the rank of $\mathcal{F}$. This contradicts the previous Lemma. Q.E.D.

Corollary. *Let X be a compact Kähler manifold with $-K_X$ hermitian semipositive. Let $\phi : X \to Y$ be a surjective holomorphic map to a normal compact Kähler space. Then $\kappa(Y) \le 0$. (Here $\kappa(Y) = \kappa(\hat{Y})$, where $\hat{Y}$ is an arbitrary desingularization of Y.)*

Proof. This follows from the inequalities $0 \ge \kappa_+(X) \ge \kappa_+(Y) \ge \kappa(Y)$. For the second inequality, which is easily checked by a pulling-back argument, see [Ca93]. Q.E.D.

3. Proof of the structure theorem

We suppose here that X is equipped with a Kähler metric g such that $\mathrm{Ricci}(g) \ge 0$, and we set $n = \dim_{\mathbb{C}} X$.

i) Let $(\widetilde{X}, g) \simeq \prod (X_i, g_i)$ be the De Rham decomposition of $(\widetilde{X}, g)$, induced by a decomposition of the holonomy representation in irreducible representations. Since the holonomy is contained in $\mathrm{U}(n)$, all factors (X_i, g_i) are Kähler manifolds with irreducible holonomy and holonomy group $H_i \subset \mathrm{U}(n_i)$, $n_i = \dim X_i$. By Cheeger-Gromoll [CG71], there is possibly a flat factor $X_0 = \mathbb{C}^q$ and the other factors X_i, $i \ge 1$, are compact. Also, the product structure shows that $-K_{X_i}$ is hermitian semipositive. It suffices to prove that $\kappa_{++}(X_i) = 0$ implies that X_i is a Calabi-Yau manifold or a symplectic manifold. In view of § 2, the condition $\kappa_{++}(X_i) = 0$ means that there is a nonzero section $u \in H^0(X_i, \Omega_{X_i}^{\otimes m})$ for some $m > 0$. Since u is parallel by the lemma, it is invariant under the holonomy action, and therefore the holonomy group H_i is not the full unitary group $\mathrm{U}(n_i)$ (indeed, the trivial representation does not occur in the decomposition of $(\mathbb{C}^{n_i})^{\otimes m}$ in irreducible $\mathrm{U}(n_i)$-representations, all weights being of length m). By Berger's classification of holonomy groups [Bg55] there are only two remaining possibilities, namely $H_i = \mathrm{SU}(n_i)$ or $H_i = \mathrm{Sp}(n_i/2)$. The case $H_i = \mathrm{SU}(n_i)$ leads to X_i being a Calabi-Yau manifold. The remaining case $H_i = \mathrm{Sp}(n_i/2)$ implies that X_i is symplectic (see e.g. [Be83]).

ii) Set $X' = \prod_{i \ge 1} X_i$. The group of covering transformations acts on the product $\widetilde{X} = \mathbb{C}^q \times X'$ by holomorphic isometries of the form $x = (z, x') \mapsto (u(z), v(x'))$.

At this point, the argument is slightly more involved than in Beauville's paper [Be83], because the group G' of holomorphic isometries of X' need not be finite (X' may be for instance a projective space); instead, we imitate the proof of ([CG72], Theorem 9.2) and use the fact that X' and $G' = \mathrm{Isom}(X')$ are compact. Let $E_q = \mathbb{C}^q \ltimes U(q)$ be the group of unitary motions of $\mathbb{C}^q$. Then $\pi_1(X)$ can be seen as a discrete subgroup of $E_q \times G'$. As G' is compact, the kernel of the projection map $\pi_1(X) \to E_q$ is finite and the image of $\pi_1(X)$ in E_q is still discrete with compact quotient. This shows that there is a subgroup Γ of finite index in $\pi_1(X)$ which is isomorphic to a crystallographic subgroup of $\mathbb{C}^q$. By Bieberbach's theorem, the subgroup $\Gamma_0 \subset \Gamma$ of elements which are translations is a subgroup of finite index. Taking the intersection of all conjugates of Γ_0 in $\pi_1(X)$, we find a normal subgroup $\Gamma_1 \subset \pi_1(X)$ of finite index, acting by translations on $\mathbb{C}^q$. Then $\widehat{X} = \widetilde{X}/\Gamma_1$ is a fibre bundle over the torus $\mathbb{C}^q/\Gamma_1$ with X' as fibre and $\pi_1(X') = 1$. Therefore $\widehat{X}$ is the desired finite étale covering of X.

iii) is an immediate consequence of ii), using the homotopy exact sequence of a fibration. Q.E.D.

Corollary 1. *Let X be a compact Kähler manifold with $-K_X$ hermitian semi-positive. If $\widetilde{X}$ is indecomposable and $\kappa_+(X) = 0$, then X is Ricci-flat.*

Corollary 2. *Let X be a compact Kähler manifold with $-K_X$ hermitian semi-positive. Then, if $\widehat{X} \to X$ is an arbitrary finite étale covering,*

$$\begin{aligned} \kappa_+(X) = -\infty &\iff \kappa_{++}(X) = -\infty \\ &\iff \forall \widehat{X} \to X,\ \forall p \geq 1, \quad H^0(\widehat{X}, \Omega^p_{\widehat{X}}) = 0. \end{aligned}$$

If $\kappa_+(X) = -\infty$, then $\chi(X, \mathcal{O}_X) = 1$ and X is simply connected.

Proof. The equivalence of all three properties is a direct consequence of the structure theorem. Now, every étale covering $\widehat{X} \to X$ satisfies $\kappa_+(\widehat{X}) = \kappa_+(X) = -\infty$, hence $\chi(\widehat{X}, \mathcal{O}_{\widehat{X}}) = \chi(X, \mathcal{O}_X) = 1$ (by Hodge symmetry we have $h^p(X, \mathcal{O}_X) = 0$ for $p \geq 1$, whilst $h^0(X, \mathcal{O}_X) = 1$). However, if d is the covering degree, the Riemann-Roch formula implies $\chi(\widehat{X}, \mathcal{O}_{\widehat{X}}) = d\,\chi(X, \mathcal{O}_X)$, hence $d = 1$ and X must be simply connected. Q.E.D.

4. Related questions for the case -K_X nef

In order to make the structure theorem more explicit, it would be necessary to characterize more precisely the manifolds for which $\kappa_+(X) = -\infty$. We expect these manifolds to be rationally connected, even when $-K_X$ is just supposed to be nef.

Conjecture. *Let X be a compact Kähler manifold such that $-K_X$ is nef and $\kappa_+(X) = -\infty$. Then X is rationally connected, i.e. any two points of X can be joined by a chain of rational curves.*

Campana even conjectures this to be true without assuming $-K_X$ to be nef.

Another hope we have is that a similar structure theorem might also hold in the case $-K_X$ nef. A small part of it would be to understand better the structure of the Albanese map. We proved in [DPS93] that the Albanese map is surjective when $\dim X \le 3$, and if $\dim X \le 2$ it is well-known that the Albanese map is a locally trivial fibration. It is thus natural to state the following

Problem. *Let X be a compact Kähler manifold with $-K_X$ nef. Is the Albanese map $\alpha : X \to \mathrm{Alb}(X)$ a smooth locally trivial fibration?*

The following simple example shows, even in the case of a locally trivial fibration, that the structure group of transition automorphisms need not be a group of isometries, in contrast with the case $-K_X$ hermitian semipositive.

Example 1 (see [DPS94], Example 1.7). Let $C = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ be an elliptic curve, and let $E \to C$ be the flat rank 2 bundle associated to the representation $\pi_1(C) \to \mathrm{GL}_2(\mathbb{C})$ defined by the monodromy matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then the projectivized bundle $X = \mathbb{P}(E)$ is a ruled surface over C with $-K_X$ nef and not hermitian semipositive (cf. [DPS94]). In this case, the Albanese map $X \to C$ is a locally trivial $\mathbb{P}_1$-bundle, but the monodromy group is not relatively compact in $\mathrm{GL}_2(\mathbb{C})$, hence there is no invariant Kähler metric on the fibre.

Example 2. The following example shows that the picture is unclear even in the case of surfaces with $\kappa_+(X) = -\infty$. Let $\mathbf{p} = (p_1, \ldots, p_9)$ be a configuration of 9 points in $\mathbb{P}_2$ and let $\pi : X_{\mathbf{p}} \to \mathbb{P}_2$ be the blow-up of $\mathbb{P}_2$ with center $\mathbf{p}$. Here some of the points p_i may be infinitely near: as usual, this means that the blowing-up process is made inductively, each p_i being an arbitrary point in the blow-up of $\mathbb{P}_2$ at $(p_1, \ldots, p_{i-1})$. There is always a cubic curve C containing all 9 points (C is even unique if $\mathbf{p}$ is general enough). The only assumption we make is that C is nonsingular, and we let $C = \{Q(z_0, z_1, z_2) = 0\} \subset \mathbb{P}_2$, $\deg Q = 3$. Then C is an elliptic curve and $-K_{X_{\mathbf{p}}} = \pi^\star\mathcal{O}(3) - \sum E_i$ where $E_i = \pi^{-1}(p_i)$ are the exceptional divisors. Clearly Q defines a section of $-K_{X_{\mathbf{p}}}$, of divisor equal to the strict transform C' of C, hence $-K_{X_{\mathbf{p}}} \simeq \mathcal{O}(C')$, and $(-K_{X_{\mathbf{p}}})^2 = (C')^2 = C^2 - 9 = 0$. Therefore $-K_{X_{\mathbf{p}}}$ is always nef.

It is easy to see that $-mK_{X_{\mathbf{p}}}$ may be generated or not by sections according to the choice of the 9 points p_i. In fact, if p'_i is the point of C' lying over p_i, we have

$$-K_{X_{\mathbf{p}}|C'} = \pi^\star(\mathcal{O}(3))_{|C'} \otimes \mathcal{O}(-\textstyle\sum p'_j) = \pi^\star\Big(\mathcal{O}(3)_{|C} \otimes \mathcal{O}(-\textstyle\sum p_j)\Big).$$

Since $C' \simeq C$ is an elliptic curve and $-K_{X_{\mathbf{p}}|C'}$ has degree 0, there are nonzero sections in $H^0(C', -mK_{X_{\mathbf{p}}|C'})$ if and only if $L_{\mathbf{p}} = \mathcal{O}(3)_{|C} \otimes \mathcal{O}(-\sum p_j))$ is a torsion point in $\mathrm{Pic}^0(C)$ of order dividing m. Such sections always extend to $X_{\mathbf{p}}$. Indeed, we may assume that m is exactly the order. Then $\mathcal{O}(-C') \otimes \mathcal{O}(-mK_{X_{\mathbf{p}}}) = \mathcal{O}((m-1)C')$ admits a filtration by its subsheaves $\mathcal{O}(kC')$, $0 \le k \le m-1$, and the H^1 groups of the graded pieces are $H^1(X_{\mathbf{p}}, \mathcal{O}_{X_{\mathbf{p}}}) = 0$ for $k = 0$ and

$$H^1(C', \mathcal{O}(kC')_{|C'}) = H^0(C', \mathcal{O}(-kC')) = 0 \quad \text{for } 0 < k < m.$$

Therefore $H^1(X_{\mathbf{p}}, \mathcal{O}(-C') \otimes \mathcal{O}(-mK_{X_{\mathbf{p}}})) = 0$, as desired. In particular, $-K_{X_{\mathbf{p}}}$ is hermitian semipositive as soon as $L_{\mathbf{p}}$ is a torsion point in $\mathrm{Pic}^0(C)$. In this case, there is a polynomial R_m of degree $3m$ vanishing of order m at all points p_i, such that the rational function R_m/Q^m defines an elliptic fibration $\varphi : X_{\mathbf{p}} \to \mathbb{P}_1$; in this fibration C is a multiple fibre of multiplicity m and we have $-mK_{X_{\mathbf{p}}} = \varphi^\star \mathcal{O}_{\mathbb{P}_1}(1)$. An interesting question is to understand what happens when $L_{\mathbf{p}}$ is no longer a torsion point in $\mathrm{Pic}^0(C)$ (this is precisely the situation considered by Ogus [Og76] in order to produce a counterexample to the formal principle for infinitesimal neighborhoods). In this situation, we may approximate $\mathbf{p}$ by a sequence of configurations $\mathbf{p}_m \subset C$ such that the corresponding line bundle $L_{\mathbf{p}_m}$ is a torsion point of order m (just move a little bit p_9 and take a suitable $p_{9,m} \in C$ close to p_9). The sequence of fibrations $X_{\mathbf{p}_m} \to \mathbb{P}_1$ does not yield a fibration $X_{\mathbf{p}} \to \mathbb{P}_1$ in the limit, but we believe that there might exist instead a holomorphic foliation on $X_{\mathbf{p}}$. In this foliation, C would be a closed leaf, and the generic leaf would be nonclosed and of conformal type $\mathbb{C}$ (or possibly $\mathbb{C}^\star$). If indeed the foliation exists and admits a smooth invariant transversal volume form, then $-K_{X_{\mathbf{p}}}$ would still be hermitian semipositive. We are thus led to the following question.

Question. *Let X be compact Kähler manifold with $-K_X$ nef and X rationally connected. Is then $-K_X$ automatically hermitian semipositive? In particular, is it always the case that $\mathbb{P}_2$ blown-up in 9 points of a nonsingular cubic curve has a semipositive anticanonical bundle?*

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