

COMPACT LEAVES WITH ABELIAN HOLONOMY

TOSHIYUKI NISHIMORI

(Received March 22, 1974)

1. Introduction and statement of the results. In this paper we investigate the behaviour of codimension-one foliations in neighborhoods of their compact leaves with abelian holonomy. If the foliations are C^2 class, we have three cases which occur independently. The following theorem is the precise formulation.

THEOREM 1. *Let \mathcal{F} be a transversely orientable codimension-one C^r foliation on an orientable C^r manifold M and F_0 a compact leaf of \mathcal{F} . Suppose that $2 \leq r \leq \infty$. Let T be a tubular neighborhood of F_0 and U_+ the union of F_0 and a connected component of $T - F_0$. We denote by $\Phi_+(F_0)$ the one-sided holonomy group of F_0 which is defined by the restricted foliation $\mathcal{F} | U_+$. Suppose that $\Phi_+(F_0)$ is an abelian group, then only one of the following three cases occurs.*

(1) *For all neighborhoods V of F_0 , the restricted foliation $\mathcal{F} | V \cap U_+$ has a compact leaf which is not F_0 .*

(2) *There is a neighborhood V of F_0 such that all leaves of $\mathcal{F} | V \cap U_+$ except F_0 are dense in $V \cap U_+$. In this case $\Phi_+(F_0)$ is a free abelian group of the rank ≥ 2 .*

(3) *There are a neighborhood V of F_0 and a connected oriented codimension-one submanifold N of F_0 as follows. We denote by F_* the compact manifold with boundary obtained by attaching two copies N_1 and N_2 of N to $F_0 - N$, so that $\partial F_* = N_1 \cup N_2$. Let $f: [0, \varepsilon) \rightarrow [0, \delta)$ be a contracting C^r diffeomorphism satisfying $f(0) = 0$. We denote by X_f the quotient manifold obtained from $F_* \times [0, \varepsilon)$ by identifying $(x, t) \in N_1 \times [0, \varepsilon)$ and $(x, f(t)) \in N_2 \times [0, \delta)$. We have a foliation \mathcal{F}_f on X_f whose leaves consist of some family of the sets $F_* \times \{t\}$, $t \in [0, \varepsilon)$. Then for some f as above, there is a C^r diffeomorphism $h: V \cap U_+ \rightarrow X_f$ which maps each leaf of $\mathcal{F} | V \cap U_+$ onto some leaf of \mathcal{F}_f . $\mathcal{F} | V \cap U_+$ determines the homology class $[N] \in H_{n-2}(F_0^{n-1}, \mathbb{Z})$ uniquely and the germ at 0 of f uniquely up to conjugation. In this case $\Phi_+(F_0)$ is an infinite cyclic group.*

We can reformulate Theorem 1 in a following weaker form. Foliations of the type in Theorem 1* were studied by Reeb [8] and Theorem 1*

can be considered as a strengthened form of the results of [8]. Theorem 1* says that Theorem 5 of [8] is valid even if supposing C^2 in place of analyticity of foliations.

THEOREM 1*. *Let \mathcal{F} be a transversely orientable codimension-one C^r foliation on $M \times [0, 1]$ where M is a connected orientable closed C^r manifold, $2 \leq r \leq \infty$. Suppose that all leaves of \mathcal{F} are transverse to $\{x\} \times [0, 1]$ for all $x \in M$ and that \mathcal{F} has only two compact leaves $M \times \{0\}$ and $M \times \{1\}$. As usual we have a homomorphism $\Phi: \pi_1(M, x_0) \rightarrow \text{Diff}_0^r([0, 1])$ where $x_0 \in M$ and $\text{Diff}_0^r([0, 1])$ is the group of all orientation-preserving C^r diffeomorphisms: $[0, 1] \rightarrow [0, 1]$. [Let $\omega: ([0, 1], \{0, 1\}) \rightarrow (M, x_0)$ be a closed path, then we have a foliation \mathcal{F}_ω on $[0, 1] \times [0, 1]$ induced from \mathcal{F} by the map $\omega \times \text{id}: [0, 1] \times [0, 1] \rightarrow M \times [0, 1]$. We define $f_\omega(t) \in [0, 1]$ so that $(t, 0)$ and $(f_\omega(t), 1)$ are the endpoints of a leaf of \mathcal{F}_ω . Let $\Phi([\omega]) = f_\omega: [0, 1] \rightarrow [0, 1]$ where $[\omega]$ means the homotopy class of ω , then Φ is a well-defined homomorphism.] If we suppose that the image of Φ is abelian, then only one of the following occurs.*

(1) *All leaves except the compact leaves are dense and Image (Φ) is a free abelian group of the rank ≥ 2 .*

(2) *All leaves are proper. All leaves except the compact leaves are mutually C^r diffeomorphic and are covering spaces of M whose covering transformation groups are infinite cyclic groups. Image (Φ) is an infinite cyclic group.*

(1) and (3) of Theorem 1 clearly occur for some foliations. In § 4 we construct a C^∞ foliation which admits (2) of Theorem 1. Pay attention to the example of Sacksteder [9]. It shows that the hypothesis that the holonomy group is abelian is a necessary condition. Furthermore we can construct a topological foliation as to which Theorem 1 and 1* are not valid (Theorem 2). But we can say nothing about the C^1 case.

THEOREM 2. *Let S_g be the closed orientable surface of genus g , then there is a transversely orientable codimension-one topological foliation \mathcal{F} on $S_g \times [0, 1]$ as follows.*

(1) *All leaves of \mathcal{F} are C^∞ submanifolds and transverse to $\{x\} \times [0, 1]$ for all $x \in S_g$ and \mathcal{F} has only two compact leaves $S_g \times \{0\}$ and $S_g \times \{1\}$.*

(2) *Image (Φ) (defined as in Theorem 1*) is a free abelian group of rank g .*

(3) *All leaves are proper.*

(4) *There is a sequence F_0, F_1, \dots, F_g of the leaves of \mathcal{F} such that F_0 is $S_g \times \{0\}$ and $\bar{F}_0 \subsetneq \bar{F}_1 \subsetneq \dots \subsetneq \bar{F}_g$.*

2. Definitions and notations.

2.1. The fundamental references on foliations are Reeb [7], Haefliger [2] and Lawson [5]. In spite of their existence, we will start from the definitions for the sake of the self-containedness and technical advantage.

Let M be a C^r n -manifold, $0 \leq r \leq \omega$. A subset \mathcal{F} of the C^r structure $\mathcal{S} \subset \{(U, f) \mid U \text{ is an open set of } M. f: U \rightarrow R^n \text{ is an injective homeomorphism.}\}$ of M is called a *codimension q C^r foliation* if \mathcal{F} satisfies the following conditions (1) and (2).

(1) $\{U \mid (U, f) \in \mathcal{F}\}$ is an open covering of M and $f(U) = R^n$ for all $(U, f) \in \mathcal{F}$.

(2) For all pairs $(U_1, f_1), (U_2, f_2) \in \mathcal{F}$ with $U_1 \cap U_2 \neq \emptyset$, the C^r diffeomorphism $f_1 \circ f_2^{-1}: f_2(U_1 \cap U_2) (\subset R^n = R^p \times R^q) \rightarrow R^n = R^p \times R^q$ has the form

$$f_1 \circ f_2^{-1}(x, y) = (\alpha(x, y), \beta(y))$$

where $x \in R^p, y \in R^q, \alpha: R^p \times R^q \rightarrow R^p$ is a C^r local submersion and $\beta: R^q \rightarrow R^q$ is a C^r local diffeomorphism.

Two foliations \mathcal{F}_1 and \mathcal{F}_2 are *equivalent* if $\mathcal{F}_1 \cup \mathcal{F}_2$ satisfies the condition (2). A foliation \mathcal{F} is called *transversely orientable* if \mathcal{F} is equivalent to another foliation \mathcal{F}_1 such that in the definition of \mathcal{F}_1 all β of the condition (2) are orientation-preserving.

Let \mathcal{F} be a foliation on M . The condition (2) guarantees that

$$\begin{aligned} f_2^{-1}(R^p \times \{y\}) \cap f_1^{-1}(R^p \times \{\beta(y)\}) &= U_1 \cap f_2^{-1}(R^p \times \{y\}) \\ &= U_2 \cap f_1^{-1}(R^p \times \{\beta(y)\}) \end{aligned}$$

and $f_2^{-1}(R^p \times \{y\}) \cup f_1^{-1}(R^p \times \{\beta(y)\})$ is a submanifold of $U_1 \cup U_2$. We say that $f_2^{-1}(R^p \times \{y\})$ and $f_1^{-1}(R^p \times \{\beta(y)\})$ are *adjacent*. Then we have an equivalence relation on the family

$$\{f^{-1}(R^p \times \{y\}) \mid (U, f) \in \mathcal{F}, y \in R^q\}$$

defined by the relation adjacent. The union F of all subsets in an equivalence class is called a *leaf* of \mathcal{F} and F has naturally a structure of C^r manifold of dimension p .

If the topology of a leaf F as a subset of M and the topology as a p -manifold coincide, F is called *proper*.

If \mathcal{F} is a codimension q C^r foliation on M_2 and $f: M_1 \rightarrow M_2$ is a C^r map which is transverse to all leaves of \mathcal{F} , then there is a codimension q C^r foliation $f^*\mathcal{F}$ whose leaves are connected components of $f^{-1}(F)$ where F runs through the leaves of \mathcal{F} . We call $f^*\mathcal{F}$ the foliation induced by f from \mathcal{F} . If f is an inclusion, we call $f^*\mathcal{F}$ the restricted foliation and

we write $f^*\mathcal{F} = \mathcal{F} | M$.

2.2. Now we restrict our attention to transversely orientable codimension-one C^r foliation \mathcal{F} on an orientable manifold M . We give a definition of holonomy groups in a style which is useful in the following arguments and which is a slight modification of that of Hirsch [3].

We suppose $r \geq 1$ and we remark that in the C^0 case the following arguments can be suitably modified by using Siebenmann [10] and Hirsch [3].

Let \mathcal{F} be a transversely orientable codimension-one C^r foliation on an orientable C^r n -manifold, $r \geq 1$. Therefore we may suppose that all β of 2.1 (2) are orientation-preserving. Transverse orientability of \mathcal{F} and the partition of unity imply the existence of a non-singular vector field X which is transverse to all leaves of \mathcal{F} . Let $\varphi: M \times R \rightarrow M$ be the associated flow of X . Then $\varphi|_{\varphi(\{x\} \times R)}$ and $f^*|_{\varphi(\{x\} \times R)}$ have derivatives at $0 \in R$ of the same sign for all $x \in M$ and $(U, f) \in \mathcal{F}$ satisfying $x \in U$ where f^* is the composition of f and the projection of $R^{n-1} \times R$ to the last factor R , or of the opposite sign. We suppose the former without the loss of generality.

Let F be a compact leaf of \mathcal{F} . Triangulate F and take its dual cell decomposition $F = \bigcup_{\lambda \in A} A_\lambda$. By restarting from another finer triangulation of F if necessary, we can suppose that for all $\lambda \in A$ there are a positive number ε_λ and a chart $(U_\lambda, f_\lambda) \in \mathcal{F}$ such that $\varphi(B_\lambda \times [-\varepsilon_\lambda, \varepsilon_\lambda]) \subset U_\lambda$ where $B_\lambda = \bigcup \{A_{\lambda'} | A_{\lambda'} \cap A_\lambda \neq \emptyset\}$. We denote by Ω_λ the subset

$$\varphi(A_\lambda \times [0, \varepsilon_\lambda]) \cap f_\lambda^{*-1}([a_\lambda, b_\lambda])$$

where

$$a_\lambda = \text{Max} \{f_\lambda^* \circ \varphi(x, -\varepsilon_\lambda) | x \in A_\lambda\} \quad \text{and} \quad b_\lambda = \text{Min} \{f_\lambda^* \circ \varphi(x, \varepsilon_\lambda) | x \in A_\lambda\}.$$

Let $S = \{\Omega_\lambda | \lambda \in A\}$ and $m = \#(A) < \infty$. An S chain means a sequence $\omega = (\Omega_0, \dots, \Omega_k)$ such that $\Omega_i \in S$ and $\Omega_i \cap \Omega_{i-1} \neq \emptyset, i = 1, \dots, k$. We call k the length of ω . An Ω_λ plaque means a set of the form $\Omega_\lambda \cap f_\lambda^{-1}(R^{n-1} \times \{y\}), y \in R$. We denote by $\Omega_\lambda/\mathcal{F}$ the set of all Ω_λ plaques. $\Omega_\lambda/\mathcal{F}$ can be identified with an interval by the map $\bar{f}_\lambda: \Omega_\lambda/\mathcal{F} \rightarrow R$ induced from $f_\lambda^*: \Omega_\lambda \rightarrow R$. An Ω_0 plaque P_0 admits an S chain $\omega = (\Omega_0, \dots, \Omega_k)$ if there are Ω_i plaques $P_i, i = 0, \dots, k$, such that $P_{i-1} \cap P_i \neq \emptyset, i = 1, \dots, k$. In this case $\Omega_{i-1} \cap \Omega_i$ is a disk and P_{i-1} determines P_i uniquely by the condition $P_{i-1} \cap P_i \neq \emptyset$. So P_0 determines P_k uniquely. Thus an S chain $\omega = (\Omega_0, \dots, \Omega_k)$ defines a local C^r map

$$G(\omega): \Omega_0/\mathcal{F} \rightarrow \Omega_k/\mathcal{F}$$

whose domain $DG(\omega) \subset \Omega_0/\mathcal{F}$ is the set of all Ω_0 plaques admitting ω .

We call $G(\omega)$ the *geometric holonomy* of ω . If $\omega_1 = (\Omega_0, \dots, \Omega_k)$ and $\omega_2 = (\Omega'_0, \dots, \Omega'_l)$ are S chains and $\Omega_k = \Omega'_0$, let $\omega_1 \# \omega_2 = (\Omega_0, \dots, \Omega_k, \Omega'_1, \dots, \Omega'_l)$. Then $G(\omega_1 \# \omega_2) = G(\omega_2) \circ G(\omega_1)$ and $DG(\omega_1 \# \omega_2) = DG(\omega_1) \cap G(\omega_1)^{-1}(DG(\omega_2))$.

A *homotopy* in S is a sequence of S chains $\omega_0, \dots, \omega_q$ such that ω_i is obtained from ω_{i-1} by inserting an element $\Omega \in S$ or ω_{i-1} from ω_i for $i = 1, \dots, q$. We call ω_0 and ω_q *homotopic*. Note that ω_0 and ω_q are coterminous. In this case $G(\omega_0) = G(\omega_q)$ on $\bigcap_{i=0}^q DG(\omega_i)$.

Choose an element $\Omega_* \in S$. Let $\pi_1(S, \Omega_*)$ be the set of homotopy classes of all S chains $\omega = (\Omega_0, \dots, \Omega_k)$ such that $\Omega_0 = \Omega_k = \Omega_*$. We can induce a group operation on $\pi_1(S, \Omega_*)$ from the operation $\#$.

Let $G_{P_*}^r$ be the group of the germs f_{P_*} at the plaque $P_* = \Omega_* \cap F$ of all orientation-preserving local C^r diffeomorphisms $f: \Omega_*/\mathcal{F} \rightarrow \Omega_*/\mathcal{F}$ satisfying $f(P_*) = P_*$. Note that Ω_*/\mathcal{F} is a compact connected one-dimensional manifold and P_* is an interior point of Ω_*/\mathcal{F} . If ω_1 and ω_2 are homotopic closed S chains at Ω_* , the germs $(G(\omega_1))_{P_*}$ and $(G(\omega_2))_{P_*}$ coincide as already seen. Thus we have a map

$$\bar{G}: \pi_1(S, \Omega_*) \rightarrow G_{P_*}^r$$

which is clearly a homomorphism. Let H_0^r be the group of the germs f_0 at 0 of all orientation-preserving local C^r diffeomorphism $f: R \rightarrow R$ satisfying $f(0) = 0$ and $\iota: \Omega_*/\mathcal{F} \rightarrow R$ a C^r imbedding satisfying $\iota(P_*) = 0$. We define a homomorphism

$$\bar{H}: \pi_1(S, \Omega_*) \rightarrow H_0^r$$

by $H([\omega]) = (\iota \circ G(\omega) \circ \iota^{-1})_0$. The image of the homomorphism H is called the *holonomy group* of F , which is unique up to conjugation. Let H_+^r be the group of the germs f_0 at 0 of all local C^r diffeomorphisms $f: [0, \infty) \rightarrow [0, \infty)$ satisfying $f(0) = 0$ and $r: H_0^r \rightarrow H_+^r$ the homomorphism induced by the restriction. The image of $r \circ \bar{H}$ is called a *one-sided holonomy group* of F and denoted by $\Phi_+(F)$.

Choose a point $x_* \in \Omega_* \cap F$. A path $u: [0, 1] \rightarrow F$ is contained in an S chain $\omega = (\Omega_0, \dots, \Omega_k)$ if there is a subdivision $0 = t_0 < \dots < t_{k+1}$ such that $u([t_i, t_{i+1}]) \subset \Omega_i$, $i = 0, \dots, k$.

LEMMA 1. (1) For all $\alpha \in \pi_1(F, x_*)$, we can find a closed path $u: ([0, 1], \{0, 1\}) \rightarrow (F, x_*)$ representing α which is contained in some S chain $\omega = (\Omega_0, \dots, \Omega_k)$ with $\Omega_0 = \Omega_k = \Omega_*$.

(2) If a path u_i is contained in an S chain ω_i , $i = 1, 2$, then u_1 and u_2 are homotopic relative to $\{0, 1\}$ if and only if ω_1 and ω_2 are homotopic.

The proof of Lemma 1 is easy and we omit it. According to Lemma

1, we have naturally an isomorphism $i: \pi_1(F, x_*) \rightarrow \pi_1(S, \Omega_*)$.

3. The proof of Theorem 1 and 1*. We will prove only Theorem 1 since Theorem 1* can be analogically proved. We consider the case when (1) of Theorem 1 does not occur and suppose that $\mathcal{F} | U_+$ has the only compact leaf F_0 throughout this section. We use the notation of 2.2 with F_0 in the place of F and we may assume that $\varphi(A_\lambda \times [0, \varepsilon_\lambda]) \subset U_+$. Let $\Omega_\lambda^+ = \Omega_\lambda \cap U_+$.

At first we show the following.

LEMMA 2. *The one-sided holonomy group $\Phi_+(F_0)$ of F_0 is a non-trivial free abelian group.*

PROOF. $\Phi_+(F_0)$ is free abelian since $\Phi_+(F_0)$ is abelian by assumption and the group H_+^1 has no torsion elements. We suppose that $\Phi_+(F_0)$ is a trivial group, out of which we will bring a contradiction. Let A_{2m} is the set of all S chains $= (\Omega_0, \Omega_1, \dots, \Omega_k)$ such that $k \leq 2m$ and $\Omega_0 = \Omega_k = \Omega_*$. Recall that $\#(S) = m < \infty$. For all $\omega \in A_{2m}$ we have a neighborhood W_ω of the plaque P_* in Ω_*^+/\mathcal{F} such that $G(\omega) | W_\omega$ is the identity map since the germ at P_* of $G(\omega) | \Omega_*^+/\mathcal{F}$ is the identity. Let

$$W = \bigcap \{W_\omega \mid \omega \in A_{2m}\},$$

then W is a neighborhood of P_* since A_{2m} is a finite set. Take a plaque $P \in \Omega_*^+/\mathcal{F} - \{P_*\}$. Let

$$F = \bigcup \{G(\omega)(P) \mid \omega = (\Omega_0, \dots, \Omega_k), \Omega_0 = \Omega_*, k \leq m\},$$

F is compact since ω runs through a finite set. Let $Q = G(\omega)(P) \subset F$ and $Q^* \in \Omega_{k+1}/\mathcal{F}$ are adjacent, that is $G(\omega_1)(Q) = Q^*$ where ω_1 is the S chain (Ω_k, Ω_{k+1}) . There is an S chain $\omega_2 = (\Omega_0^*, \dots, \Omega_j^*)$ such that $j < m, \Omega_0^* = \Omega_*$ and $\Omega_j^* = \Omega_{k+1}$. Since $\omega \# \omega_1 \# \omega_2^{-1} \in A_{2m}$ where $\omega_2^{-1} = (\Omega_j^*, \dots, \Omega_0^*)$,

$$\begin{aligned} P &= G(\omega \# \omega_1 \# \omega_2^{-1})(P) \\ &= G(\omega_2)^{-1} \circ G(\omega_1) \circ G(\omega)(P) \\ &= G(\omega_2)^{-1}(Q^*) \end{aligned}$$

and then $Q^* = G(\omega_2)(P) \subset F$. Therefore F is a leaf of $\mathcal{F} | U_+$. Furthermore $F \cap \Omega$ is an Ω plaque for all $\Omega \in S$ and F is C^r diffeomorphic to F_0 . This is a contradiction to the assumption that $\mathcal{F} | U_+$ has no compact leaf except F_0 . This completes the proof of Lemma 2.

We need the theorem of Kopell.

THEOREM (Kopell [4]). *Let $f: [0, a_1] \rightarrow [0, b_1]$ and $g: [0, a_2] \rightarrow [0, b_2]$ be orientation-preserving C^2 diffeomorphisms. Suppose that there is $0 < a < \text{Min}\{a_1, a_2\}$ satisfying the following.*

- (1) There is $t_0 \in (0, a)$ such that $f(t_0) = t_0$.
- (2) $g(t) < t$ for all $t \in (0, a)$.
- (3) $f \circ g$ and $g \circ f$ can be defined on $[0, a)$ and coincide there.

Then $f|_{[0, a)}$ is the identity map.

By using this theorem we prove the following lemma, which shows some character of the elements of abelian holonomy groups.

LEMMA 3. For all closed S chain $\omega_0 = (\Omega_0, \dots, \Omega_k)$ at Ω_* , that is $\Omega_0 = \Omega_k = \Omega_*$, such that the germ $(G(\omega_0) | \Omega_*^+/\mathcal{S})_{P_*}$ is not the identity, there is a neighborhood W_{ω_0} of P_* in Ω_*^+/\mathcal{S} such that $G(\omega_0) | W_{\omega_0}$ has no fixed point except P_* .

PROOF. Let $\iota: \Omega_*^+/\mathcal{S} \rightarrow [0, a)$ be a C^r diffeomorphism and let $H(\omega) = \iota \circ G(\omega) \circ \iota^{-1}$ for all closed S chains ω at Ω_* . Since the group $\Phi_+(F_0)$ is abelian and A_{2m} , defined in the proof of Lemma 2, is a finite set, there is a positive number ε such that

- (1) $[0, \varepsilon] \subset (DG(\omega))$ for all $\omega \in A_{2m} \cup \{\omega_0\}$;
- (2) $H(\omega) \circ H(\omega_0)$ and $H(\omega_0) \circ H(\omega)$ are defined on $[0, \varepsilon]$ and coincide there for all $\omega \in A_{2m}$.

We are going to show that $\iota^{-1}([0, \varepsilon])$ has the property of W_{ω_0} in Lemma 3. We suppose that there is $t_0 \in (0, \varepsilon)$ satisfying that $H(\omega_0)(t_0) = t_0$, out of which we will bring a contradiction.

We may suppose that there is t_1 such that $0 < t_1 < t_0$ and $f(t) \neq t$ for all $t \in (t_1, t_0)$, without loss of generality. In fact let

$$K = \{t \in (0, t_0) \mid H(\omega_0)(t) \neq t\},$$

then K is a non-empty open subset since the germ at 0 of $H(\omega_0)$ is not the identity. Let (b, c) be a connected component of K . Then $H(\omega_0)(b) = b$ and $H(\omega_0)(c) = c$. It is sufficient to take c in place of t_0 and let $t_1 = b$.

Now we see that there is a closed S chain $\omega_1 \in A_{2m}$ such that $H(\omega_1)(t_0) \neq t_0$. In fact if not, we can show that

$$F = \bigcup \{G(\omega_0)(P_0) \mid \omega = (\Omega_0, \dots, \Omega_k), \Omega_0 = \Omega_*, k \leq m\}$$

is a compact leaf, where $P_0 = \iota^{-1}(t_0)$, by the same argument as in the proof of Lemma 2. This is a contradiction.

We may suppose that $H(\omega_1)(t_0) < t_0$ by taking ω_1^{-1} in place of ω_1 if necessary. Let $f = H(\omega_0)$ and $g = H(\omega_1)$. Since $f([0, t_0]) = [0, t_0]$ and $g([0, t_0]) = [0, t_0]$, $f^{\mu_1} \circ g^{\nu_1} \circ \dots \circ f^{\mu_j} \circ g^{\nu_j}(t) \in [0, t_0]$ if $t \in [0, t_0]$, $\mu_i \geq 0$ and $\nu_i \geq 0$. Then

$$f^{\mu_1} \circ g^{\nu_1} \circ \dots \circ f^{\mu_j} \circ g^{\nu_j}(t) = f^{\mu_1 + \dots + \mu_j} \circ g^{\nu_1 + \dots + \nu_j}(t)$$

since f and g are commutative on $[0, t_0]$. Let $t_2 = \lim_{n \rightarrow \infty} g^n(t_0)$, which

exists because the sequence $t_0, g(t_0), \dots$ is monotonely decreasing and bounded. Then

$$g(t_2) = \lim_{n \rightarrow \infty} g(g^n(t_0)) = \lim_{n \rightarrow \infty} g^{n+1}(t_0) = t_2 .$$

$$f(t_2) = \lim_{n \rightarrow \infty} f(g^n(t_0)) = \lim_{n \rightarrow \infty} g^n(f(t_0)) = \lim_{n \rightarrow \infty} g^n(t_0) = t_2 .$$

Since $g(t) < t$ for all $t \in (t_2, t_0]$, there is t_3 such that $t_0 < t_3 < \varepsilon$ and $g| [t_2, t_3]$ is a contraction. Note that $f(t_0) = t_0$ and f and g are C^2 and commutative on $[t_2, t_3] \subset [0, \varepsilon]$. By applying the theorem of Kopell to f and g in the interval $[t_2, t_3]$, we see that $f| [t_2, t_3]$ is the identity map. This is a contradiction, which completes the proof of Lemma 3.

Now we need the well-known theorem of Denjoy-Siegel.

THEOREM (Denjoy [1], Siegel [11]). *Let $f: S^1 \rightarrow S^1$ be a C^2 diffeomorphism. Then only one of the following occurs.*

- (1) $\{f^\nu(x) \mid \nu: \text{positive integer}\}$ is dense in S^1 for all $x \in S^1$.
- (2) There are $x \in S^1$ and a positive integer ν such that $f^\nu(x) = x$.

By using this theorem we show the following.

LEMMA 4. *Let ω_1 and ω_2 be closed S chains at Ω_* such that the germs at P_* of $G(\omega_1)$ and $G(\omega_2)$ are not the identity. Then only one of the following occurs.*

- (1) There is a neighborhood V of F_0 such that all leaves of $\mathcal{F} \mid V \cap U_+$ except F_0 are dense in $V \cap U_+$.
- (2) There are integers μ and ν such that the germs at P_* of $G(\omega_1)^\mu$ and $G(\omega_2)^\nu$ coincide.

PROOF. Let $f = H(\omega_1)$ and $g = H(\omega_2)$. By Lemma 3, there is $\varepsilon > 0$ such that

- (1) $f| [0, \varepsilon]$ and $g| [0, \varepsilon]$ have no fixed points except 0.
- (2) $f \circ g$ and $g \circ f$ are defined on $[0, \varepsilon]$ and coincide there.

It is sufficient to consider the case where f and g are contractions. Since $f([0, \varepsilon]) \subset [0, \varepsilon)$ and $g([0, \varepsilon]) \subset [0, \varepsilon)$, $f^{\mu_1} \circ g^{\nu_1} \circ \dots \circ f^{\mu_j} \circ g^{\nu_j}$ are defined on $[0, \varepsilon]$ if $\mu_i \geq 0$ and $\nu_i \geq 0$, and coincide there with $f^{\mu_1 + \dots + \mu_j} \circ g^{\nu_1 + \dots + \nu_j}$.

Consider the equivalence relation \sim on $(0, \varepsilon)$ defined by $t \sim f(t)$, $t \in (0, \varepsilon)$. The quotient space $(0, \varepsilon)/\sim$ is a C^r manifold and C^r diffeomorphic to S^1 . We denote by $[t]$ the equivalence class of $t \in (0, \varepsilon)$. If $t_1 \sim t_2$, then $g(t_1) \sim g(t_2)$ because of the commutativity of f and g . Thus we have a C^r diffeomorphism

$$g_*: (0, \varepsilon)/\sim \rightarrow (0, \varepsilon)/\sim$$

such that $g_*([t]) = [g(t)]$.

By the theorem of Denjoy-Siegel, only one of the following occurs.

- (1) $\{g^{\nu}_*(x) \mid \nu \geq 0\}$ is dense for all $x \in (0, \varepsilon)/\sim$.
- (2) There are $x \in (0, \varepsilon)/\sim$ and a positive number ν such that $g^{\nu}_*(x) = x$.

At first we consider the case (1). Let V be a neighborhood of F_0 such that $V \cap U_+$ consists of the leaves of $\mathcal{F} \mid U_+$ which contains a plaque $\in W = \iota^{-1}([0, \varepsilon]) \subset \Omega_*/\mathcal{F}$. We are going to show all leaves of $\mathcal{F} \mid V \cap U_+$ except F_0 are dense in $V \cap U_+$. Let $P = \iota^{-1}(t)$, $t \in (0, \varepsilon)$ and F be a leaf of $\mathcal{F} \mid V \cap U_+$ which contains P . Let $K = \{Q \in W \mid Q \subset F\}$. It is sufficient to show that K is dense in W . Let W^* be an arbitrary open set of W . There is a positive integer ν such that $g^{\nu}(t) \sim t^* \in \iota(W^*)$. There is an integer μ such that $t^* = f^{\mu} \circ g^{\nu}(t)$ by the definition of \sim . Then

$$\begin{aligned}
 P^* &= \iota^{-1}(t^*) \\
 &= G(\omega_1)^{\mu} \circ G(\omega_2)^{\nu}(P) \\
 &= G(\omega_1^{\mu} \# \omega_2^{\nu})(P)
 \end{aligned}$$

and $P^* \subset F$. Therefore $P^* \in K \cap W^*$, which implies that K is dense in W .

Secondly we consider the case (2). Let $x = [t]$. Since f and g are contractions and $g^{\nu}(t) \sim t$, there is a positive integer μ such that $g^{\nu}(t) = f^{\mu}(t)$. Note that $f^{-\mu} \circ g^{\nu}$ is defined at least on $[0, t]$. Since $g^{\nu}(f^k(t)) = f^k(g^{\nu}(t)) = f^k(f^{\mu}(t)) = f^{\mu}(f^k(t))$, there is a sequence $t, f(t), f^2(t), \dots$ of fixed points of $f^{-\mu} \circ g^{\nu}$ which converges to 0. Therefore by Lemma 3, the germ at P_* of $G(\omega_1^{-\mu} \# \omega_2^{\nu})$ is the identity. This completes the proof of Lemma 4.

If the one-sided holonomy group $\Phi_+(F_0)$ is a free abelian group of rank ≥ 2 , there are $G(\omega_1)_{P_*}$ and $G(\omega_2)_{P_*}$ which do not admit (2) of Lemma 4. Then (2) of Theorem 1 occurs.

The rest is the case where (1) of Theorem 1 does not occur and $\Phi_+(F_0)$ is an infinite cyclic group. Let $j: \Phi_+(F_0) \rightarrow Z$ be the isomorphism which maps contractions to positive integers. Since $\Phi_+(F_0)$ is abelian, there is an epimorphism $\alpha: H_1(F_0, Z) \rightarrow \Phi_+(F_0)$ such that $\alpha \circ h = r \circ \bar{H} \circ i$ where $h: \pi_1(F_0, x_*) \rightarrow H_1(F_0, Z)$ is the Hurewicz homomorphism. Thus we have a diagram

$$\begin{array}{ccccccc}
 \pi_1(F_0, x_*) & \xrightarrow[\cong]{i} & \pi_1(S, \Omega_*) & \xrightarrow{r \cdot \bar{H}} & \Phi_+(F_0) & \xrightarrow[\cong]{j} & Z \\
 \downarrow h & & & \nearrow \alpha & & & \\
 H_1(F_0, Z) & & & & & &
 \end{array}$$

In the exact sequence

$$0 \rightarrow \text{Ext}(H_0(F_0, Z), Z) \rightarrow H^1(F_0, Z) \xrightarrow{\beta} \text{Hom}(H_1(F_0, Z), Z) \rightarrow 0$$

β is an isomorphism since $H_0(F_0, Z)$ is free and $\text{Ext}(H_0(F_0, Z), Z)$ is trivial. Let $\theta = d \circ \beta^{-1}(j \circ \alpha) \in H_{n-2}(F_0, Z)$ where $d: H^1(F_0, Z) \rightarrow H_{n-2}(F_0, Z)$ is the Poincaré duality isomorphism.

Now we need the theorem of Nakatsuka which is on the line of Thom's representation theorem of codimension-one integral homology classes [12].

THEOREM (Nakatsuka [6]). *Let M be a compact connected orientable manifold of dimension $n \geq 3$ and $\theta \in H_{n-1}(M, Z)$. Then there is a connected orientable $(n - 1)$ -submanifold $N \subset M$ such that $\theta = [N]$ if and only if there is a homology class $\alpha \in H_1(M, Z)$ such that the intersection number $\theta \cdot \alpha$ is 1.*

Since $j \circ \alpha \in \text{Hom}(H_1(F_0, Z), Z)$ is an epimorphism, θ satisfies the condition of the theorem of Nakatsuka. Thus we have a connected oriented submanifold $N \subset F_0$ such that $\theta = [N] \in H_{n-2}(F_0, Z)$.

Retriangulate F_0 so that N is a subcomplex. We use the same notation as 2.2. Let $S' = \{\Omega_\lambda \mid \Omega_\lambda \cap N \neq \emptyset\}$ and $E = \bigcup \{A_\lambda \mid \Omega_\lambda \in S'\}$. Clearly E is homotopy equivalent to N . We may assume that $\Omega_* \in S'$ and $x_* \in N$. Let $\omega = (\Omega_0, \dots, \Omega_k)$ be a closed S' chain at Ω_* , that is $\Omega_0 = \Omega_k = \Omega_*$ and all $\Omega_i \in S'$. There is a closed path $c: ([0, 1], \{0, 1\}) \rightarrow (N, x_*)$ contained by ω . Since $j \circ \alpha \circ h([c])$ is the intersection number of $h([c])$ and θ in F_0 and N has a trivial normal bundle in F_0 , $j \circ \alpha \circ h([c]) = 0$. Then

$$\begin{aligned} H(\omega)_0 &= r \circ \bar{H} \circ i([c]) \\ &= \alpha \circ h([c]) \\ &= j^{-1} \circ j \circ \alpha \circ h([c]) \\ &= j^{-1}(0) \\ &= \text{the identity.} \end{aligned}$$

Let A'_{2m} be the set of all closed S' chains at Ω_* of length $\leq 2m$. For each $\omega \in A'_{2m}$, there is a neighborhood D_ω of P_* in Ω_*^*/\mathcal{F} such that $G(\omega) \mid D_\omega$ is the identity. Let $D = \bigcap_{\omega \in A'_{2m}} D_\omega$. By the same argument as the proof of Lemma 2, we can show that $F = \bigcup_{\omega \in J} G(\omega)(P)$ is a compact leaf which is diffeomorphic to E for each $P \in D$ where J is the set of all S' chains $\omega = (\Omega_0, \dots, \Omega_k)$ satisfying that $k \leq m$ and $\Omega_0 = \Omega_k$. Furthermore there is a C^∞ diffeomorphism $\xi: E \times [0, 1] \rightarrow K = \bigcup_{\omega \in J} G(\omega)(D)$ such that

- (1) $\xi \mid E \times \{0\}$ is the identity,
- (2) $\xi(\{x\} \times [0, 1]) \subset \varphi(\{x\} \times R)$ for all $x \in E$,

(3) $\xi(E \times \{t\})$ is a leaf of $\mathcal{F} \mid K$ for all $t \in [0, 1]$.

Since for all closed path c in $F_0 - \text{Int}(E)$ the intersection number of c and θ is 0, there is a C^r imbedding $\eta: (F_0 - \text{Int}(E)) \times [0, 1] \rightarrow U_+$ such that

- (1) $\eta \mid (F_0 - \text{Int}(E)) \times \{0\}$ is the identity,
- (2) $\eta(\{x\} \times [0, 1]) \subset \mathcal{P}(\{x\} \times R)$ for all $x \in F_0 - \text{Int}(E)$,
- (3) $\eta((F_0 - \text{Int}(E)) \times \{t\})$ is a leaf of U_+ for all $t \in [0, 1]$.

By using ξ , we can extend η to a C^r imbedding $\bar{\eta}: F_* \rightarrow U_+$ satisfying (1), (2) and (3) of η when $F_0 - \text{Int}(E)$ replaced by F_* , where F_* is defined in (3) of Theorem 1. We denote by $x_i \in N_i$ ($i = 1, 2$) the copies of $x \in N$. We may assume that $\bar{\eta}(N_1 \times [0, 1]) \subset \bar{\eta}(N_2 \times [0, 1])$. We define $f_x(t) \in [0, 1]$ by $\bar{\eta}(x_1, t) = \bar{\eta}(x_2, f_x(t))$ for all $t \in [0, 1]$. f_x does not depend on $x \in N$. Let $f = f_x$. It is clear that f is a C^r contraction. This completes the proof of Theorem 1.

4. The proof of Theorem 2 and examples. At first we give a method to construct a C^r foliation \mathcal{F} on $S_g \times [0, 1]$, whose leaves are all C^∞ submanifold and transverse to $\{x\} \times [0, 1]$ for all $x \in S_g$, from given orientation-preserving C^r diffeomorphisms $f_1, \dots, f_g: [0, 1] \rightarrow [0, 1]$, $0 \leq r \leq \infty$.

Take disjoint circles C_1, \dots, C_g in S_g such that $S_g - (C_1 \cup \dots \cup C_g)$ is connected. Let U_1, \dots, U_g be disjoint closed tubular neighborhoods of C_1, \dots, C_g . There are C^∞ diffeomorphisms $h_i: C_i \times [-1, 1] \rightarrow U_i$, $i = 1, \dots, g$. Let $\alpha: [-1, 1] \rightarrow [0, 1]$ be a C^∞ map such that

- (1) $\alpha(t) = 0$ in a neighborhood of -1 ,
- (2) $\alpha(t) = 1$ in a neighborhood of 1 .

Let \mathcal{F}_i be the foliation on $U_i \times [0, 1]$ whose leaves are

$$\{(h(x, s), t) \mid x \in C_i, s \in [-1, 1], t = \alpha(s)t_0 + (1 - \alpha(s))f_i(t_0)\},$$

$t_0 \in [-1, 1]$. Let \mathcal{F}_0 be the foliation on $(S_g - \bigcup_{i=1}^g \text{Int}(U_i)) \times [0, 1]$ whose leaves are $(S_g - \bigcup_{i=1}^g \text{Int}(U_i)) \times \{t\}$, $t \in [0, 1]$. By connecting $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_g$ together, we have a foliation \mathcal{F} on $S_g \times [0, 1]$ which clearly has the desired property. Furthermore $\text{Image}(\Phi)$ is the subgroup of $\text{Diff}_0^r([0, 1])$ generated by f_1, \dots, f_g .

PROOF OF THEOREM 2. Choose numbers $a_1, \dots, a_g, b_1, \dots, b_g$ such that

$$0 = a_g < a_{g-1} < \dots < a_2 < a_1 < b_1 < b_2 < \dots < b_g = 1.$$

By induction we construct homeomorphisms $f_1, \dots, f_g: [0, 1] \rightarrow [0, 1]$ as follows.

(1) Take $f_1 \mid [a_1, b_1]$ such that

$$f_1(a_1) = a_1, f_1(b_1) = b_1, f_1(t) < t \quad \text{for all } t \in (a_1, b_1).$$

(2) Suppose that f_1, \dots, f_k are already defined on $[a_k, b_k]$. Take $f_{k+1} | [a_{k+1}, b_{k+1}]$ such that

$$f_{k+1}(a_{k+1}) = a_{k+1}, f_{k+1}(b_{k+1}) = b_{k+1}, f_{k+1}(b_k) = a_k, f_{k+1}(t) < t$$

for all $t \in (a_{k+1}, b_{k+1})$.

For each $t \in (a_{k+1}, b_{k+1}) - [a_k, b_k]$, there is the unique integer ν satisfying $f_{k+1}^\nu(t) \in (a_k, b_k]$. Let $f_i(t) = f_{k+1}^{-\nu} \circ f_i \circ f_{k+1}^\nu(t)$ and $f_i(a_{k+1}) = a_{k+1}, f_i(b_{k+1}) = b_{k+1}$ for $i = 1, \dots, k$.

Note that f_1, \dots, f_g are mutually commutative.

Let \mathcal{F} be the foliation on $S_g \times [0, 1]$ obtained by using f_1, \dots, f_g and F_t the leaf containing $(S_g - \bigcup_{i=1}^g \text{Int}(U_i)) \times \{t\}$. Image (Φ) is a free abelian group of rank g . By considering the action of f_1, \dots, f_g , we see that

(1) $F_{a_g} = S_g \times \{0\}, F_{b_g} = S_g \times \{1\}, F_{a_{g-1}} = F_{b_{g-1}}, \dots, F_{a_1} = F_{b_1}$, and $F_t, t \in (a_1, b_1)$, are all leaves of \mathcal{F} .

(2) $\bar{F}_{a_{g-1}} = F_{a_{g-1}} \cup F_{a_g} \cup F_{b_g},$

(3) $\bar{F}_{a_{g-2}} = F_{a_{g-2}} \cup F_{a_{g-1}} \cup F_{a_g} \cup F_{b_g},$

.....

(g) $\bar{F}_{a_1} = F_{a_1} \cup F_{a_2} \cup \dots \cup F_{a_{g-1}} \cup F_{a_g} \cup F_{b_g},$

(g + 1) $\bar{F}_t = F_t \cup \bar{F}_{a_1}$ for all $t \in (a_1, b_1)$,

(g + 2) F_t is non-compact and proper for all $t \in (0, 1)$.

This completes the proof of Theorem 2.

REMARK. Although $f_i | [a_i, b_i]$ can be taken C^∞ , the differentiability may be broken at a_{i+1} and b_{i+1} .

EXAMPLE 1.*) Let the genus $g = 2$. Let $f_1, f_2: [0, 1] \rightarrow [0, 1]$ be C^∞ diffeomorphisms such that

$$f_1(t) = t/2 \quad \text{and} \quad f_2(t) = \left(\frac{1}{2}\right)^\alpha t \quad \text{for all } t \in [0, 1/2]$$

where α is a positive irrational number $\in (0, 1)$. Let \mathcal{F} be the foliation on $S_2 \times [0, 1]$ obtained by using f_1 and f_2 . In a neighborhood of the compact leaf $S_g \times \{0\}$, (2) of Theorem 1 occurs.

REMARK. This foliation may be not an example of (1) of Theorem 1*. For the case (1) of Theorem 1* we need $f \in \mathcal{S}$ such that $\mathcal{E}(f)$ is isomorphic to R^1 where the notations \mathcal{S} and $\mathcal{E}(f)$ are in Lemma 3 of Kopell [4]. We do not know whether such f exists or not.

EXAMPLE 2 (Sacksteder [9]). There are orientation-preserving C^∞ diffeomorphisms $f_1, f_2: [0, 1] \rightarrow [0, 1]$ such that the C^∞ foliation on $S_2 \times [0, 1]$

*) The author thanks Professor Y. Saito for his correction.

obtained by using f_1 and f_2 has an exceptional leaf F (that is, neither locally dense nor proper). Of course f_1 and f_2 are not commutative and $\text{Image}(\Phi)$ is a non-commutative group.

EXAMPLE 3. Denjoy [1] constructed a C^1 diffeomorphism $g: S^1 \rightarrow S^1$ such that

- (1) $\{g^\nu(x) \mid \nu \in Z\}$ is neither locally dense nor compact for all $x \in S^1$,
- (2) $x \in \bigcap_{i=1}^{\infty} \overline{\{g^\nu(x) \mid |\nu| \geq n\}}$ for some $x \in S^1$,
- (3) $x \notin \bigcap_{i=1}^{\infty} \overline{\{g^\nu(x) \mid |\nu| \geq n\}}$ for some $x \in S^1$.

Let $p: R \rightarrow S^1$ be the universal covering and $f: R \rightarrow R$ a generator of the covering transformation group of $p: R \rightarrow S^1$. Since $(g \circ p)_* \pi_1(R) = \{1\} \subset p_* \pi_1(R) = \{1\}$, there is a C^1 diffeomorphism $\tilde{g}: R \rightarrow R$ such that $p \circ \tilde{g} = g \circ p$ and $\tilde{g}(0) = \text{Min} \{t \in p^{-1}(g \circ p(0)) \mid 0 < t\}$.

Then $\tilde{g}(s) = \text{Min} \{t \in p^{-1}(g \circ p(s)) \mid s < t\}$ for all $s \in R$. Really let $A = \{s \in R \mid \tilde{g}(s) = \text{Min} \{t \in p^{-1}(g \circ p(s)) \mid s < t\}\}$. Let $s \in R$, then there is a connected neighborhood U of $p(s)$ satisfying $U \cap g(U) = \emptyset$. Let \tilde{U} be the connected component, of $p^{-1}(U)$, containing s . Then $\tilde{g}(\tilde{U})$ is the connected component, of $p^{-1}(g(U))$, containing $\tilde{g}(s)$. If $s \in A$, then $[s, \tilde{g}(s)] \cap (p^{-1}(g(U)) - \tilde{g}(\tilde{U})) = \emptyset$ and $\tilde{U} \subset A$. Therefore A is open. If $s \in R - A$, then $[s, g(s)]$ contains a connected component of $p^{-1}(g(U)) - \tilde{g}(\tilde{U})$ and $\tilde{U} \subset R - A$. Therefore A is closed. Since $0 \in A$, A is a non-empty closed open subset of R . Therefore $A = R$.

Furthermore $f \circ \tilde{g} = \tilde{g} \circ f$. In fact suppose that $f \circ \tilde{g}(s) < \tilde{g} \circ f(s)$ for some $s \in R$. Since $s < \tilde{g}(s)$ and f preserves the relation $<$, $f(s) < f \circ \tilde{g}(s) < \tilde{g}(f(s))$. Since

$$p \circ f \circ \tilde{g}(s) = p \circ \tilde{g}(s) = g \circ p(s) = g \circ p \circ f(s), \quad f \circ \tilde{g}(s) \in p^{-1}(g \circ p(f(s))) .$$

This contradicts to the fact that $\tilde{g}(f(s)) = \text{Min} \{t \in p^{-1}(g \circ p(f(s))) \mid f(s) < t\}$. Suppose that $f \circ \tilde{g}(s) > \tilde{g} \circ f(s)$ for some $s \in R$. Since $f(s) < \tilde{g} \circ f(s)$, $s < f^{-1} \circ \tilde{g} \circ f(s) < f^{-1} \circ f \circ \tilde{g}(s) = \tilde{g}(s)$. Since $f^{-1} \circ \tilde{g} \circ f(s) \in p^{-1}(g \circ p(s))$, this is a contradiction.

Choose a homeomorphism $\alpha: (0, 1) \rightarrow R$ and let $f_1, f_2: [0, 1] \rightarrow [0, 1]$ be the homeomorphisms defined by

- (1) $f_1(0) = f_2(0) = 0, f_1(1) = f_2(1) = 1,$
- (2) $f_1(s) = \alpha^{-1} \circ f \circ \alpha(s), f_2(s) = \alpha^{-1} \circ \tilde{g} \circ \alpha(s)$ for all $s \in (0, 1)$.

The topological foliation on $S_2 \times [0, 1]$, obtained by using f_1 and f_2 , has exceptional leaves. $\text{Image}(\Phi) \cong Z \oplus Z$.

REFERENCES

[1] A. DENJOY, Sur les courbes définies par les equations différentielles à la surface du tore, J. Math. Pures Appl. 9 (1932), 333-375.

- [2] A. HAEFLIGER, Variétés feuilletées, *Ann. Scuola Norm. Sup. Pisa* 16 (1962), 367-397.
- [3] M. HIRSCH, Stability of compact leaves of foliations, *Proceedings of the International Conference on Dynamical Systems, Salvador, Brazil, (1971)*, 135-153.
- [4] N. KOPELL, Commuting diffeomorphisms, *Proc. Symp. Pure Math. Vol. XIV, A.M.S.*, 1970.
- [5] B. LAWSON, Foliations, *Bull. A.M.S.* 80 (1974), 369-418.
- [6] H. NAKATSUKA, On representations of homology classes, *Proc. Japan Acad.* 48 (1972), 360-364.
- [7] G. REEB, Sur certaines propriétés topologiques des variétés feuilletées, *Actual. Sci. Ind. No. 1183, Hermann, Paris*, 1952.
- [8] G. REEB, Sur les structures feuilletées de codimension un et sur un théorème de M. A. Denjoy, *Ann. Inst. Fourier*, 11 (1961), 185-200.
- [9] R. SACKSTEDER, On the existence of exceptional leaves in foliations of codimension-one, *Ann. Inst. Fourier* 14 (1964), 221-226.
- [10] L. C. SIEBENMANN, Deformation of homeomorphisms on stratified sets, I, II, *Comment. Math. Helv.* 47 (1972), 123-163.
- [11] C. L. SIEGEL, Note on differential equations on the torus, *Ann. of Math.* 46 (1945), 423-428.
- [12] R. THOM, Quelques propriétés globales des variétés différentielles, *Comm. Math. Helv.* 28 (1954), 17-86.

MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN