Compact metric spaces have binary bases

by

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Abstract. O'Connor has shown that every compact metric space is supercompact. However his proof is valid only for spaces dense-in-itself.

This result is strengthened here, namely by proving that every compact metric space has a binary base.

A family $\mathcal{R}$ of subsets of a topological space is said to be *binary* (see [7]) if every subfamily $\mathcal{R}'$ of $\mathcal{R}$ such that $\bigcap\{\mathrm{cl}A\colon A \in \mathcal{R}'\} = \emptyset$ contains two elements with disjoint closures.

A space X is said to be *supercompact* (see de Groot [2]) if there exists an open subbase $\mathcal{S}$, called *supersubbase*, such that in every cover of X by means of elements of $\mathcal{S}$ there exist two elements which cover X.

It is clear that if X is a compact Hausdorff space and $\mathcal{B}$ is a binary base of open sets in X, then the family $\mathcal{S}$ consisting of sets of the form $X\setminus\mathrm{cl}U$, where $U \in \mathcal{B}$, is a supersubbase on X.

In the paper [4] O'Connor proved that every compact metric space is supercompact; his proof consists on construction of a special embedding of a given compact metric space into Hilbert cube. However his proof is valid only in the case when the space has no isolated points. In fact, the assertion ([4], p. 32) that the points $T_nD_n \ldots T_1D_1(a)$ and $T_nD_n \ldots \ldots T_1D_1(b)$ lie on opposite sides of π_n for $a, b \in M$, whenever $D_n \ldots T_1D_1(a)$ and $D_n \ldots T_1D_1(b)$ lie, does not follow from Lemma 2, because that lemma assures this only for $a, b \in K$, where K is a dense-in-itself subset of a given uncountable compact metric space M.

In this paper we prove a theorem (Theorem 2) which asserts that every compact metric space has a binary base. Clearly, this result contains O'Connor's one. Our proof is based on the Freudenthal's theorem on inverse expansions [1].

The question of the existence of supersubbases or binary bases for arbitrary compact Hausdorff spaces are still open. The first of these questions was raised by de Groot [2].

§ 1. Natural projections, pseudopolyhedra and non-tangent sets. By a *polyhedron* we mean a compact Euclidean polyhedron. The symbol $|P|$

denotes a polyhedron with a triangulation P. If we say about a triangulation of a simplex, then we mean on the standard triangulation consisting of all faces of that simplex.

Let $x \in |P|$. Then we define the *carrier* and the *star* of x:

$$\mathrm{car}_P x = \bigcap \{s \in P\colon\ x \in s\},$$

$$\mathrm{st}_P x = \bigcup \{s \in P\colon\ x \in s\} \setminus \bigcup \{s \in P\colon\ x \notin s\}.$$

Clearly, the carrier of x is a simplex and x belongs to the geometrical interior of it.

The following definition and lemma are taken from Rogers's paper [6].

If $|P|$ is a polyhedron, a *simple subdivision* of P is a complex P' whose vertices consist of just one point p_s from the geometrical interior of each simplex s of P, such that the simplex determined by a set V of vertices of P' belongs to P' if and only if there is a sequence $s_0, \ldots, s_i$ of simplexes of P, each except the last is a face of the next, such that $V = \{p_{s_0}, \ldots, p_{s_i}\}$. If P is of dimension n and k is a positive integer, then P' is said to be of *order* k if the barycentric coordinate of p_s on each vertex of s is not smaller than $(n+1)^{-k}$ for each s of P.

LEMMA 1. *If P' is a simple subdivision of the n-dimensional complex P of order k, then* $\mathrm{mesh}\,P' \leqslant (1-(n+1)^{-k})\mathrm{mesh}\,P$.

Let S be a simplex. If p is a vertex of S, then $|S(p)|$ denotes the opposite to p face of S. Let q be a vertex from $S(p)$. Then by a *natural projection* $\varphi_q\colon |S| \to |S(p)|$ we mean a linear map which identifies the vertex p with the vertex q and is identity on $|S(p)|$.

Remark 1. Each linear onto map between two simplexes can be represented as a composition of natural projections.

Let $|P|$ be a polyhedron and let Q be contained in P. Then $\bigcup \{\mathrm{intgeom}\, s\colon\ s \in Q\}$ is called a *pseudopolyhedron*.

Using the Lefschetz's construction ([3], Ch. 8, § 1, (5.2)) we obtain following two lemmas.

LEMMA 2. *If $|P_1|, \ldots, |P_k|$ are polyhedra contained in a polyhedron $|P|$ (there is no dependence between triangulations $P_1, \ldots, P_k$ and the triangulation P), then there exists a subdivision P' of P which induces subdivisions on each P_i.*

LEMMA 3. *The union and the intersection of two pseudopolyhedra (polyhedra) is also a pseudopolyhedron (polyhedron). The prism over pseudopolyhedron (polyhedron) is a pseudopolyhedron (polyhedron).*

The following corollary is a consequence of Remark 1 and Lemma 3.

COROLLARY 1. *The counterimage of pseudopolyhedron (polyhedron) under a simplicial map is a pseudopolyhedron (polyhedron).*

A subset M of a polyhedron $|P|$ is said to be *non-tangent in* P if $\mathrm{cl}(s \cap M) = s \cap \mathrm{cl}\, M$ for each s of P, or equivalently, if $x \in \mathrm{cl}\, M$, then $x \in \mathrm{cl}(s \cap M)$, where s is the carrier of x. A family consisting of non-tangent in P sets will be called *non-tangent family*.

LEMMA 4. *Let π be a simplicial map of a polyhedron $|P|$ onto a polyhedron $|P'|$ and let M be non-tangent in P'. Then $\pi^{-1}(M)$ is non-tangent in P and $\pi^{-1}(\mathrm{cl}\, M) = \mathrm{cl}\,\pi^{-1}(M)$.*

Proof. Note, that the both conclusions follow from the following implication:

$$x \in \pi^{-1}(\mathrm{cl} M) \Rightarrow x \in \mathrm{cl}\big(\pi^{-1}(M) \cap s\big) \quad \text{where} \quad s = \mathrm{car}_P x.$$

To prove this implication let x belongs to $\pi^{-1}(\mathrm{cl}\, M)$, let s be the carrier of x and let s' be the carrier of $\pi(x)$. Since M is non-tangent in P', hence $\pi(x)$ belongs to $\mathrm{cl}(M \cap s')$. Then there exists a sequence $\{y_n\colon n = 1, 2, \ldots\}$ of points of $M \cap s'$ converging to $\pi(x)$. We claim that there exists a sequence $\{x_n\colon n = 1, 2, \ldots\}$ of points of $\pi^{-1}(M) \cap s$ converging to x. Consider π as a map from s onto s', which is sufficient for further considerations; so we can assume that s' is a face of s.

1. If $\dim s = \dim s'$, then let $x_n = y_n$.

2. If $\dim s = \dim s' + 1$, then we can assume that π is a natural projection which identifies vertices p and q, where q belongs to s'. Let H be the $(\dim s)$-plane which contains the point x and the face of s' opposite to q. Then let x_n be a (unique) point of H such that $\pi(x_n) = y_n$.

Passing to the general situation, the proof reduces in view of Remark 1 to the cases 1 and 2.

Now we infer that $x \in \mathrm{cl}\big(\pi^{-1}(M) \cap s\big)$.

§ 2. Construction of some special binary bases on simplexes. Let $|S|$ be a simplex and let p and q be different vertices of S. A symbol s_q^p denotes the one face of S which contains p and q. If a point x belongs to s_q^p, then H^x denotes the minimal hyperplane in $|S|$ passing through x and $|S(p)(q)|$. If a point x belongs to s_q^p, and $p \neq x \neq q$, then H_p^x denotes the intersection of $|S|$ with the open half-space determined by H^x to which p belongs (do the same with q). If x and y are different points in s_q^p such that $p \neq x \neq q$ and $p \neq y \neq q$, then H_y^x denotes the intersection of H_p^y and H_q^x. The sets H_p^y, H_q^x will be called *strata* of $|S|$ (with respect to p and q). We shall omit indices in the symbols H_r^z if misunderstanding is excluded.

Let $\varphi_q\colon |S| \to |S(p)|$ be the natural projection and let $\mathfrak{R}$ be an arbitrary family of subsets of $|S(p)|$.

By a *lift* of the family $\mathcal{R}$ to $|S|$ by means of φ_q we mean the family of subsets of $|S|$ of the form:

1. $\varphi_q^{-1}(A) \cap H$, if A is disjoint with $|S(p)(q)|$,
2. $\varphi_q^{-1}(A)$, if $A \cap |S(p)(q)| \neq \emptyset$,

A being member of $\mathcal{R}$, H being a stratum.

LEMMA 5. *If A is a non-tangent in $S(p)$ subset of $|S(p)|$ disjoint with $|S(p)(q)|$, then* $\mathrm{cl}(\varphi_q^{-1}(A) \cap H) = \mathrm{cl}(\varphi_q^{-1}(A)) \cap \mathrm{cl}H$ *for every stratum H.*

Proof. Note that $\mathrm{cl}(\varphi_q^{-1}(A) \cap H) = \mathrm{cl}(\varphi_q^{-1}(A) \cap \mathrm{cl}H)$, H being open. Therefore it suffices to prove only the following inclusion:

$$\mathrm{cl}\varphi_q^{-1}(A) \cap \mathrm{cl}H \subset \mathrm{cl}(\varphi_q^{-1}(A) \cap \mathrm{cl}H) .$$

Since the map $\varphi_q|H^x\colon H^x \to |S(p)|$ is a homeomorphism and, by Lemma 4, $\mathrm{cl}\varphi_q^{-1}(A) = \varphi_q^{-1}(\mathrm{cl}A)$ hence

$$\begin{aligned}\mathrm{cl}\varphi_q^{-1}(A) \cap H^x &= \varphi_q^{-1}(\mathrm{cl}A) \cap H^x = (\varphi_q|H^x)^{-1}(\mathrm{cl}A)\\ &= \mathrm{cl}_{H^x}(\varphi_q|H^x)^{-1}(A) = \mathrm{cl}(\varphi_q^{-1}(A) \cap H^x).\end{aligned}$$

The desidered inclusion follows now from the observation that $\mathrm{cl}H$ is the union of some sets H^x.

LEMMA 6. *If A is a non-tangent in $S(p)$ subset of $|S(p)|$ disjoint with $|S(p)(q)|$, then*

$$\mathrm{cl}(\varphi_q^{-1}(A) \cap H \cap s) = \mathrm{cl}\varphi_q^{-1}(A) \cap \mathrm{cl}H \cap s = \mathrm{cl}(\varphi_q^{-1}(A) \cap H) \cap s,$$

for each $s \in S$ and every stratum H.

Proof. Let s be a face of S. First let us consider the following two cases:

1. $p \in s$ and $q \notin s$; if H is H_q^x or H_y^x, then $\mathrm{cl}H \cap s \subset |S(p)(q)|$ and therefore $\mathrm{cl}\varphi_q^{-1}(A) \cap \mathrm{cl}H \cap s = \emptyset$; if H is H_p^x, then $s \subset \mathrm{cl}H_p^x$ and therefore $\mathrm{cl}\varphi_q^{-1}(A) \cap \mathrm{cl}H \cap s = \mathrm{cl}\varphi_q^{-1}(A) \cap s = \mathrm{cl}(\varphi_q^{-1}(A) \cap s) = \mathrm{cl}(\varphi_q^{-1}(A) \cap H \cap s)$ (the second equality follows from Lemma 4).

2. $p \in s$ and $q \in s$; then observe that $\varphi_q^{-1}(\varphi_q(s)) = s$ and then Lemmas 4 and 5 imply equalities:

$$\begin{aligned}\mathrm{cl}\varphi_q^{-1}(A) \cap \mathrm{cl}H \cap s &= \varphi_q^{-1}(\mathrm{cl}A) \cap \varphi_q^{-1}(\varphi_q(s)) \cap \mathrm{cl}H = \varphi_q^{-1}(\mathrm{cl}A \cap \varphi_q(s)) \cap \mathrm{cl}H\\ &= \varphi_q^{-1}(\mathrm{cl}(A \cap \varphi_q(s))) \cap \mathrm{cl}H = \mathrm{cl}\varphi_q^{-1}(A \cap \varphi_q(s)) \cap \mathrm{cl}H\\ &= \mathrm{cl}(\varphi_q^{-1}(A \cap \varphi_q(s)) \cap H) = \mathrm{cl}(\varphi_q^{-1}(A) \cap \varphi_q^{-1}(\varphi_q(s)) \cap H)\\ &= \mathrm{cl}(\varphi_q^{-1}(A) \cap H \cap s) .\end{aligned}$$

The proof of the required equality in the remaining two cases, is analogous to that of the case 1, or obvious.

COROLLARY 2. *The property to be non-tangent is preserved by the lift operation.*

THEOREM 1. *In every simplex $|S|$ there exists a binary base $\mathcal{B}$ non-tangent in S, consisting of open pseudopolyhedra the intersections of which with each face of $|S|$ form a binary family and such that*

(*) *if $U, V \in \mathcal{B}$, $s \in S$ and the sets $\mathrm{cl}U \cap \mathrm{cl}V$, $\mathrm{cl}U \cap s$ and $\mathrm{cl}V \cap s$ are non-empty, then the set $\mathrm{cl}U \cap \mathrm{cl}V \cap s$ is so.*

Proof. The construction of such a base will be given by the induction on the dimension of $|S|$.

If $\dim|S| = 1$, then the family of all non-tangent in S open in $|S|$ intervals is the desidered one.

Let us assume that there exists a binary base $\mathcal{B}'$ consisting of open pseudopolyhedra on an $(n-1)$-simplex $|\tilde{s}|$ of S being non-tangent in $\tilde{s}$, which induces a binary family on each face of $|\tilde{s}|$ and fulfils the condition (*).

Let p and q be vertices of S such that $p \notin |\tilde{s}|$ and $q \in |\tilde{s}|$. Let $\varphi_q\colon |S| \to |S(p)| = |\tilde{s}|$ be the corresponding natural projection. Now we prove that the lift $\mathcal{B}$ of the base $\mathcal{B}'$ to $|S|$ by means of φ_q is the family in question.

It follows from Corollary 2 that $\mathcal{B}$ is non-tangent in S.

Clearly $\mathcal{B}$ is a base, having sets of arbitraliry small diameters.

To prove that $\mathcal{B}$ is binary and induces a binary family on each face of $|S|$ let s be a simplex of S and let $U_1, \ldots, U_k \in \mathcal{B}$ be such that $\mathrm{cl}(U_1 \cap s) \cap \ldots \cap \mathrm{cl}(U_k \cap s) = \emptyset$. We can assume that $U_i = \varphi_q^{-1}(V_i) \cap H_i$, $i = 1, \ldots, l$, $l \leqslant k$ and $U_j = \varphi_q^{-1}(V_j)$, $j = l+1, \ldots, k$, where V_i belong to $\mathcal{B}'$ and H_i are strata. Non-trivial is only the case when $p, q \in s$ and there exists H^x such that $H^x \subset \mathrm{cl}H_1 \cap \ldots \cap \mathrm{cl}H_1$. By Lemma 6 we get

$$\mathrm{cl}\varphi_q^{-1}(V_1) \cap \ldots \cap \mathrm{cl}\varphi_q^{-1}(V_k) \cap \mathrm{cl}H_1 \cap \ldots \cap \mathrm{cl}H_1 \cap s = \emptyset .$$

Using the equality $\varphi_q^{-1}(\varphi_q(s)) = s$ we have

$$\varphi_q^{-1}(\mathrm{cl}V_1 \cap \ldots \cap \mathrm{cl}V_k \cap \varphi_q(s)) \cap \mathrm{cl}H_1 \cap \ldots \cap \mathrm{cl}H_1 = \emptyset .$$

The assumption $H^x \subset \mathrm{cl}H_1 \cap \ldots \cap \mathrm{cl}H_1$ implies that $\mathrm{cl}V_1 \cap \ldots \cap \mathrm{cl}V_k \cap \varphi_q(s) = \emptyset$. Since $\varphi_q(s)$ is a face of $|S(p)|$ and $\mathcal{B}'$ induces a binary family on each face of $|S(p)|$ hence there exist V_i and V_j from $\{V_1, \ldots, V_k\}$ such that $\mathrm{cl}V_i \cap \mathrm{cl}V_j \cap \varphi_q(s) = \emptyset$. Consequently, $\mathrm{cl}U_i \cap \mathrm{cl}U_j \cap s = \emptyset$, and all the more $\mathrm{cl}(U_i \cap s) \cap \mathrm{cl}(U_j \cap s) = \emptyset$.

In order to prove the condition (*) let U, V and s be such that $U, V \in \mathcal{B}$, $s \in S$ and the sets $\mathrm{cl}U \cap \mathrm{cl}V$, $\mathrm{cl}U \cap s$, $\mathrm{cl}V \cap s$ are non-empty. We can assume that p and q belong to s (other cases are trivial). Since U and V are in $\mathcal{B}$, hence $U = \varphi_q^{-1}(F) \cap E_1$ and $V = \varphi_q^{-1}(G) \cap E_2$, where F and G belong to $\mathcal{B}'$ and E_1, E_2 are strata of $|S|$. Let, on the contrary, $\mathrm{cl}U \cap \mathrm{cl}V \cap s = \emptyset$. Using Lemma 6 and the formula $\varphi_q^{-1}(\varphi_q(s)) = s$ we get

$$\varphi_q^{-1}(\mathrm{cl}F \cap \mathrm{cl}G \cap \varphi_q(s)) \cap \mathrm{cl}E_1 \cap \mathrm{cl}E_2 = \emptyset .$$

Since $\operatorname{cl}U \cap \operatorname{cl}V \neq \emptyset$ hence there exists a hyperplane H^x which is contained in $\operatorname{cl}E_1 \cap \operatorname{cl}E_2$. But in this case $\operatorname{cl}F \cap \operatorname{cl}G \cap \varphi_q(s) = \emptyset$; a contradiction with the fact that $F, G \in \mathcal{B}'$ which fulfils (*).

The fact that $\mathcal{B}$ consists of pseudopolyhedra follows immediately from Lemma 3.

§ 3. Further lemmas.

LEMMA 7. *For every simplex $|S|$ and a positive δ there exist a finite number of points of $|S|$, say $x_1, \ldots, x_l$, and open (in $|S|$) neighbourhoods of that points, say $U_1, \ldots, U_l$, which cover $|S|$ and such that*

(1) *U_i is a pseudopolyhedron non-tangent in S and* $\operatorname{diam} U_i < \delta$,

(2) $$\operatorname{cl}U_i \subset \operatorname{st}_S x_i,$$

(3) $\operatorname{cl}U_i \cap \operatorname{cl}U_j \neq \emptyset$ *implies that* $\operatorname{car}_S x_i$ *is a face of* $\operatorname{car}_S x_j$ *or conversely,*

(4) *$\{U_1, \ldots, U_l\}$ is a binary family which induces a binary family on each face of S,*

(5) *if $s \in S$ and all the sets $\operatorname{cl}U_i \cap \operatorname{cl}U_j$, $\operatorname{cl}U_i \cap s$ and $\operatorname{cl}U_j \cap s$ are non-empty, then the set $\operatorname{cl}U_i \cap \operatorname{cl}U_j \cap s$ is so.*

Proof. Let $\mathcal{B}$ be a base constructed in Theorem 1. Let $\{x_1, \ldots, x_{n_0}\}$ be the 0-skeleton of S. Since $\mathcal{B}$ is a base hence there exist elements $U_1, \ldots, U_{n_0}$ of $\mathcal{B}$ with disjoint closures satysfying the conditions (1)–(5) (the conditions (3)–(5) in vacuum).

Let $S^{(k)}$ be the k-skeleton of S. Let us assume that we have points $x_1, \ldots, x_{n_0}, \ldots, x_{n_k}$ in $|S^{(k)}|$ and sets $U_1, \ldots, U_{n_0}, \ldots, U_{n_k}$ from $\mathcal{B}$ which satisfy conditions (1)–(5) and such that $|S^{(k)}| \subset U_1 \cup \ldots \cup U_{n_k}$.

Now let s^{k+1} be a $(k+1)$-simplex of $S^{(k+1)}$. The compactness of the set $D = s^{k+1} \setminus (U_1 \cup \ldots \cup U_{n_k})$ implies that there exist points $x_{n_k+1}, \ldots, x_p$ in D and sets $U_{n_k+1}, \ldots, U_p$ in $\mathcal{B}$ which cover D, which satisfy (1) and such that

$$x_i \in U_i \subset \operatorname{cl}U_i \subset \operatorname{st}_S x_i \quad \text{and} \quad \operatorname{diam} U_i < \delta \quad \text{for all } i$$

and if a set V from $\{U_1, \ldots, U_{n_k}\}$ is such that $\operatorname{cl}V \cap s^{k+1} = \emptyset$, then $\operatorname{cl}V \cap \operatorname{cl}(U_{n_k+1} \cup \ldots \cup U_p) = \emptyset$.

It is easy to see that the points $x_1, \ldots, x_{n_k}, \ldots, x_p$ and the sets $U_1, \ldots, U_{n_k}, \ldots, U_p$ satisfy conditions (1)–(5). Applying this construction succesively to remaining $(k+1)$-simplexes of $S^{(k+1)}$ we obtain, in finitely many steps, the points $x_1, \ldots, x_{n_{k+1}}$ in $|S^{(k+1)}|$ and the sets $U_1, \ldots, U_{n_{k+1}}$ in $\mathcal{B}$ which satisfy conditions (1)–(5) and such that $|S^{(k+1)}| \subset U_1 \cup \ldots \cup U_{n_{k+1}}$.

Now the lemma follows by the induction.

LEMMA 8. *Let $|S|$ be a simplex and let $|P'|$ be a polyhedron such that $P' \subset S$. Let $\mathcal{C}$ be a finite binary family of subpolyhedra of $|P'|$ such that*

(6) $$\text{if} \quad |V| \in \mathcal{C}, \quad \text{then} \quad V \subset P'.$$

Let Q be a binary family on $|S|$ constructed as in Lemma 7.

Then the intersections of elements of $\mathcal{Q} \cup \mathcal{C}$ with $|P'|$ form a binary family on $|P'|$ non-tangent in P'.

Proof. The fact that the family in question is non-tangent in P' holds, because $\mathcal{Q}$ is non-tangent in P' and elements of $\mathcal{C}$ are closed.

In order to prove the binarity let $U_1, \ldots, U_j \in \mathcal{Q}$ and $|V_1|, \ldots, |V_k| \in \mathcal{C}$ be such that the closures of each two members of $\{U_1, \ldots, U_j, |V_1|, \ldots, |V_k|\}$ have non-empty intersection and each of $\operatorname{cl}U_i$ has non-empty intersection with $|P'|$.

Let $x_1, \ldots, x_j$ be points corresponding to sets $U_1, \ldots, U_j$ as in Lemma 7.

It follows from the fact that $\operatorname{cl}U_p \cap |V_q| \neq \emptyset$ and from (2) that $x_p \in |V_q|$, for each p and q. Hence

$$x_1, \ldots, x_j \in |V_1| \cap \ldots \cap |V_k| \cap |P'|.$$

It is easy to prove using (3), by the induction on j, that all the points x_i, $i = 1, \ldots, j$, lie in one simplex s from $V_1 \cap \ldots \cap V_k \cap P'$, s being the carrier of one of them. This, together with (5), imply that

$$\operatorname{cl}U_1 \cap \ldots \cap \operatorname{cl}U_j \cap \operatorname{cl}|V_1| \cap \ldots \cap \operatorname{cl}|V_k| \cap |P'| \neq \emptyset.$$

In the case of the lack of $V-s$ the proof holds with obvious simplifications. In the case of the lack of $U-s$ conclusion follows immediately from the hypotheses.

LEMMA 9. *Let $|P|$ be a polyhedron, let $\mathcal{R}$ be a finite binary family consisting of pseudopolyhedra non-tangent in P and let δ be a positive number.*

Then there exists a finite open covering $\mathcal{F}$ of $|P|$ consisting of open pseudopolyhedra such that $\operatorname{mesh}\mathcal{F} < \delta$ *and $\mathcal{F} \cup \mathcal{R}$ is a binary family non-tangent in P.*

Proof. It follows from Lemma 2 that there exists a subdivision P' of P which induces a triangulation on each nonempty intersection of closure of elements of each subfamily of $\mathcal{R}$, the elements of $\mathcal{R}$ are pseudopolyhedra. We can assume that $P' \subset S$, where $|S|$ is a simplex. Let $\mathcal{C} = \{\operatorname{cl}A\colon A \in \mathcal{R}\}$. Define $\mathcal{F}$ to be the family of all intersections of elements of Q with $|P'|$, where Q is a δ-covering taken for S according to Lemma 7. Lemma 8 assures that $\mathcal{F} \cup \mathcal{C}$ is binary. Thus $\mathcal{F} \cup \mathcal{R}$ is binary. The same Lemma 8 assures that $\mathcal{F} \cup \mathcal{R}$ is non-tangent in P.

LEMMA 10. *Let $|P|$ be a polyhedron and let $W_1, \ldots, W_m$ be open in $|P|$ pseudopolyhedra non-tangent in P. Then there exists a simple subdivision $P^{(1)}$ of P of order 2 such that all $W_1, \ldots, W_m$ are non-tangent in $P^{(1)}$.*

Proof. For each $s \in P$ we denote by $U(s)$ the set of all points of s having all the barycentric coordinates in s not smaller than $(\dim P+1)^{-2}$. Now the thesis of our lemma may be stated as follows.

For each s of P there exists $p_s \in U(s)$ such that for each sequence

$$s_0 \subsetneq s_1 \subsetneq \ldots \subsetneq s_k \subsetneq s \tag{7}$$

and for each face Δ of $(p_{s_0}, \ldots, p_{s_k}, p_s)$, the simplex determined by vertices $p_{s_0}, \ldots, p_{s_k}, p_s$, there is for each i, $i \leqslant m$

$$\mathrm{cl}(\Delta \cap W_i) = \Delta \cap \mathrm{cl}W_i . \tag{8}$$

The proof of our lemma will be done by the induction on $\dim s$.

If $\dim s = 0$, then p_s equals s and (8) follows easily from the assumption that W_i are non-tangent in P.

Let p_s be already defined for simplexes of the n-skeleton of P and let $s \in P$ be a $(n+1)$-simplex. Now we are going to define p_s such that (8) holds for each Δ of each simplex $(p_{s_0}, \ldots, p_{s_k}, p_s)$, where $s_0, \ldots, s_k$ satisfy (7).

Take on each polyhedron $\mathrm{cl}(s \cap W_i)$ a triangulation T_i such that T_i induces on $s \cap \mathrm{bd}W_i$ (bd stands for the topological boundary in $|P|$) a triangulation which we denote T'_i (the existence of such T'_i follows from Lemma 2). Note that $s \cap \mathrm{bd}W_i$ is equal, in virtue of the non-tangence of W_i in P, to $\mathrm{bd}_s(s \cap W_i)$ (bd_s stands for the topological boundary in s).

Consider all the hyperplanes in s with the dimension not greater than n determined by arbitrary families of simplexes of $T'_1, \ldots, T'_m$ and of points $p_{s_0}, \ldots, p_{s_k}$ already defined which satisfy (7). Let A be the union of all such hyperplanes. It follows from the fact that A is nowhere dense in s that there exists a point p_s such that $p_s \in U(s) - A$.

Let Δ be a face of $(p_{s_0}, \ldots, p_{s_k}, p_s)$, where $s_0, \ldots, s_k$ be such that (7) holds. To prove (8), let $p \in \Delta \cap \mathrm{cl}W_i$.

To prove that $p \in \mathrm{cl}(\Delta \cap W_i)$ it suffices, in virtue of the inductive hypothesis, to consider only the case when $p \in \mathrm{intgeom}\, s$, i.e. when $s = \mathrm{car}_P p$. In consequence $p_s \in \mathrm{car}_\Delta p$. Clearly we can assume that $p \in \mathrm{bd}W_i \cap \mathrm{bdgeom}\Delta$, W_i being non-tangent in P.

If M is a subset of $|P|$, then by $H(M)$ we denote the hyperplane determined by M.

Let $t' = \mathrm{car}_{T'_i} p$ and let $t'' \in T_i$ be such that $t' \subset t''$ and $t'' \cap W_i \cap s \neq \emptyset$. Let $t = \mathrm{intgeom}\, t''$. Clearly, $t \subset W_i \cap s$, $t' \subset \mathrm{cl}\, t$ and $\dim H(t) > \dim H(t')$.

Let $p_{s_{i_0}}, \ldots, p_{s_{i_r}}$ be all points from $\{p_{s_0}, \ldots, p_{s_k}\}$ which lie in $\mathrm{car}_\Delta p$. We have $H(\mathrm{car}_\Delta p \cup t') = H(\{p_{s_{i_0}}, \ldots, p_{s_{i_r}}\} \cup t')$, because p_s belongs to the set on the right side. Then, by the definition of p_s we infer that $\dim H(\mathrm{car}_\Delta p \cup t') = n+1$. This implies, in virtue of $p \in H(\mathrm{car}_\Delta p) \cap H(t') \subset H(\mathrm{car}_\Delta p) \cap H(t)$,

$$\dim\big(H(\mathrm{car}_\Delta p)\big) + \dim H(t') = \dim\big(H(\mathrm{car}_\Delta p) \cap H(t')\big) + n + 1 ,$$

$$\dim H(\mathrm{car}_\Delta p) + \dim H(t) = \dim\big(H(\mathrm{car}_\Delta p) \cap H(t)\big) + n + 1 .$$

But $\dim H(t) > \dim H(t')$. So

$$\dim\big(H(\mathrm{car}_\Delta p) \cap H(t)\big) > \dim\big(H(\mathrm{car}_\Delta p) \cap H(t')\big) .$$

Let

$$q \in H(\mathrm{car}_\Delta p) \cap t \setminus H(\mathrm{car}_\Delta p) \cap H(t') .$$

Then the open interval (p, q) is contained in $H(\mathrm{car}_\Delta p) \cap t$. So there exists an $r \in (p, q)$ such that $(p, r) \subset t \cap \Delta \subset W_i \cap \Delta$ and hence $p \in \mathrm{cl}(W_i \cap \Delta)$ what ends the proof of the equality (8). Hence the inductive thesis is proved.

Lemma 11. *Let $\mathfrak{R}$ be a finite family of open in $|P|$ pseudopolyhedra in a polyhedron $|P|$ non-tangent in P and let ε be a positive number. Then there exists a subdivision P' of P such that* $\mathrm{mesh}\, P' < \varepsilon$ *and each element of $\mathfrak{R}$ is non-tangent in P'.*

Proof. By Lemma 10, there exists a simple subdivision $P^{(1)}$ of P of order 2 such that all elements of $\mathfrak{R}$ are non-tangent in $P^{(1)}$. By Lemma 1, $\mathrm{mesh}\, P^{(1)} \leqslant \big(1 - (\dim P+1)^{-2}\big) \cdot \mathrm{mesh}\, P$. This implies that if we shall iterate the process described above, then we can find an r such that $P^{(r)}$ is a subdivision in question.

§ 4. Construction of a binary base on a compact metric space.

Theorem 2. *Every compact metric space X has a binary base.*

Proof. We shall construct an inverse sequence of polyhedra $|P_n|$ (the metric d_n on $|P_n|$ let be such that $d_n(|P_n|) \leqslant 1$) with simplicial onto bonding maps π^n_{n-1}, whose inverse limit Y with the standard metric

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n\big(\pi_n(x), \pi_n(y)\big) ,$$

where π_n denotes the standard projection of Y onto P_n, is homeomorphic to X, and has a binary base $\mathfrak{B} = \{\pi_n^{-1}(A)\colon A \in \mathfrak{R}_n,\ n = 1, 2, \ldots\}$, where $\mathfrak{R}_n$ is a certain finite open covering of $|P_n|$. To do this it suffices in the procedure of the proof of Freudenthal's theorem, [1] (on the existence of such an expansion without the existence of a binary base on Y) in the form of Pasynkov [5] (proof of Prop. 2, pp. 97–98; for μ being the class of all polyhedra; we assume that the technique of maps of X into nerves of open finite coverings of X is known to the reader), which consists on the inductive construction of $|P_n|$ and π^n_{n-1}, to take into consideration the following two observations:

(1) if we have been already defined sequences $|P_1| \xleftarrow{\pi^2_1} |P_2| \leftarrow \ldots \ldots \leftarrow |P_{n-1}| \xleftarrow{\pi^n_{n-1}} |P_n|$, $\mathfrak{R}_1, \ldots, \mathfrak{R}_n$ and $\delta_1, \ldots, \delta_n$, such that, for $1 < i \leqslant n$,

(a) $|P_i|$ is a polyhedron such that $\operatorname{diam}|P_i| \leqslant 1$ and π^i_{i-1} is a simplicial map of $|P_i|$ onto $|P'_{i-1}|$, where P'_{i-1} is a certain subdivision of P_{i-1},

(b) $\mathfrak{R}_i$ is a finite family consisting of pseudopolyhedra open in $|P_i|$, non-tangent in P'_i for $i < n$ and in P_n for $i = n$, such that $\operatorname{mesh}\mathfrak{R}_i < \delta_i$, $\mathfrak{R}_i$ covers $|P_i|$ and $\mathfrak{R}_i \cup \{(\pi^i_k)^{-1}(A_k)\colon A_k \in \mathfrak{R}_k,\ k < i\}$ is a binary family,

(c) δ_i is a positive number such that if $A \subset |P_i|$ and $\operatorname{diam} A < \delta_i$, then for each $j \leqslant i$, $\operatorname{diam}\pi^i_j(A) < 1/2^i$, then for every positive number ε there exists a subdivision P'_n of P_n such that $\operatorname{mesh} P'_n < \varepsilon$ and each element of $\mathfrak{R}_n \cup \{(\pi^n_k)^{-1}(A_k)\colon A_k \in \mathfrak{R}_k,\ k < n\}$ is non-tangent in P'_n, and

(2) if we have been already defined a simplicial map π^{n+1}_n of $|P_{n+1}|$ onto $|P'_n|$, where P'_n satisfies conditions from (1), then we can find

(e) a positive number δ_{n+1} such that if $A \subset |P_{n+1}|$ and $\operatorname{diam} A < \delta_{n+1}$, then for each $j \leqslant n+1$ $\operatorname{diam}\pi^{n+1}_j(A) < 1/2^{n+1}$, and

(f) an open (in $|P_{n+1}|$) covering $\mathfrak{R}_{n+1}$ of $|P_{n+1}|$ with $\operatorname{mesh}\mathfrak{R}_{n+1} \leqslant \delta_{n+1}$ consisting of pseudopolyhedra such that family

$$\mathfrak{R}_{n+1} \cup \{(\pi^{n+1}_k)^{-1}(A_k)\colon A_k \in \mathfrak{R}_k,\ k < n+1\}$$

is binary and each element of that family is non-tangent in P_{n+1}.

In fact, (1) assures (see the note of Pasynkov loco cit.) the existence of polyhedron $|P_{n+1}|$ and a simplicial map $\pi^{n+1}_n\colon |P_{n+1}| \to |P'_n|$, where P'_n is as in (1) (for sufficiently small ε), such that they satisfy the conclusions of the Pasynkov's construction (in order to get a homeomorphism of X with the inverse limit of the inverse sequence $\{|P_n|;\ \pi^n_{n-1}\}$). In virtue of Lemma 4, the family $\{(\pi^{n+1}_k)^{-1}(A_k)\colon A_k \in \mathfrak{R}_k,\ k < n+1\}$ is non-tangent in P_{n+1}. Now take $\mathfrak{R}_{n+1}$ as in (2).

By the induction we have constructed an inverse sequence $\{|P_n|;\ \pi^n_{n-1}\}$ with inverse limit Y, a homeomorphism of X onto Y (by the procedure of Pasynkov loco cit.) and a sequence $\mathfrak{R}_n$ satisfying the conditions of (1) and (2).

Now, easy calculation with projections π_n and maps π^k_l leads, in virtue of (b), to the formula

$$\pi_n^{-1}(\operatorname{cl} A_k) = \operatorname{cl}(\pi_n^{-1}(A_k)) \quad \text{for} \quad A_k \in \mathfrak{R}_k,\ k \leqslant n.$$

Using this formula and applying (b) once again we infer that $\mathfrak{B}$ is a binary family on Y.

The fact that $\mathfrak{B}$ is a base follows easily from (b) and (c).

To complete the proof observe that the assertions (1) and (2) follows immediately from Lemma 11 and Lemma 9, respectively.

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