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COMPACT MODULI SPACES OF DEL PEZZO SURFACES AND KÄHLER–EINSTEIN METRICS

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Abstract

We prove that the Gromov–Hausdorff compactification of the moduli space of Kähler–Einstein Del Pezzo surfaces in each degree agrees with certain algebro-geometric compactification. In particular, this recovers Tian's theorem on the existence of Kähler–Einstein metrics on smooth Del Pezzo surfaces and classifies all the degenerations of such metrics. The proof is based on a combination of both algebraic and differential geometric techniques.

1. Introduction

For each positive integer d, we denote by M_d^{GH} the *Gromov–Hausdorff compactification* of the moduli space of degree d Kähler–Einstein Del Pezzo surfaces, and we denote by M_d^0 the dense subset that parametrizes those smooth surfaces. It is well known that for $d \geq 5$ the moduli space is just a single point, so in this paper we will always assume $d \in \{1, 2, 3, 4\}$. By Tian and Yau [**75**] we know that M_d^0 is at least a non-empty set. By general theory, M_d^{GH} is a compact Hausdorff space under the Gromov–Hausdorff topology. By [**4**, **7**, **70**], points in $M_d^{GH} \setminus M_d^0$ parametrize certain Kähler–Einstein log Del Pezzo surfaces, and a famous theorem of Tian [**70**] says that, every smooth Del Pezzo surface admits a Kähler–Einstein metric so that it is actually parametrized in M_d^0.

In this paper for each d we identify M_d^{GH} with a certain explicit algebro-geometric moduli space of log Del Pezzo surfaces. The latter is a compact Moishezon analytic space M_d, which, roughly speaking, parametrizes isomorphism classes of certain $\mathbb{Q}$-Gorenstein smoothable log Del Pezzo surfaces of degree d. Notice that there are a priori several possibilities of such algebro-geometric compactifications of the moduli varieties. On the other hand, the Gromov–Hausdorff compactification is clearly canonical but very non-algebraic and just topological in nature. We refer to [**10**] as an introductory textbook for those who are not familiar with the Gromov–Hausdorff topology. The following main theorem of the present article builds a bridge between the two notions of moduli spaces.

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Theorem 1.1. *For each integer d, there is a compact moduli algebraic space M_d, which will be constructed explicitly in later sections, and a homeomorphism*

$$\Phi\colon M_d^{GH} \to M_d,$$

such that $[X]$ and $\Phi([X])$ parametrize isomorphic log Del Pezzo surfaces for any $[X] \in M_d^{GH}$. Moreover, M_d contains a (Zariski) open dense subset that parametrizes all smooth degree d Del Pezzo surfaces.

For the precise formulation, see Section 3.4. We remark here that, for $d \neq 1$, it follows from the construction that M_d is actually a *projective variety.*

Theorem 1.1 immediately implies the above mentioned theorem of Tian, and also classifies all degenerations of Kähler–Einstein Del Pezzo surfaces which was posed as a problem in [**71**]. When $d = 4$ Theorem 1.1 was proved by Mabuchi-Mukai [**45**], and we shall provide a slightly different proof based on our uniform strategy. For other degrees, there have been partial results [**15**, **17**, **27**, **65**, **77**] on the existence of Kähler–Einstein metrics on some canonical Del Pezzo surfaces, by calculating α-invariant.

A minor point is that the Gromov–Hausdorff topology defined here is slightly different from the standard definition, in that we also remember the complex structure when we talk about convergence. See [**23**]; [**66**, Chapters 1 and 4] and Section 2 for a related discussion on this. The standard Gromov–Hausdorff compactification is homeomorphic to the quotient of M_d by the involution which conjugates the complex structures.

For the proof of Theorem 1.1, we do not need to assume the existence of Kähler–Einstein metrics on all the smooth Del Pezzo surfaces. The only assumption that we need, and which has been originally proved by Tian and Yau [**75**], is the following:

Hypothesis 1.2. *For each $d \in \{1, 2, 3, 4\}$, M_d^0 is non-empty as a set.*

Given this, the main strategy of proving Theorem 1.1 is as follows:

1) For each d, we construct a natural moduli variety M_d with a Zariski open subset M_d^{sm} parametrizing all smooth degree d Del Pezzo surfaces. Moreover, there is a well-defined continuous map $\Phi\colon M_d^{GH} \to M_d$, where we use the Gromov–Hausdorff distance in the domain and the local analytic topology in the target, so that $[X]$ and $\Phi([X])$ parametrize isomorphic log Del Pezzo surfaces for any $[X] \in M_d^{GH}$.
2) Φ is injective. This follows from the uniqueness theorem of Bando and Mabuchi [**8**] and its extension to orbifolds.

3) Φ is surjective. This follows from the fact that the image of Φ is open in M_d^{sm} (by the implicit function theorem, see, for example, [**42**]) and closed in M_d (by the continuity of Φ in (1)).
4) Since M_d^{GH} is compact and M_d is Hausdorff, then Φ is a homeomorphism.

The main technical part lies in Step (1). For this we need first to investigate Gromov–Hausdorff limits of Kähler–Einstein Del Pezzo surfaces and then construct a moduli space that includes all the possible limits. The difficulty increases as the degree goes down. When $d = 3, 4$, we take the classical GIT moduli space on the anti-canonical embedding. For $d = 2$ we take the moduli space constructed in [**52**] (based on Shah's idea [**64**], which blows up a certain GIT quotient). For $d = 1$ we need to combine Shah's method with further modifications suggested by the differential geometric study of Gromov–Hausdorff limits. As far as we are aware, this moduli space is new. We should mention that in the last two cases, M_d (and thus M_d^{GH}) contains points that parametrize non-canonical log Del Pezzo surfaces. This disproves a conjecture of Tian in [**70**], see Remark 5.14. We also remark that Gromov–Hausdorff limits of Kähler–Einstein Del Pezzo surfaces were first studied by Tian in [**70**], but as we shall see, there are some inaccuracies in [**70**], see Remark 2.8 and Example 5.8.

Finally, we remark that for each $d \in \{1, 2, 3, 4\}$, it is easy to find explicit examples of singular degree d $\mathbb{Q}$-Gorenstein smoothable Kähler–Einstein log Del Pezzo surface by a global quotient construction (see the examples in later sections). Thus one way to avoid assuming Hypothesis 1.2 would be to find a smooth Kähler–Einstein Del Pezzo surface by a gluing construction. For example, it has been proved in [**67**] that for a Kähler–Einstein log Del Pezzo surface with only nodal singularities and discrete automorphism group, one can glue model Eguchi–Hanson metrics to obtain nearby Kähler–Einstein metrics in the smoothing. This can be applied when $d = 3$, since the Cayley cubic (see Section 4) satisfies these assumptions.

The organization of this paper is as follows. In Section 2 we collect the main results that we need on the structure of Gromov–Hausdorff limits, focusing on the two-dimensional case. In Section 3 we make an algebro-geometric study of the Gromov–Hausdorff limits and define precisely the notion of moduli spaces that we use in this paper. Then we reduce the proof of Theorem 1.1 to the construction of moduli spaces in each degree. In later sections we treat the cases $d \geq 3$ and $d \leq 2$ separately. We also investigate the relation with moduli space of curves, in subsections 5.2.1 and 5.3.5. In Section 6 we make some further discussions.

Notation. A *Del Pezzo surface* is a smooth projective surface with ample anti-canonical bundle. A *log Del Pezzo surface* is a normal projective surface with quotient singularities (or equivalently, with log terminal singularities) and ample anti-canonical divisor. For a log Del Pezzo surface X, its degree $\deg(X)$ is the intersection number K_X^2. In general dimensions, a $\mathbb{Q}$-*Fano variety* means a normal projective variety with log terminal singularities and with $-rK_X$ ample for some positive integer r. Smallest such r will be called the *index* or *Gorenstein index*.

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2. General results on the Gromov–Hausdorff limits

The main differential geometric ingredient involved in the proof of the main theorem is the study of the structure of Gromov–Hausdorff limits of Kähler–Einstein Del Pezzo surfaces. The following orbifold compactness theorem is well known.

Proposition 2.1 ([**4**, **7**, **70**]). *Given a sequence of degree d Kähler–Einstein Del Pezzo surfaces (X_i, ω_i, J_i), then, by passing to a subsequence, it converges in the Gromov–Hausdorff sense to a Kähler–Einstein log Del Pezzo surface $(X_\infty, \omega_\infty, J_\infty)$, and* $\deg(X_\infty) = d$.

In [**70**] Tian found further constraints on the possible singularities that could appear in X_∞. We will state a more general theorem and give an alternative proof. First we have (compare also [**74**]):

Proposition 2.2 ([**23**]). *Given a sequence of n-dimensional Kähler–Einstein Fano manifolds (X_i, ω_i, J_i), by passing to a subsequence, it converges in the Gromov–Hausdorff sense to a $\mathbb{Q}$-Fano variety (X_∞, J_∞) endowed with a weak Kähler–Einstein metric ω_∞ (cf. [**24**]). Moreover, there exist integers k and N, depending only on n, so that we could embed X_i $(i \in \mathbb{N} \cup \{\infty\})$ into $\mathbb{P}^N$ using the orthonormal basis of $H^0(X_i, -kK_{X_i})$ with respect to the Hermitian metric defined by ω_i, and X_i converges to X_∞ as varieties in $\mathbb{P}^N$.*

Here one can think of the convergence of varieties in $\mathbb{P}^N$ as the convergence of defining polynomials. Notice that the orbifold property in Proposition 2.1 also follows naturally from Proposition 2.2, since by Kawamata's theorem [**36**] a two-dimensional log terminal singularity is a quotient singularity.

We will treat singular varieties that come from certain limits of smooth ones. The following algebro-geometric notion is very natural from the point of view of the minimal model program, and will be shown to be also naturally satisfied by the above limit X_∞.

Definition 2.3. Let X be a $\mathbb{Q}$-Fano variety. We say X is $\mathbb{Q}$-*Gorenstein smoothable* if there exists a deformation $\pi : \mathcal{X} \to \Delta \ni 0$ of X over a smooth curve germ Δ such that $\mathcal{X}_0 = X$, the general fibre is smooth and $K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier.

Lemma 2.4. *X_∞ is $\mathbb{Q}$-Gorenstein smoothable.*

Proof. By Proposition 2.2 and general theory we can find a family of varieties $\pi_2 : \mathcal{X} \subset \mathbb{P}^N \times \Delta \to \Delta$ in $\mathbb{P}^N$ where for $t \neq 0$ $\mathcal{X}_t$ is smooth and $\mathcal{X}_0$ is the variety X_∞. Indeed, for a morphism from Δ to the Hilbert scheme that sends 0 to X_∞ (embedded by $|-kK_{X_\infty}|$) and contains X_i (embedded by $|-kK_{X_i}|$ as well) for one sufficiently large i, we can construct the required family by pulling back the total space and take its normalization if necessary. Denote the other projection map by $\pi_1 \colon \mathbb{P}^N \times \Delta \to \mathbb{P}^N$; then $-rK_{\mathcal{X}}$ and $\pi_1^*\mathcal{O}(1)$ agrees up to a pull back from the base. Thus $-rK_{\mathcal{X}}$ is Cartier, and so X_∞ is $\mathbb{Q}$-Gorenstein smoothable. q.e.d.

Note that the above proof does not use the $\mathbb{Q}$-Gorenstein property of the normal central fiber, although in our case, we knew it by Proposition 2.2. We only need a relatively ample linear line bundle and the normality assumption of the total space and the central fiber. The definition of $\mathbb{Q}$-Gorenstein smoothability can be obviously defined also for local singularities, and for a $\mathbb{Q}$-Gorenstein smoothable $\mathbb{Q}$-Fano manifold all its singularities must also be $\mathbb{Q}$-Gorenstein smoothable. On the other hand, it is proved in [**30**] that a log Del Pezzo surface with $\mathbb{Q}$-Gorenstein smoothable singularities is $\mathbb{Q}$-Gorenstein smoothable. In dimension two, $\mathbb{Q}$-Gorenstein smoothable quotient singularities are also commonly called *"T-singularities."* The classification of T-singularities is well known; see [**39**], [**47**] for example. So combining the above discussions we obtain:

Theorem 2.5 ([**70**]). *The Gromov–Hausdorff limit (X_∞, J_∞) of a sequence of Kähler–Einstein Del Pezzo surfaces is a Kähler–Einstein log Del Pezzo with either canonical (ADE) singularities or cyclic quotient singularities of type $\frac{1}{dn^2}(1, dna-1)$ with $(a,n)=1$ $(1 \leq a < n)$.*

Remark 2.6. *For the sake of completeness, even if we will not use it in our proof, we should remark that it is known that local smoothings of T-singularities admit asymptotically conical Calabi–Yau metrics* [**40**, **69**]. *It is then natural to expect from a metric perspective the following picture: given a sequence (X_i, ω_i) of degree d Kähler–Einstein Del Pezzo surfaces Gromov–Hausdorff converging to a singular*

$(X_\infty, \omega_\infty)$ and choose $p_\infty \in Sing(X_\infty)$, then there exists a sequence of points $p_i \in X_i \to p_\infty \in X_\infty$ and scaling parameters $\lambda_i \to +\infty$ such that $(X_i, p_i, \lambda_i \omega_i)$ converges in the pointed Gromov–Hausdorff sense to an asymptotically conical Calabi–Yau metric on a smoothing of the T-singularity at p_∞.

Next, we can use the Bishop–Gromov volume comparison theorem to control the order of the orbifold group at each point.

Theorem 2.7 ([**70**]). *Let (X, ω) be a Kähler–Einstein log Del Pezzo surface, and let $\Gamma_p \subseteq U(2)$ be the orbifold group at a point $p \in X$. Then*

$$|\Gamma_p| \deg(X) < 12. \tag{2.1}$$

Proof. Without loss of generality we may normalize the metric so that $Ric(\omega) = 3\omega$. The Bishop–Gromov volume comparison extends without difficulty to orbifolds [**12**], so for all $p \in X$ the function $\frac{Vol(B(p,r))}{Vol(\overline{B}(r))}$ is decreasing in r, where $\overline{B}(r)$ is the ball of radius r in the standard four sphere $S^4(1)$. As r tends to zero the function converges to $1/|\Gamma_p|$, and for sufficiently large r the function is constant $Vol(X, \omega)/Vol(S^4(1))$. So $Vol(X, \omega)|\Gamma_p| \leq Vol(S^4(1))$. The normalization condition $Ric(g) = 3g$ implies that $[\omega] = \frac{2\pi}{3} c_1(X)$. So $Vol(X, \omega) = \int_X \frac{\omega^2}{2} = \frac{2\pi^2}{9} \deg(X)$. Then, using the fact that $Vol(S^4(1)) = \frac{8}{3}\pi^2$, it is easy to see $|\Gamma_p| \deg(X) \leq 12$. If the equality is achieved, then X must have constant curvature. But since X is Kähler, we have $S(\omega)^2 = 24|W^+|^2$, and so the scalar curvature vanishes. Contradiction. q.e.d.

Remark 2.8. *The two theorems above were essentially known to Tian* [**70**]. *For the inequality* 2.1*, the constant on the right-hand side was* 48 *in* [**70**].

By Theorem 2.5 and Theorem 2.7, we have the constraints on the possible singularities that could appear on the Gromov–Hausdorff limit X_∞. (Recall that the order of the finite Klein group yielding an A_k singularity is $k+1$, a D_k singularity is $4(k-2)$, an E_6 singularity is 24, an E_7 singularity is 48, and an E_8 singularity is 120.)

- $\deg = 4$, X_∞ is canonical and can have only A_1 singularities.
- $\deg = 3$, X_∞ is canonical and can have only A_1 or A_2 singularities.
- $\deg = 2$, X_∞ can have only A_1, A_2, A_3, A_4, and $\frac{1}{4}(1,1)$ singularities.
- $\deg = 1$, X_∞ can have only $\frac{1}{4}(1,1)$, $\frac{1}{8}(1,3)$, and $\frac{1}{9}(1,2)$ singularities besides A_i $(i \leq 10)$ and D_4 singularities.

For the case when $d \geq 3$, the above classification is already sufficient for our purposes, as canonical log Del Pezzo surfaces are classified (see the next section). When $d \leq 2$ we will make a further study in Section 5. Now we make a side remark about the Gromov–Hausdorff topology used in this paper. In [**66**] it is proved that if two Kähler–Einstein log Del

Pezzo surfaces are isometric, then the complex structures could be the same or conjugate. For this reason the standard Gromov–Hausdorff distance cannot distinguish two conjugate complex structures in general. So in our case there is an easy modification, where we say a sequence (X_i, J_i, ω_i) converges to $(X_\infty, J_\infty, \omega_\infty)$ if it converges in the Gromov–Hausdorff topology and in the sense of Anderson and Tian, i.e., smooth convergence of both the metric and complex structure away from the singularities. The spaces M_d appearing in Theorem 1.1 admit an involution given by conjugating the complex structure, and we will identify explicitly this involution for each d.

3. Algebro-geometric properties of log Del Pezzo surfaces

Here we continue to study the algebro-geometric properties of X_∞ appearing in the last section. These constraints help the construction of the desired moduli spaces in later sections.

3.1. Classification of mildly singular log Del Pezzo surfaces. We first recall some general classification results for log Del Pezzo surfaces with mild singularities. The following is classical.

Theorem 3.1 ([**31**]). *A degree d log Del Pezzo surface with canonical singularities is*

- *a complete intersection of two quadrics in $\mathbb{P}^4$, if $d = 4$;*
- *a cubic hypersurface in $\mathbb{P}^3$, if $d = 3$;*
- *a degree 4 hypersurface in $\mathbb{P}(1, 1, 1, 2)$ not passing $[0 : 0 : 0 : 1]$, if $d = 2$;*
- *a degree 6 hypersurface in $\mathbb{P}(1, 1, 2, 3)$ not passing $[0 : 0 : 1 : 0]$ and $[0 : 0 : 0 : 1]$, if $d = 1$.*

Although we will not use them, log Del Pezzo surfaces with Gorenstein index 2 are also classified, by [**3**] and [**55**]. In the case the degree is 1 or 2, we have:

Theorem 3.2 ([**37**]). *A degree 2 log Del Pezzo surface with Gorenstein index at most 2 is either a degree 4 hypersurface in $\mathbb{P}(1, 1, 1, 2)$, or a degree 8 hypersurface in $\mathbb{P}(1, 1, 4, 4)$. A degree 1 log Del Pezzo surface with Gorenstein index at most two is a degree 6 hypersurface in $\mathbb{P}(1, 1, 2, 3)$.*

Notice that by the restrictions on Gromov–Hausdorff limits of Kähler–Einstein Del Pezzo surfaces discussed in the previous section, we know that the Gorenstein index of such limits is less than or equal to 2 for degree ≥ 2, and at most 6 in the degree 1 case.

3.2. CM line bundle comparison. In this subsection we study GIT stability of Kähler–Einstein log Del Pezzo surfaces. For smooth Kähler–Einstein manifolds, it is known that they are K-polystable (cf. [**73**, **68**,

44]). This has been generalized to the singular setting in [**11**], and we state the two-dimensional case here:

Theorem 3.3 ([**11**]). *A log Del Pezzo surface admitting a Kähler–Einstein metric is K-polystable.*

Next, we state a general theorem relating K-polystability and usual GIT stabilities, using the CM line bundle of Paul and Tian [**60**]. Recall that the CM line bundle is a line bundle defined on the base scheme of each flat family of polarized varieties in terms of the Deligne pairing, and that if the family is G-equivariant with an algebraic group G, the line bundle naturally inherits the group action. It gives a GIT weight interpretation to the Donaldson–Futaki invariant whose positivity is roughly the K-stability. A point is that the CM line bundle is *not* even nef in general so that we cannot apply GIT straightforward. We refere to [**60**, **61**] for more details.

Theorem 3.4. *Let G be a reductive algebraic group without non-trivial characters. Let $\pi\colon (\mathcal{X}, \mathcal{L}) \to S$ be a G-equivariant polarized projective flat family of equidimensional varieties over a projective scheme. Here "polarized" means that $\mathcal{L}$ is a relatively ample line bundle on $\mathcal{X}$, and "equidimensional" means that all the irreducible components have the same dimension. Suppose that*

1) *the Picard rank $\rho(S)$ is 1;*
2) *there is at least one K-polystable $(\mathcal{X}_t, \mathcal{L}_t)$ that degenerates in S via a one-parameter subgroup λ in G, i.e., the corresponding test configuration is not of product type.*

Then a point $s \in S$ is GIT (poly, semi)stable if $(\mathcal{X}_s, \mathcal{L}_s)$ is K-(poly, semi)stable.

Proof. Let Λ_{CM} be the CM line bundle [**60**] over S associated to π. In general, this is a G-linearized $\mathbb{Q}$-line bundle. Let Λ_0 be the positive generator of $Pic(S)$; then there exists integers $r > 0$ and k, so that $\Lambda_{CM}^{\otimes r} \cong \Lambda_0^{\otimes k}$. The isomorphism is G-equivariant by the condition that G has no non-trivial character. On the other hand, from the condition (2), we know that the degree of CM line along the closure of the λ-orbit is positive. This is because by [**78**] the degree is the sum of the Donaldson-Futaki invariant on the two degenerations along λ and λ^{-1}. This implies that the integer k is positive. Therefore, $\Lambda_{CM}^{\otimes r}$ is ample.

If $\pi\colon \mathcal{X} \to S$ is the universal polarized family over a Hilbert scheme, and G is the associated special linear group SL, then it is known [**60**] that for any $s \in S$ and one-parameter subgroup $\lambda\colon \mathbb{C}^* \to G$, the associated Donaldson–Futaki invariant [**22**] $DF((\mathcal{X}_s, \mathcal{L}_s); \lambda)$ is the GIT weight in the usual sense with respect to the CM line bundle $\Lambda_{CM}^{\otimes r}$, up to a positive multiple. This fact can be extended to our general family $\pi\colon (\mathcal{X}, \mathcal{L}) \to S$ in a straightforward way by considering a G-equivariant morphism into a certain Hilbert scheme defined by $(\mathcal{X}, (\pi^*\Lambda_0)^{\otimes l} \otimes \mathcal{L}^{\otimes m})$ for $l \gg m \gg 0$. If $\mathcal{X}_s$ is reduced, from our equidimensionality assumption on all fibers, we cannot get *almost trivial* test configurations from a one-parameter subgroup of G (in the sense of [**43**, **59**]). This is because the central fiber of an almost trivial test configuration for a reduced equidimensional variety should have an embedded component. Summarizing up, the conclusion follows from the Hilbert–Mumford numerical criterion. q.e.d.

We believe Theorem 3.4 should have more applications in the explicit study of general cscK metrics beyond our study of log Del Pezzo surfaces in this paper. For instance, there are many examples of equivariant families of polarized varieties parametrized by a projective space or Grassmanian through various covering constructions. In these situations one can always apply Theorem 3.4. We remark that in the above proof what we really need is the CM line bundle to be ample. For example, the following was known to Paul and Tian a long time ago:

Corollary 3.5 ([**72**]). *A hypersurface $X \subseteq \mathbb{P}^N$ is Chow polystable (resp. Chow semistable) if $(X, \mathcal{O}_X(1))$ is K-polystable (resp. K-semistable).*

Hence, in particular, combined with [**58**, Theorem 1.2], it follows that semi-log-canonical hypersurfaces with ample canonical classes and log-terminal Calabi–Yau hypersurfaces are GIT stable. This is just one of the easiest examples of applications of Theorem 3.4.

We also state the following *local* version of Theorem 3.4, which we also believe to be a fundamental tool for future developments.

Lemma 3.6. *Let S be an affine scheme, and let G be a reductive algebraic group acting on S fixing $0 \in S$. Let $\pi\colon (\mathcal{X}, \mathcal{L}) \twoheadrightarrow S$ be a G-equivariant polarized flat projective deformation of a K-polystable reduced polarized variety $(\mathcal{X}_0, \mathcal{L}_0)$, and suppose all fibers $\mathcal{X}_s$ are equidimensional varieties. We assume that if $(\mathcal{X}_{s_1}, \mathcal{L}_{s_1})$ is isomorphic to $(\mathcal{X}_{s_2}, \mathcal{L}_{s_2})$, then $s_2 \in Gs_1$ and that the G-action on S is faithful. Then there is an affine neighborhood S' of 0 such that the CM line bundle is equivariantly trivial over S'. For such S', it holds that a point $s \in S'$ is GIT (poly)stable if $(\mathcal{X}_s, \mathcal{L}_s)$ is K-(poly)stable.*

This follows from similar arguments as in the proof of Theorem 3.4, and we write a detailed proof here for the convenience of readers.

Proof. First, let us prove that the CM line bundle λ_{CM} is locally G-equivariant trivial around $0 \in S$. Pick an arbitrary but sufficiently ample G-equivariant line bundle λ_B and put $\lambda_A := \lambda_{CM} \otimes \lambda_B$. We can assume λ_B is also ample. By multiplying an appropriate character of G to change the G-linearizations on λ_A and λ_B if necessary, we can assume the G-action at $\lambda_A|_0$ and $\lambda_B|_0$ is trivial. It is possible since, from our assumption, G-action on $\lambda_{CM}|_0$ is trivial. Embed S via λ_A into projective space and take a compactification simply as the Zariski closure $\bar{S}_A$. Then applying the Hilbert-Mumford criterion to $(\bar{S}_A, \lambda_A)$, we can see that there is some G-invariant section s_1 which does not vanish at 0. It is indeed the original definition of GIT semi stability. Note this implies the equivariant triviality of λ_A at the locus where s_1 does not vanish. Do the same for λ_B and we get s_2, a G-invariant section of B non-vanishing around 0. Then we let S' be the locus where both s_1 and s_2 do not vanish. This is of course affine again. Over S', A and B are both equivariantly trivial, as well as it is their difference that is exactly equal to our CM line bundle.

Suppose $s \in S'$ is not polystable in the GIT sense, although $(\mathcal{X}_s, \mathcal{L}_s)$ is K-polystable. The non-polystability implies that it degenerates to $s' \in (S' \setminus Gs)$ via one parameter subgroup $\lambda \colon \mathbb{C}^* \to G$. This gives a non-product test configuration but its Donaldson-Futaki invariant vanishes, due to the equivariant triviality of the CM line bundle over S'. This contradicts. So we proved the assertion. q.e.d.

One can often apply this to the versal deformation family, as we do in the Section 3.3. We remark that in general if the base S is replaced by an open analytic subset of S, we can also work locally analytically provided S is smooth at 0. Let K be a maximal compact subgroup of G, and let A be the tangent space of S at 0. Since G fixes 0, it induces a linear G action on A. We fix a K-invariant Hermitian metric on A. Then by standard slice theory for compact group actions we can find a K-invariant analytic neighborhood U of 0 in S, a ball $B_r(0)$ in A, and a K-equivariant biholomorphic map from U to $B_r(0)$. So we can identify U with $B_r(0)$—in particular, the CM line bundle λ_{CM} restricts to $B_r(0)$. Our statement then becomes that a point $s \in B_r(0)$ is GIT (poly)stable if $(\mathcal{X}_s, \mathcal{L}_s)$ is K-(poly)stable. To prove this, by making r small we may choose a non-vanishing holomorphic section s of λ_{CM} over $B_r(0)$. Now define $\tilde{s}(x) = \int_K g^{-1}.s(g.x)dg$, where dg is a bi-invariant Harr measure on K. Then $\tilde{s}$ is a K-invariant holomorphic section over $B_r(0)$, and hence G- invariant (more precisely, it is invariant in the Lie algebra level since $Lie(G) = Lie(K)^{\mathbb{C}}$). On the other hand, since G fixes zero, and $\mathcal{X}_0$ is K-polystable, G acts trivially on the fiber of λ_{CM} over 0. So $\tilde{s}(0) = s(0) \neq 0$. By making r smaller again we may assume $\tilde{s}$ is also nowhere vanishing over $B_r(0)$. Now suppose $x \in B_r(0)$ is not polystable in the GIT sense; then there is a unique polystable orbit $G.x'$ in the closure of the G orbit of x. Moreover by the Kempf-Ness theorem it is easy to see that we may assume x' is also in $B_r(0)$ (for example, we can choose x' to satisfy the moment map equation $\mu(x') = 0$), and there is a one-parameter subgroup $\lambda \colon \mathbb{C}^* \to G$ that degenerates x to x'. Since $\tilde{s}$ is non-vanishing and $Lie(G)$-invariant, it follows that the Donaldson–Futaki invariant as the weight of the action of the $\mathbb{C}^*$ action on the fiber of λ_{CM} over x' must vanish. By assumption, $\mathcal{X}_{x'}$ is not isomorphic to $\mathcal{X}_s$; this implies $\mathcal{X}_x$ is not K-polystable.

3.3. Semi-universal $\mathbb{Q}$-Gorenstein deformations. In this subsection we provide some general theory on $\mathbb{Q}$-Gorenstein deformations, continuing Section 2. Most of the general theory we review in the former half of this subsection should be well-known to experts of deformation theory, but we review them for convenience. First, the following is well known; see, for example, the Main Theorem in page 2 of [**47**]:

Lemma 3.7 ([**39**, **47**])**.** *A T-singularity has a smooth semi-universal $\mathbb{Q}$-Gorenstein deformation.*

A T-singularity is either Du Val (ADE type), which is a hypersurface singularity in $\mathbb{C}^3$ and has a smooth semi-universal $\mathbb{Q}$-Gorenstein deformation space, or a cyclic quotient of type $\frac{1}{dn^2}(1, dna-1)$ with $(a, n) = 1$. The latter is the quotient of the Du Val singularity A_{dn-1} by the group $\mathbb{Z}/n\mathbb{Z}$. More precisely, an A_{dn-1} singularity embeds as a hypersurface $z_1 z_2 = z_3^{dn}$ in $\mathbb{C}^3$. The generator ζ_n of $\mathbb{Z}/n\mathbb{Z}$ acts on $\mathbb{C}^3$ by $\zeta_n.(z_1, z_2, z_3) = (\zeta_n z_1, \zeta_n^{-1} z_2, \zeta_n^a z_3)$, where ζ_n is the nth root of unity. One can explicitly write down a semi-universal $\mathbb{Q}$-Gorenstein deformation as the family of hypersurfaces in $\mathbb{C}^3$ given by $z_1 z_2 = z_3^{dn} + a_{d-1} z_3^{(d-1)n} + \cdots + a_0$; see [**47**]. Then its dimension is d.

Moreover, it is also known that T-singularities are the only quotient surface singularities which admit $\mathbb{Q}$-Gorenstein smoothings.

Furthermore, for log Del Pezzo surface X which only has T-singularities, since $H^2(T_X) = 0$ (cf., e.g., [**30**, Proposition 3.1]), we know that X has global $\mathbb{Q}$-Gorenstein smoothing. Summing up, we have the following:

Lemma 3.8. [**39**, Proposition 3.10], *and* [**30**, Proposition 2.2]. *Let X be a log Del Pezzo surface. Then X is $\mathbb{Q}$-Gorenstein smoothable if and only if it has only T-singularities.*

We give a more precise structure of the deformations as follows, which should be certainly known to experts.

Lemma 3.9. *Let X be a $\mathbb{Q}$-Gorenstein smoothable log Del Pezzo surface with singularities $p_1, \cdots, p_n$. Then for the $\mathbb{Q}$-Gorenstein deformation tangent space Def(X) of X, we have*

$$0 \to Def'(X) \to Def(X) \to \bigoplus_{i=1}^{n} Def_i \to 0,$$

where $Def'(X)$ is the subspace of $Def(X)$ corresponding to equisingular deformations and Def_i is the $\mathbb{Q}$-Gorenstein deformation tangent space of the local singularity p_i. Notice $Aut(X)$ naturally acts on $Def'(X)$ as well as on $Def(X)$.

Moreover, if Aut(X) is a reductive group, there is an affine algebraic scheme $(Kur(X), 0)$ with tangent space Def(X) at 0, and a semi-universal $\mathbb{Q}$-Gorenstein family $\mathcal{U} \to (Kur(X), 0)$ that is Aut(X)-equivariant, and the induced action on Def(X) is the natural one as above. Here Aut(X) denotes the automorphism group of X.

We give a sketch of the proof here. In general, there is a tangent-obstruction theory for deformation of singular reduced varieties, with tangent space $\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$ and obstruction space $\mathrm{Ext}^2(\Omega_X, \mathcal{O}_X)$. Since X has only isolated singularities and $H^2(X, T_X) = 0$ (cf. [**30**, Proposition 3.1]) in which the local-to-global obstructions lie in general, we have the following natural exact sequnce due to the local-to-global spectral sequence of Ext:

$$0 \to H^1(T_X) \hookrightarrow Ext^1(\Omega_X, \mathcal{O}_X) \twoheadrightarrow \oplus_{x \in X} \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X) \to 0.$$

It is well-known that $H^1(T_X) = Def'(X)$; i.e., it is the first-order deformation tangent space of equisingular deformations. The local obstruction for deforming singularities lies in the map

$$\begin{aligned} H^0(\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) = \bigoplus_{i=1}^{n} \mathcal{E}xt^1_{p_i}(\Omega_X, \mathcal{O}_X) &\to H^0(\mathcal{E}xt^2(\Omega_X, \mathcal{O}_X)) \\ &= \bigoplus_{i=1}^{n} \mathcal{E}xt^2_{p_i}(\Omega_X, \mathcal{O}_X). \end{aligned}$$

Hence, restricting the above exact sequence to the subspace of $\oplus_{x \in X} \mathcal{E}xt^1 (\Omega_X, \mathcal{O}_X)$ that corresponds to $\mathbb{Q}$-Gorenstein deformation (cf. Lemma 3.7, [**39**, 3.9(i)]), we are done.

Now let us argue the construction Kur(X). It follows from general algebraic deformation theory (or the Grauert's construction of analytic semi-universal deformation [**28**]) that there exists a formal semi-universal family $\mathcal{X} \to Spec(R)$, where R is the completion of an essentially finite-type local ring. By using the Grothendieck existence theorem [**25**] and the Artin algebraicity theorem [**5**], we obtain a semi-universal deformation. Moreover, in the semi-universal deformation, it follows from [**39**, Theorem 3.9(i)] that $\mathbb{Q}$-Gorenstein deformation corresponds to one irreducible component.

As we stated, we can even take semi-universal $\mathbb{Q}$-Gorenstein deformation space Kur(X) as an Aut(X)-equivariant affine scheme; i.e., Aut(X) acts on both the total space of the semi-universal deformation above Kur(X) and Kur(X) equivariantly while the projection is equivariant. Indeed, it follows from Luna étale slice theorem which we apply to Hilbert schemes (see [**2**, especially Section 2]) combined with [**39**, Theorem 3.9(i)].

We remark that for our main applications in this paper we will only need the existence of the versal deformation as an analytic germ, in which case we do not have the action of Aut(X), but only a holomorphic action of K for a maximal compact subgroup of Aut(X). This follows from the equivariant version of the construction of Grauert [**28**] (cf. also [**62**]).

Now we study a particular example, which we will use in Section 5.

Example 3.10. *Let X_1^T be the quotient of $\mathbb{P}^2$ by $\mathbb{Z}/9\mathbb{Z}$, where the generator ξ of $\mathbb{Z}/9\mathbb{Z}$ acts by $\zeta_9.[z_1 : z_2 : z_3] = [z_1 : \zeta_9 z_2 : \zeta_9^{-1} z_3]$ and ζ_9 is the primitive ninth root of unity. Then X_1^T is a degree one log Del Pezzo surface, with one A_8 singularity at $[1 : 0 : 0]$ and two $\frac{1}{9}(1,2)$ singularities at $[0 : 1 : 0]$ and $[0 : 0 : 1]$. In particular, it is $\mathbb{Q}$-Gorenstein smoothable and has Gorenstein index 3. Note that the Fubini-Study metric on $\mathbb{P}^2$ descends to a Kähler–Einstein metric. Since this metric has constant positive bisectional curvature, the cohomology group $H^1_{orb}(X, T_X)$(the space of harmonic T_X-valued $(0,1)$ forms on the orbifold) vanishes (see, for example,* [**19**, Proposition 9.4]*), so by the obvious orbifold generalization of the Kodaira–Spencer theory X_1^T has no equisingular deformations. By the above general theory and a dimension counting using the Main Theorem in* [**47**]*, we have a decomposition*

$$Def(X_1^T) = Def_1 \oplus Def_2 \oplus Def_3,$$

where Def_i is the $\mathbb{Q}$-Gorenstein deformation tangent space of the local singularity p_i. It is not hard to see that the connected component of the automorphism group is $Aut^0(X_1^T) = (\mathbb{C}^*)^2$. We want to identify its action on $Def(X_1^T)$. We first choose coordinates on $Aut^0(X_1^T)$ so that $\lambda = (\lambda_1, \lambda_2)$ acts on $X_1^t = \mathbb{P}^2/(\mathbb{Z}/9\mathbb{Z})$ by $\lambda.[z_1 : z_2 : z_3] = [\lambda_1 z_1 : \lambda_2 z_2 : z_3]$. Around p_3 we may choose affine coordinates $y_1 = z_1/z_3$ and $y_2 = z_2/z_3$. So the action of $\mathbb{Z}/9\mathbb{Z}$ is given by $\xi.(y_1, y_2) = (\zeta_9 y_1, \zeta_9^2 y_2)$, which is the standard model for the $\frac{1}{9}(1,2)$ singularity. The action of $(\mathbb{C}^*)^2$ is then $\lambda.(y_1, y_2) = (\lambda_1 y_1, \lambda_2 y_2)$. Now a local deformation of the affine singularity $\frac{1}{9}(1,2)$ can be seen as follows. We embed $\mathbb{C}^2/(\mathbb{Z}/9\mathbb{Z})$ into $\mathbb{C}^3/(\mathbb{Z}/3\mathbb{Z})$ by sending (y_1, y_2) to $(u, v, w) = (y_1^3, y_2^3, y_1 y_2)$. A versal deformation is given by $uv - w^3 = s$. The induced action of $(\mathbb{C}^*)^2$ is then $\lambda.s = \lambda_1^{-3}\lambda_2^{-3}s$. This is then the weight of the action on Def_3. Similarly, one can see the weight on Def_2 is given by $\lambda_1^{-3}\lambda_2^6$. To see the weight on Def_1, we can embed X_1^T into $\mathbb{P}(1,2,9,9)$ as a hypersurface $x_3 x_4 = x_2^9$, by sending $[z_1 : z_2 : z_3]$ to $[x_1 : x_2 : x_3 : x_4] = [z_1 : z_2 z_3 : z_2^9 : z_3^9]$. One can easily write down a space of deformations of X_1^T as $x_3 x_4 = x_2 \Pi_{i=1}^8 (x_2 + a_i x_i)$. This deformation only partially smoothes the A_8 singularity, so that Def_1 can be identified with the space of all vectors $(a_1, \cdots, a_8)$. It is then easy to see the weight of the action of λ on Def_1 is $\lambda_1 \lambda_2^{-1}$. So we have arrived at the following:

Lemma 3.11. *The action of $Aut^0(X_1^T)$ on $Def(X_1^T)$ is given by*

$$\lambda.(v_1, v_2, v_3) = (\lambda_1 \lambda_2^{-1} v_1, \lambda_1^{-3}\lambda_2^6 v_2, \lambda_1^{-3}\lambda_2^{-3} v_3).$$

From Lemma 3.9 we have a linear action of a group $\mathrm{Aut}(X)$ on $\mathrm{Def}(X)$. If $\mathrm{Aut}(X)$ is reductive (e.g., when X admits a Kähler–Einstein metric, by Matsushima's theorem [**50**]), one can take a GIT quotient $\mathrm{Def}(X)//\mathrm{Aut}(X)$. So we are in the situation of the remark after Lemma 3.6, and locally analytically this can be viewed as a "local" coarse moduli space of $\mathbb{Q}$-Gorenstein deformations of X. The following lemma provides a more precise link between the Gromov–Hausdorff convergence and algebraic geometry.

Lemma 3.12. *Let X_∞ be the Gromov–Hausdorff limit of a sequence of Kähler–Einstein Del Pezzo surfaces X_i; then for i sufficiently large we may represent X_i by a point u_i in an open neighborhood of the GIT quotient $Kur(X_\infty)//Aut(X_\infty)$ (analytically interpreted above as $Def(X)//Aut(X)$) so that $u_i \to 0$ as i goes to infinity.*

Proof. From Section 2, we know there are integers m, N such that by passing to a subsequence the surface X_i converges to X_∞, under the projective embedding into $\mathbb{P}^N$ defined by orthonormal section of $H^0(X_i, -mK_{X_i})$. Since X_∞ has reductive automorphism group, we can choose a Luna slice S in the component of the Hilbert scheme corresponding to $\mathbb{Q}$-Gorenstein smoothable deformations of X_∞. Hence for i large enough, X_i is isomorphic to a surface parametrized by $s_i (\in S) \to 0$. By the versality, shrinking S, we have a map $F\colon S \to \mathrm{Kur}(X_\infty)$ so that s and $F(s)$ represent isomorphic surfaces. Let $v_i = F(s_i)$. Then $v_i \to 0$. Moreover, by the remark after Lemma 3.6, the corresponding point to v_i in $\mathrm{Def}(X)$ is polystable for i large, and thus its image $u_i \in \mathrm{Kur}(X_\infty)//\mathrm{Aut}(X_\infty)$ represents the same surface X_i. The conclusion then follows. q.e.d.

3.4. Moduli spaces.

In this section we will define precisely what "moduli of Kähler–Einstein $\mathbb{Q}$-Fano varieties" means to us in this paper.

Definition 3.13 (KE moduli stack)**.** We call a moduli algebraic (Artin) stack $\mathcal{M}$ of $\mathbb{Q}$-Gorenstein family of $\mathbb{Q}$-Fano varieties a *KE moduli stack* if:

1) It has a categorical moduli M in the category of algebraic spaces;
2) There is an étale covering of $\mathcal{M}$ of the form $\{[U_i/G_i]\}$ with affine algebraic schemes U_i and reductive groups G_i, where there is a G_i-equivariant $\mathbb{Q}$-Gorenstein flat family of $\mathbb{Q}$-Fano varieties.
3) Closed orbits of $G_i \curvearrowright U_i$ correspond to geometric points of M, and parametrize $\mathbb{Q}$-Gorenstein smoothable Kähler–Einstein $\mathbb{Q}$-Fano varieties.

We call the categorical moduli in the category of algebraic space M a *KE moduli space*. If it is an algebraic variety, we also call it *KE moduli variety*.

For an introduction to the theory of algebraic stacks, one may refer to [**9**]. For the general conjecture and for more details on the existence of

KE moduli stacks, seee Section 6. For our main purposes in proving Theorem 1.1, we only need a much weaker notion.

Definition 3.14 (Analytic moduli space)**.** An *analytic moduli space* of degree d log Del Pezzo surfaces is a compact analytic space M_d with the following structures:

1) We assign to each point in M_d a unique isomorphism class of $\mathbb{Q}$-Gorenstein smoothable degree d log Del Pezzo surfaces. For simplicity of notation, we will denote by $[X] \in M_d$ a point that corresponds to the isomorphism class of the log Del Pezzo surface X.
2) For each $[X] \in M_d$ with $\mathrm{Aut}(X)$ reductive, there is an analytic neighborhood U and a quasi-finite locally biholomorphic map Φ_U from U onto an analytic neighborhood of $0 \in \mathrm{Def}(X)//\mathrm{Aut}(X)$ (where as in the remark after Lemma 3.6, we have chosen a K-equivariant identification between analytic neighborhoods in $\mathrm{Kur}(X)$ and $\mathrm{Def}(X)$) such that $\Phi_U^{-1}(0) = [X]$ and for any $u \in U$, the surfaces parametrized by u and $\Phi_U(u)$ are isomorphic.

Definition 3.15. We say that an analytic moduli space has *property (KE)* if every surface parametrized by M_d^{GH} is isomorphic to one parametrized by some point in M_d.

Theorem 3.16. *For any analytic moduli space M_d that has property (KE), there is a homeomorphism from M_d^{GH} to M_d, under the obvious map.*

Proof. To carry out the strategy described in the introduction, we just need the natural map from M_d^{GH} to M_d to be continuous. It suffices to show that if we have a sequence $[X_i] \in M_d^0$ converges to a point $[X_\infty] \in M_d^{GH}$, then $\Phi([X_i])$ converges to $\Phi([X_\infty])$. Unwrapping the definitions, this is exactly Lemma 3.12. q.e.d.

In Sections 4 and 5, we will construct the analytic moduli space M_d for $\mathbb{Q}$-Gorenstein smoothable cases one by one. We will show that these M_d's satisfy property (KE). Moreover, they are actually categorical moduli of moduli stacks $\mathcal{M}_d$, with a Zariski open subset parametrizing all smooth degree d Del Pezzo surfaces. Thus Theorem 1.1 follows.

4. The cases of degree 4 and 3

4.1. Degree 4 case. In this case Theorem 1.1 has already been proved in [**45**]. Following the general strategy outlined in the introduction, we give a partially new proof here. Recall that smooth degree 4 Del Pezzo surfaces are realized by the anti-canonical embedding as intersections of two quadrics in $\mathbb{P}^4$. So in order to construct a moduli space, it is natural to consider the GIT picture

$$PGL(5;\mathbb{C}) \curvearrowright H_4 = Gr(2, Sym^2(\mathbb{C}^5)) \hookrightarrow \mathbb{P}_*(\Lambda^2 Sym^2(\mathbb{C}^5)),$$

with a linearization induced by the Plücker embedding (where $\mathbb{P}_*(V) = \mathbb{P}(V)$ is the covariant projectivization and $\mathbb{P}^*(V) = \mathbb{P}(V^*)$ is the contravariant projectivization). Here, Gr stands for the Grassmanian.

Theorem 4.1 (Mabuchi and Mukai [**45**]). *An intersection X of two quadrics in $\mathbb{P}^4$ is*

- *stable $\Longleftrightarrow$ X is smooth;*
- *semistable $\Longleftrightarrow$ X has at worst A_1 singularities (nodes);*
- *polystable $\Longleftrightarrow$ the two quadrics are simultaneously diagonalizable, i.e., X is isomorphic to the intersection of quadrics*

$$\begin{cases} x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \\ \lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 = 0 \end{cases}$$

and no three of the λ_is are equal (or equivalently, X is either smooth or has exactly two or four A_1 singularities).

Since two degree 4 Del Pezzo surfaces as considered above are abstractly biholomorphic if and only if their equations in the above embeddings are transformed by the natural action of an element of $PGL(5;\mathbb{C})$ (this follows by the very ampleness of the anti-canonical line bundle), the GIT quotient

$$M_4 := H_4^{ss}//PGL(5;\mathbb{C})$$

parametrizes (abstract) isomorphism classes of polystable intersections of two quadrics. We remark here that the same reasoning should be applied later in the other degree cases.

Moreover, since M_4 is naturally coarsely isomorphic to the moduli space of binary quintics on $\mathbb{P}^1$, choosing invariants as in [**21**], Chapter 10.2, we see that M_4 is isomorphic to $\mathbb{P}(1,2,3)$ and that the smooth surfaces are parametrized by the Zariski open subset $M_4^{\mathrm{sm}} \cong \mathbb{P}(1,2,3) \setminus D$, where D is an ample divisor cut out by the equation $z_1^2 = 128 z_2$.

The $d = 4$ case of Theorem 1.1 then follows from the following:

Theorem 4.2. *The above-constructed M_4 is an analytic moduli space with property (KE).*

Proof. First, we verify that our degenerations of Del Pezzo surfaces in $\mathbb{P}^4$ can *not* give any "pathological test configuration" in the sense of [**43**] (called "almost trivial" test configuration in [**59**]) whose normalization is trivial. This is due to the following reason. The central fibers of such pathological test configurations are not equidimensional, so they are especially *not* Cohen–Macaulay. However, the degenerations in $\mathbb{P}^4$ that we considered here are all Cohen–Macaulay. This is because, in general, weighted projective spaces only have quotient singularities, and so are Cohen–Macaulay (cf., e.g., [**34**]). Then the finite times cut by Cartier divisors (hypersurfaces) are inductively Cohen–Macaulay (cf., e.g., [**49**,

p. 105]). Later, we will use the similar reasoning for other degrees as well.

To check that M_4 is an analytic moduli space, observe that item (1) is obvious and item (2) follows from the construction of M_4 as a GIT quotient (the versal family is the universal one over H_4). To see that M_4 has property (KE), we first use Theorem 3.1 to see that any $[X] \in M_4^{GH}$ is parametrized by H_4. Then we apply Theorem 3.3 and Theorem 3.4 (since Picard rank of H_4 is 1, and it is easy to verify the assumptions are satisfied in this case) to see that $[X]$ is parametrized by M_4. q.e.d.

Clearly $\mathcal{M}_4 := [H_4^{ss}/PGL(5;\mathbb{C})]$ is a quotient stack, so we conclude that it is indeed a KE moduli stack. We make a few remarks here. First of all, the above arguments actually prove that all degree 4 Kähler–Einstein log Del Pezzo surfaces are parametrized by M_4. By Theorem 4.1 the Gromov–Hausdorff limits of smooth Del Pezzo quartics have only an even number of A_1 singularities. The maximum number of such singularities is four. There is exactly one such surface X_4^T, which is defined by the equations $x_0x_1 = x_2^2 = x_3x_4$. It is isomorphic to the quotient $\mathbb{P}^1 \times \mathbb{P}^1/(\mathbb{Z}/2\mathbb{Z})$, where the generator ξ of $\mathbb{Z}/2\mathbb{Z}$ acts as $\xi.(z_1, z_2) = (-z_1, -z_2)$. So it admits an obvious Kähler–Einstein metric.

It is also easy to see that the action of complex conjugation, which sends a Del Pezzo quartic to its complex conjugate, coincides with the natural complex conjugation on $\mathbb{P}(1, 2, 3)$.

4.2. Degree 3 case. Recall that smooth degree 3 Del Pezzo surfaces are cubic hypersurfaces in $\mathbb{P}^3$. Note that the anti-canonical bundle is very ample. We recall the following classical GIT picture. The group $PGL(4;\mathbb{C})$ acts naturally on the space $H_3 = \mathbb{P}_*(Sym^3(\mathbb{C}^4)) \cong \mathbb{P}^{19}$ of cubic polynomials.

Theorem 4.3 (Hilbert)**.** *A cubic surface X in $\mathbb{P}^3$ is*

- *stable $\iff$ X has at most singularities of type A_1;*
- *semistable $\iff$ X has at worst singularities of type A_1 or A_2;*
- *strictly polystable $\iff$ X is isomorphic to the cubic X_3^T defined by equation $x_1x_2x_3 = x_0^3$. It is not hard to see that X_3^T has exactly three A_2 singularities and is isomorphic to the quotient $\mathbb{P}^2/(\mathbb{Z}/3\mathbb{Z})$, where the generator ξ of $(\mathbb{Z}/3\mathbb{Z})$ acts by $\xi.[z_1 : z_2 : z_3] = [z_1 : e^{2\pi i/3}z_2 : e^{-2\pi i/3}z_3]$.*

Define the quotient stack $\mathcal{M}_3 := [H_3^{ss}/PGL(4;\mathbb{C})]$ and the corresponding GIT quotient (or, in other words, categorical moduli)

$$M_3 := H_3^{ss}//PGL(4;\mathbb{C})$$

which parametrizes isomorphism classes of polystable cubics. The above Theorem is classical. It was proved by D. Hilbert in his doctoral dissertation [**32**]. For a modern proof consult [**53**]. Moreover, by looking at the ring of invariants [**63**], it is known that

$$M_3 \cong \mathbb{P}(1,2,3,4,5),$$

and that $M_3^{\mathrm{sm}} \cong \mathbb{P}(1,2,3,4,5) \setminus D$, where D is the ample divisor of equation $(z_1^2 - 64z_2)^2 - 2^{11}(8z_4 + z_1 z_3) = 0$. So M_3^{sm} is Zariski open and parametrizes all smooth cubic surfaces.

Note that we can apply Theorem 3.4 for a universal family over H_3. Thus it follows that $\mathcal{M}_3$ is a KE moduli stack and M_3 is a KE moduli variety.

Observe that a Gromov–Hausdorff limit of smooth Kähler–Einstein cubic surfaces has either exactly three A_2 singularities or at most four A_1 singularities. In the former case, it is isomorphic to X_3^T. In the latter case, this is the Cayley's cubic X_3^C defined by $x_0x_1x_2+x_1x_2x_3+x_2x_3x_0+x_3x_0x_1 = 0$. It is not hard to see it is isomorphic to the quotient of $X_6/(\mathbb{Z}/2\mathbb{Z})$, where X_6 is the degree 6 Del Pezzo surface and the action of $\mathbb{Z}/2\mathbb{Z}$ is induced by the standard Cremona transformation on $\mathbb{P}^2$, i.e., $[z_1 : z_2 : z_3] \mapsto [z_1^{-1} : z_2^{-1} : z_3^{-1}]$. The existence of Kähler–Einstein metrics on X_3^T and X_3^C can also be easily seen using the above quotient description.

We remark that it was proved in [**20**] that a Kähler–Einstein cubic surface must be GIT semistable, and our application of Theorem 3.4 sharpens this. The existence of Kähler–Einstein metrics on cubic surfaces with exactly one A_1 singularity was proved by [**77**], using Kähler–Ricci flow on orbifolds and a certain calculation of α-invariants. In [**67**], by the glueing method we know the existence of Kähler–Einstein metrics on a partial smoothing of the Cayley cubic X_3^C. For general cubics with two or three A_1 singularities this was previously unknown. Here we actually know that all degree 3 $\mathbb{Q}$-Gorenstein smoothable Kähler–Einstein log Del Pezzo surfaces are parametrized by M_3.

As in the degree 4 case, the action of complex conjugation on M_3 is also given by the natural anti-holomorphic involution.

5. The cases of degree 2 and degree 1

5.1. More detailed study of Gromov–Hausdorff limits. When the degree is 1 or 2, there are new difficulties as non-canonical singularities could appear in the Gromov–Hausdorff limits. So the classification of canonical Del Pezzo surfaces (Theorem 3.1) is not enough for our purpose. In degree 2, by Theorem 2.7 we only need to deal with index 2 log del Pezzo surfaces, which have been classified in [**3**], [**55**], and [**37**]. We could simply use these classification results directly, but since our assumption is much more restricted, we provide a more elementary approach that treats both $d = 1$ and $d = 2$ cases.

A common feature of the two cases is the existence of a holomorphic involution. For a degree 2 Del Pezzo surface X, it is well-known that the anti-canonical map defines a double cover of X to $\mathbb{P}^2$. Therefore, X admits an involution σ ("Geiser involution") that is simply the deck transformation of the covering map. The fixed locus of σ is smooth quartic curve. If X admits a Kähler–Einstein metric ω, then by [**8**] ω must be invariant under any such σ. Similarly for a degree 1 Del Pezzo surface X, the linear system $|-2K_X|$ defines a double cover of X to $\mathbb{P}(1,1,2) \subset \mathbb{P}^3$. So X also admits an involution σ ("Bertini involution"). Again, any such σ must preserve the Kähler–Einstein metric if X admits one. The fixed locus of σ consists of the point $[0:0:1]$ and a sextic in $\mathbb{P}(1,1,2)$.

Lemma 5.1. *Suppose a sequence of degree 1 (or 2) Kähler–Einstein Del Pezzo surfaces (X_i, ω_i, J_i) converges to a Gromov–Hausdorff limit $(X_\infty, \omega_\infty, J_\infty)$, then by passing to a subsequence one can take a limit σ_∞, which is a holomorphic involution on X_∞.*

Proof. This is certainly well known. We include a proof here for the convenience of readers. Let $p_1, \cdots, p_n$ be the singular points of X_∞. We denote $\Omega_r = X_\infty \setminus \cup_{j=1}^n B(p_j, r)$. For any $r > 0$ small, from the convergence theorem 2.1 we know that for i sufficiently large, there are σ_i invariant open subsets $\Omega_i \subset X_i$ and embeddings $f_i : \Omega_i \to X_\infty \setminus \{p_1, \cdots, p_n\}$ such that Ω_r is contained in the image of each f_i and $(f_i^{-1})^*(\omega_i, J_i)$ converges to $(\omega_\infty, J_\infty)$ smoothly. Then, by passing to a subsequence, the isometries $(f_i^{-1})^*\sigma_i$ converge to a limit $\sigma_{r,\infty} : \Omega_r \to X_\infty$ with $\sigma_{r,\infty}^*(\omega_\infty, J_\infty) = (\omega_\infty, J_\infty)$. Then we can let r tend to zero and choose a diagonal subsequence so that $\sigma_{r,\infty}$ converges to a holomorphic isometry σ_∞ on $X_\infty \setminus \{p_1, \cdots, p_n\}$. Then by the Hartog's extension theorem, σ_∞ extends to a holomorphic isometry on the whole X_∞. It is also clear σ_∞^2 is the identity. q.e.d.

Theorem 5.2. *In the degree 2 case, X_∞ is either a double cover of* $\mathbb{P}^2$ *branched along a quartic curve, or a double cover of* $\mathbb{P}(1,1,4)$ *branched along a degree 8 curve not passing through the vertex* $[0:0:1]$*. In the degree 1 case, X_∞ is either a double cover of* $\mathbb{P}(1,1,2)$ *branched along the point* $[0:0:1]$ *and a sextic , or a double cover of* $\mathbb{P}(1,2,9)$ *branched along the point* $[0:1:0]$ *and a degree 18 curve not passing through the vertex* $[0:0:1]$.

Proof. We first treat the case of degree 1. The proof of the degree 2 case is essentially the same, and we will add some remarks later. Denote by Y_i the quotient of X_i by σ_i, so the quotient $Y_\infty = X_\infty/\sigma_\infty$ is the Gromov–Hausdorff limit of Y_i's. For each integer m we have an orthogonal decomposition $H^0(X_i, -mK_{X_i}) = V_i \oplus W_i$ with V_i being the $+1$

eigenspace and W_i the -1 eigenspace. Then we have a corresponding decomposition $H^0(X_\infty, -mK_{X_\infty}) = V_\infty \oplus W_\infty$ on X_∞. Now, by constructing orthonormal σ_∞-invariant sections of $-kK_{X_\infty}$ for some k large divisible, one can show that there is a well-defined map $\iota_\infty : X_\infty \to \mathbb{P}^*(V_\infty)$, which induces a projective embedding of Y_∞. By an adaption of the Hörmander technique ([**70**, **23**]), this implies that the orthonormal σ_i-invariant sections of $-kK_{X_i}$(equivalent to sections of $-lK_{Y_i}$ for some integer l) define an embedding (Tian's embedding) of Y_i into $\mathbb{P}^*(V_i)$ for i sufficiently large. Moreover, we may assume Y_i converges to Y_∞ as normal varieties in $\mathbb{P}^N$ for some integer N. Since Y_i's are all isomorphic to $\mathbb{P}(1,1,2)$, we see that Y_∞ is $\mathbb{Q}$-Gorenstein smoothable and there is a partial $\mathbb{Q}$-Gorenstein smoothing of Y_∞ to $\mathbb{P}(1,1,2)$.

Claim 5.3. *Y_∞ is isomorphic to* $\mathbb{P}(1,2,9)$.

Since the degree is preserved in the limit, we have $K^2_{Y_\infty} = 8$, and thus we may apply [**30**]. Notice the full proof in [**30**] relies on the classification theorem of Alexeev and Nikulin [**3**], but in our case we only need the more elementary part [**29**], without use of [**3**]. So we know Y_∞ is either a toric log Del Pezzo surface $\mathbb{P}(a^2, b^2, 2c^2)$ with $a^2 + b^2 + 2c^2 = 4abc$ or its partial smoothings. Since the orbifold structure group of X_∞ always has order less than 12, the order of all the orbifold structure groups of Y_∞ must be less than or equal to 22. Then, by an easy investigation of the above Markov equation, we see that Y_∞ must have two singularities, one of type A_1 and one of type $\frac{1}{9}(1,2)$. It could be possible that Y_∞ is a partial smoothing of $\mathbb{P}(9, b^2, 2c^2)$, but we claim then it must be $\mathbb{P}(1,2,9)$. For this we need to go back to the proof in [**29**]. For the minimal resolution $\pi : \widetilde{Y}_\infty \to Y_\infty$, let n be the largest number such that there is a birational morphism μ_n from $\widetilde{Y}_\infty$ to the nth Hirzebruch surface $\mathbb{F}_n$. Let B' be the proper transform of the negative section B in $\mathbb{F}_n$, and let $p : \widetilde{Y}_\infty \to \mathbb{P}^1$ be the composition of μ_n with the projection map on $\mathbb{F}_n$. Then by a theorem of Manetti [**47**, Theorem 11] (see also [**29**, Theorem 5.1]) we know that $n \geq 2$, and the exceptional locus E of π is the union of B' and the components of degenerate fibers of p with self-intersection at most -2; furthermore, each degenerate fiber of p contains a unique -1 curve. Moreover, by the proof of Theorem 18 in [**47**] (see also Theorem 5.7 in [**29**]), there are only two possible types for the dual diagram of the degenerate fiber: one type is that two strings of curves of self-intersection at most -2 joined by a (-1)-curve, and the other type is that we join a string of (-2)-curves though a (-1) curve to the middle of a string of curves of self-intersection at most -2. In our case we know Y_∞ has exactly one A_1 and one $\frac{1}{9}(1,2)$ singularity. By general theory on the resolution of cyclic quotient singularities we know E is the disjoint union of a (-2)-curve and a string of a (-2)-curve and a (-5)-curve. Then one easily sees that the only possibility is that there is exactly one degenerate fiber of $\widetilde{Y}_\infty$ which consists of a string of (-2)-(-1)-(-2)-curve, and one of the (-2)-curves in the string intersects the horizontal section B' which is a (-5)-curve. Clearly $\widetilde{Y}_\infty$ is then a toric blown-up of $\mathbb{F}_5$ and then Y_∞ is toric that must be $\mathbb{P}(1,2,9)$. This completes the proof of the claim.

The degree of the branched locus follows from the Hurwitz formula for coverings. The degree 18 curve cannot pass through the point $[0:0:1]$, for otherwise the equation would be $a_0 x_3 f_9(x_1,x_2) + a_1 f_{18}(x_1,x_2) = 0$. Then by Lemma 5.6, below, the singularity on the branched cover is not quotient, so cannot be X_∞ by Theorem 2.1. This finishes the proof of Theorem 5.2 for the degree 1 case.

In the degree 2 case we can follow exactly the same arguments, noticing that, in this case, Y_∞ must have degree equal to 9 and that the associated Markov-type equation to be satisfied is now $a^2+b^2+c^2 = 3abc$ (corresponding to the weighted projective space $\mathbb{P}(a^2,b^2,c^2)$). Thus it follows, by inspection as above, that the only possibility is that $Y_\infty = \mathbb{P}(1,1,4)$ (since the standard projective plane is the only $\mathbb{Q}$-Gorenstein smoothing of the above weighted projective space). q.e.d.

Noticing that in the above situation our double cover can be realized as a hypersurface in a one-dimensional or higher weighted projective space, in terms of equations we have the following:

Corollary 5.4. *In degree 1 case X_∞ is either a sextic hypersurface in $\mathbb{P}(1,1,2,3)$ of the form $x_4^2 = f_6(x_1,x_2,x_3)$ or a degree 18 hypersurface in $\mathbb{P}(1,2,9,9)$ of the form $x_3^2 + x_4^2 = f_{18}(x_1,x_2)$.*

Corollary 5.5. *In degree 2 case X_∞ is either a quartic hypersurface in $\mathbb{P}(1,1,1,2)$ of the form $x_4^2 = f_4(x_1,x_2,x_3)$ or an octic hypersurface in $\mathbb{P}(1,1,4,4)$ of the form $x_3^2 + x_4^2 = f_8(x_1,x_2)$.*

Lemma 5.6. *Suppose f is a polynomial and the surface $w^2 = f(x,y)$ in $\mathbb{C}^3$ or its $\mathbb{Z}/2\mathbb{Z}$ quotient by $(x,y,w) \mapsto (-x,-y,-w)$ has a quotient singularity at the origin; then f must contain a monomial with degree at most 3.*

Proof. If the singularity is a quotient singularity, then singularity $w^2 = f(x,y)$ in $\mathbb{C}^3$ is canonical since the finite map does not have branch divisor. Then the statement follows from a criterion of canonicity in terms of Newton polygon (cf., e.g., [**35**, Corollary 1.7]).

q.e.d.

The idea of using involutions to study X_∞ was previously used in [**70**], where some partial results were claimed. For example, in Proposition 6.1 of [**70**], it was stated that in degree 2 case X_∞ can have at most $\frac{1}{4}(1,1)$ singularities besides canonical singularities and $|-2K_{X_\infty}|$ is base point free. This agrees with the above result. But, as one can see from the following example, the claims in Proposition 6.2 of [**70**] that in the degree 1 case X_∞ can have at most one non-canonical singularity and $|-2K_{X_\infty}|$ is base point free, are both incorrect.

Now we show explicit examples of Kähler–Einstein log Del Pezzo surfaces with non-canonical singularities in both degree 1 and 2. (We are indebted to A. Kasprzyk for discussions related to these examples [**38**].) In the next two subsections it will be proved that both are parametrized in the moduli spaces.

Example 5.7. *Let X_2^T be the quotient of $\mathbb{P}^1\times\mathbb{P}^1$ by the action of $\mathbb{Z}/4\mathbb{Z}$, where the generator ξ of $\mathbb{Z}/4\mathbb{Z}$ acts by $\xi.([z_1:z_2],[w_1:w_2]) = ([\sqrt{-1}z_1:z_2],[-\sqrt{-1}w_1:w_2])$. Then it is easy to see that X_2^T is a degree 2 log Del Pezzo surface, with two A_3 singularities and two $\frac{1}{4}(1,1)$ singularities. The standard product of round metrics on $\mathbb{P}^1\times\mathbb{P}^1$ descends to a Kähler–Einstein metric on X_2^T. The space $H^0(X_2^T,-K_{X_2^T})$ is spanned by the sections $z_1^2w_1^2$, $z_2^2w_2^2$, and $z_1z_2w_1w_2$. So a generic divisor in $|-K_{X_2^T}|$ is given by the union of two curves $z_1w_1+az_2w_2=0$ and $z_1w_1+bz_2w_2=0$ for $a\neq b$, and is thus reducible. The space $H^0(X_2^T,-2K_{X_2^T})$ is spanned by sections $z_1^4w_1^4, z_2^4w_2^4, z_1^2z_2^2w_1^2w_2^2, z_1^3z_2w_1^3w_2, z_1z_2^3w_1w_2^3, z_1^4w_2^4, z_2^4w_1^4$. The subspace U spanned by the first five sections is generated by $H^0(X_2^T,-K_{X_2^T})$. The involution σ maps $([z_1:z_2],[w_1:w_2])$ to $([w_1:w_2],[z_1:z_2])$. The $+1$ eigenspace V_1 is still six dimensional, spanned by U and the element $z_1^4w_2^4+z_2^4w_1^4$. It is easy to see that the image of X under the projection to V is the cone over the rational normal curve of degree 4, i.e., $\mathbb{P}(1,1,4)$. The branch locus is defined by $z_1^4w_2^4=z_2^4w_1^4$, with singularies exactly at the two A_3 singularities. We can also see directly that X_2^T is the hypersurface in $\mathbb{P}(1,1,4,4)$ defined by $x_1^4x_2^4=x_3x_4$. The map is given by*

$$([z_1:z_2],[w_1:w_2])\mapsto(z_1w_1,z_2w_2,z_1^4w_2^4,z_2^4w_1^4).$$

Make a change of variable $x_3'=x_3+x_4$ and $x_4'=x_3-x_4$; then the projection to the (x_1,x_2,x_3') plane realizes X_2^T as a double cover of $\mathbb{P}(1,1,4)$.

Example 5.8. *Let X_1^T be the example studied in Section 3. It is a toric degree one Kähler–Einstein log Del Pezzo surface with one A_8 singularity and two $\frac{1}{9}(1,2)$ singularities. It can be viewed as a hypersurface in $\mathbb{P}(1,2,9,9)$ given by the equation $x_3x_4=x_2^9$. The embedding is defined by*

$$[z_1:z_2:z_3]\mapsto[z_1:z_2z_3:z_2^9:z_3^9].$$

The projection map $\mathbb{P}(1,2,9,9)\to\mathbb{P}(1,2,9)$ sending $[x_1:x_2:x_3:x_4]$ to $[x_1:x_2:x_3+x_4]$ realizes X_1^T as the double cover of $\mathbb{P}(1,2,9)$, branched along the rational curve $x_2^9=x_3^2$. On X_1^T the holomorphic involution σ simply exchanges z_2 with z_3. One can see the pluri-anticanonial linear systems on X_1^T. $H^0(X_1^T,-K_{X_1^T})$ is spanned by z_1^3 and

*$z_1z_2z_3$, so it has a fixed component $z_1 = 0$. $H^0(X_1^T, -2K_{X_1^T})$ is spanned by $z_1^6, z_1^4z_2z_3, z_1^2z_2^2z_3^2, z_2^3z_3^3$, so it has two base points $[0:1:0]$ and $[0:0:1]$. We will show below that X_1^T is the Gromov–Hausdorff limit of a sequence of Kähler–Einstein degree one Del Pezzo surfaces. This implies that the Proposition 6.2 in [**70**] is incorrect. Similarly, it is easy to see that $|-3K_{X_1^T}|$ is base point free. As before we have an eigenspace decomposition $H^0(X_1^T, -mK_{X_1^T}) = V_m \oplus W_m$ for σ. Then $|V_6|$ is base point free, and it defines the embedding of $\mathbb{P}(1,2,9)$ into $\mathbb{P}^{15}$ by sections of $\mathcal{O}(18)$.*

5.2. Degree 2 case. We first recall the moduli space constructed in [**52**]. For a smooth Del Pezzo surface X of degree 2 the anti-canonical map is a double covering to $\mathbb{P}^2$ branched along a smooth quartic curve F_4. The geometric invariant theory for quartic curves is well-understood (cf. [**53**]) as follows. (Note that Mukai's citation [**52**, 9.3] misses one case.)

Lemma 5.9 ([**33**], Theorem 2)**.** *For a quartic curve F_4 in $\mathbb{P}^2$ we have:*

- *F_4 is stable if and only if F_4 has only rational double points of type A_1 or A_2;*
- *F_4 is strictly polystable if and only if F_4 is one of the following: either a double conic or a union of two reduced conics that are tangential at two points and at least one is smooth (called cateye and ox in [**33**]).*

It follows that the quotient $Q := \mathbb{P}_*(Sym^4\mathbb{C}^3)^{ss}//PGL(3;\mathbb{C})$ parametrizes certain canonical log Del Pezzo surfaces of degree 2, away from the double conic. The stable curves parametrize surfaces with at worst A_1 or A_2 singularities, the double conic parametrize a non-normal surface with non-orbifold singularities (note in fact that the variety of equation $t^2 = (x^2+y^2+z^2)^2$ can be decomposed into irreducible factors $(t+x^2+y^2+z^2)(t-x^2+y^2+z^2) = 0$), and the other polystable curves parametrize surfaces with exactly $2A_3$ singularities. As in [**52**], we blow up the point corresponding to the double conic to obtain a new variety, denoted by M_2. Let E be the exceptional divisor. Then, as in [**64**], we know E is isomorphic to the GIT moduli space $\mathbb{P}_*(Sym^8\mathbb{C}^2)^{ss}//PGL(2;\mathbb{C})$, parametrizing binary octics $f_8(x,y)$. Moreover,

Theorem 5.10. *M_2 is an analytic moduli space of log Del Pezzo surfaces of degree 2. For any $[s] \notin E$, X_s is the double cover of $\mathbb{P}^2$ branched along the polystable quartic defined by $[s]$, and for $[s] \in E$, X_s is the double cover of $\mathbb{P}(1,1,4)$ (i.e., the cone over the rational normal curve in $\mathbb{P}^5$) branched along the hyperelliptic curve $z_3^2 = f_8(z_1,z_2)$, where f_8 is the polystable binary octic defined by $[s]$.*

The proof uses some ideas of [**64**] as written in [**52**], but note that the proof in [**64**] is incomplete regarding the existence of the moduli algebraic stack since no family has been constructed. The argument in [**64**] is curve-wise and only verifies the properness criterion formally.

Proof. Let H_4 be the Hilbert scheme of quartics in $\mathbb{P}^2$, and fix a non-degenerate conic $C = \{q = 0\}$. We identify the automorphism group of C with $PGL(2;\mathbb{C})$ (The notation $PGL(2;\mathbb{C})$ appears only in this context of this proof, so should not be confusing). Denote by Ψ the (nine-dimensional) $PGL(2;\mathbb{C})$-invariant subspace of $H^0(\mathbb{P}^2, \mathcal{O}(4))$ that corresponds to $H^0(C, \mathcal{O}(4)|_C)$. Take an affine space $\mathbb{A} \simeq \mathbb{C}^9$ in H_4 that represents $\{q^2 + f_4(x, y, z)\}$ for all quartics $f_4 \in \Psi$. From the construction, this gives a Luna étale slice. Note that the blow up $\mathbb{B}$ of $\mathbb{A}$ at 0 is a closed subvariety of $\mathbb{A} \times \mathbb{P}_*(\mathbb{A})$, and let $\mathbb{E}$ be its exceptional divisor. Let $\mathcal{B} \subset \mathbb{A} \times (\mathbb{A} \setminus \{0\})$ be the cone over $\mathbb{B}$, and let $\mathcal{E} = \{0\} \times (\mathbb{A} \setminus \{0\})$ be the cone over $\mathbb{E}$. For each point $(a, b) \in \mathcal{B}$, we can associate the curve $q^2 + b = 0$ in $\mathbb{P}^2$. These form a flat projective family $\mathcal{Q}$ over $\mathcal{B}$.

On the other hand, consider the trivial family of $(\mathbb{P}^2, C)$ over $\mathcal{B}$. We blow up $C \times \mathcal{E}$ and contract the strict transform of $\mathbb{P}^2 \times \mathcal{E}$. It is possible because $\mathcal{E}$ is a Cartier divisor in $\mathcal{B}$ and the classical degeneration (deformation to the normal cone of C) of $\mathbb{P}^2$ to $\mathbb{P}(1, 1, 4)$ over a smooth curve is constructed in the same way, so we can do it locally and glue the contraction morphism. Denote the family constructed in this way $\mathcal{P} \to \mathcal{B}$. The generic fibers are $\mathbb{P}^2$ and special fibers (those over $\mathcal{E}$) are $\mathbb{P}(1, 1, 4)$. We also obtain a natural family of conics $C_{\mathcal{P}} \subset \mathcal{P}$ over $\mathcal{B}$.

All the above process is $PGL(2;\mathbb{C}) \times \mathbb{C}^*$-equivariant. Thus we can construct $PGL(2;\mathbb{C})$-invariant complement of $\mathbb{C}q^2$ in $H^0(\mathcal{P}_u, \mathcal{O}(2C_{\mathcal{P}}))$ $(u \in \mathcal{B})$ in a continuous way, and extend the family of quartics $\mathcal{Q}|_{(\mathcal{B}\setminus\mathcal{E})}$ to the whole $\mathcal{B}$. We denote the new total space by $\mathcal{D}$. Notice that over $\mathbb{E}$ this is a family of binary octics. Then construct $\mathcal{S}$ as the double of $\mathcal{P}$ branched along $\mathcal{D}$. As everything is again $PGL(2;\mathbb{C}) \times \mathbb{C}^*$-equivariant, we can first divide by $\mathbb{C}^*$ and obtain a $\mathbb{Q}$-Gorenstein flat family S of degree two log Del Pezzo surfaces over $\mathbb{B}$.

There is still an action of $PGL(2;\mathbb{C})$ on $\mathbb{B}$. We consider GIT with respect to this action and with the $PGL(2;\mathbb{C})$-linearized line bundle $\mathcal{O}_{\mathbb{B}}(-\mathbb{E})$. The natural morphism $\mathbb{B}^{ss}//PGL(2;\mathbb{C}) \to \mathbb{A}//PGL(2;\mathbb{C})$ is an isomorphic away from $\mathbb{E} \subset B$ and $0 \in \mathbb{A}$. So this is a blow up with exceptional divisor $\mathbb{E}^{ss}//PGL(2;\mathbb{C})$. By the local picture of GIT ([**64**, Prop. 5.1]), we can see that $\mathbb{A}//PGL(2;\mathbb{C}) \to H_4^{ss}//PGL(3;\mathbb{C})$ is étale (or in differential geometric language, local biholomorphism) around 0. This follows completely the same way as in [**64**, Prop 5.1] or the proof of the famous Luna étale slice theorem. Hence, the blow up $\mathbb{B}^{ss}//PGL(2;\mathbb{C}) \to \mathbb{A}//PGL(2;\mathbb{C})$ induces blow up M_2 of $H_4//PGL(3;\mathbb{C})$.

To see that M_2 is an analytic moduli space for degree 2 log Del Pezzo surfaces, we only need to check the item (2) in the definition. For this, one simply notices that, by construction, for any $[s] \in M_2$ there is a Luna's slice V in H_4 or in $\mathbb{B}$ (depending on whether $[s]$ is in E or not). Then by versality there is an $\mathrm{Aut}(X_s)$ equivariant analytic map Ψ_U from a small analytic neighborhood $U = V//\mathrm{Aut}(X_s)$ of $[s]$ to the GIT quotient $\mathrm{Kur}(X_s)//\mathrm{Aut}(X_s)$ so that $\Phi_U^{-1}(0) = 0$. Then it follows that Ψ_U is a finite map onto an open neighborhood of 0.

In terms of étale topology one can also directly check the versality by going through our construction. We only need to check that our $(H_4^{ss} \setminus PGL(3;\mathbb{C})q^2) \coprod \mathbb{B}^{ss}$ is versal in étale topology. That is, given a $\mathbb{Q}$-Gorenstein projective family $f\colon \mathcal{X} \to S$ of our log del Pezzo surfaces of degree 2, there is a morphism $\tilde{S} \to (H_4^{ss} \setminus PGL(3;\mathbb{C})q^2) \coprod \mathbb{B}^{ss}$ compatible with fibers where $\tilde{S} \to S$ is an étale cover. For this, we can first construct a degenerating family of $\mathbb{P}^2$ to $\mathbb{P}(1,1,4)$ over S and from the $\mathbb{Q}$-Gorenstein deformation theory of $\mathbb{P}(1,1,4)$ (with one-dimensional smooth semi-universal deformation space) we know that the locus of $\mathbb{P}(1,1,4)$ should be a Cartier divisor so that we can convert the process to obtain a family of reduced quartics of $\mathbb{P}^2$. Thus we have a compatible morphism to $(H_4^{ss} \setminus PGL(3;\mathbb{C})q^2) \coprod B^{ss}$ locally in étale topological sense. q.e.d.

Remark 5.11. *In terms of algebro-geometric language, M_2 coarsely represents the algebraic stack $\mathcal{M}_2$ constructed by gluing together the quotient stacks* $[\mathbb{B}^{ss}/PGL(2;\mathbb{C})]$ *and* $[(H_4^{ss} \setminus PGL(3;\mathbb{C})q^2)/PGL(3;\mathbb{C})]$.

Remark 5.12. *Replacing the blow up and its cone as above by the weighted blow up and its quasi-cone, the argument in* [**64**] *can be completed to prove that the blow up is a coarse moduli scheme of degree two K3 surfaces and its degenerations.*

The proof of Theorem 1.1 follows from the fact that all smooth degree 2 Del Pezzo surfaces are parametrized by M_2 and from the following:

Theorem 5.13. *M_2 has property (KE).*

Proof. By Theorem 5.2 there are two possibilities for $X \in M_2^{GH}$: it is either a double cover of $\mathbb{P}^2$ branched along a quartic $f_4(x_1, x_2, x_3) = 0$ or a double cover of $\mathbb{P}(1,1,4)$ branched along a hyperelliptic octic curve $x_3^2 - f_8(x_1, x_2) = 0$. It suffices to show f_4 and f_8 are polystable. For this we use Theorem 3.3 and Theorem 3.4. When applying Theorem 3.4, in the first case we choose $S = \mathbb{P}_*(Sym^4(\mathbb{C}^3))$; in the second case we choose $S = \mathbb{P}_*(Sym^8(\mathbb{C}^2))$. Note that both these parameter spaces have Picard rank 1. Also recall the first paragraph of the proof of (4.2), which asserts that there is no pathological test configurations in the sense of [**43**] in this situation. Thus we can apply Theorem 3.16 and conclude that all the points in this glued moduli are indeed GH limits. q.e.d.

So we also conclude that $\mathcal{M}_2$ is a KE moduli stack. As it is immediately clear from the proof, the complex conjugation acts on M_2 by the natural anti-holomorphic involution.

Remark 5.14. *In* [**70**] *it is conjectured that degenerations of Kähler–Einstein Del Pezzo surfaces should have canonical singularities. In this section we have seen that this conjecture is in general false, as all the surfaces parametrized by the exceptional divisor E have exactly two non-canonical singularities of type* $\frac{1}{4}(1,1)$*. In general, dimension 1 expects the compact moduli space of smoothable* $\mathbb{Q}$*-Fano varieties to have log terminal singularities; see* [**23**]*. This type of singularities also appear to be the worst singularities allowed for K-semistability of Fano varieties, see* [**58**].

We finish this subsection with a discussion on the surfaces parametrized by the ox and cateyes, which will be used in our study of the degree 1 case. These are defined by equations in $\mathbb{P}(1,1,1,2)$ parametrized by $\lambda = [\lambda_1 : \lambda_2]$ in $(\mathbb{P}^1 \setminus \{[1:1]\})$ which we denote by X_2^λ. The equation of X_2^λ is

$$w^2 = (\lambda_1 z^2 + xy)(\lambda_2 z^2 + xy).$$

It is clear that when we interchange λ_1 and λ_2 we get isomorphic surfaces. When λ is $[1:0]$ or $[0:1]$, the branch locus is an *ox* and the surface $X_2^\infty = X_2^\lambda$ with exaclty two A_3 plus one A_1 singularities; otherwise, the branch locus is a *cateye* and X_2^λ with exactly two A_3 singularities. By Theorem 5.13 this family of surfaces all admit Kähler–Einstein metrics. As λ tends to $[1:1]$ these Kähler–Einstein surfaces converge to X_2^T, with the obvious Kähler–Einstein metric.

One can see that X_2^∞ is a global quotient of $\mathbb{P}^1 \times \mathbb{P}^1$, as follows. Consider the action of $\mathbb{Z}/4\mathbb{Z}$ on $\mathbb{P}^1 \times \mathbb{P}^1$, where the generator ξ acts by

$$\xi.([z_1 : z_2], [w_1 : w_2]) = ([-w_1 : w_2], [z_1 : z_2]).$$

Then there are exactly four points with non-trivial isotropy. Let Y be the quotient. Then the points $([0:1],[0:1])$ and $([1:0],[1:0])$ are A_3 singularities and $([1:0],[0:1])$ and $([0:1],[1:0])$ are A_1 singularities. One can see that the anti-canonical map p from Y to $\mathbb{P}^2$ is given by

$$([z_1 : z_2], [w_1 : w_2]) \mapsto (z_1^2 w_1^2 : z_2^2 w_2^2 : z_1^2 w_2^2 + z_2^2 w_1^2),$$

and that the corresponding involution to the double covering structure is

$$\sigma.([z_1 : z_2], [w_1 : w_2]) = ([w_1 : w_2], [z_1 : z_2]).$$

The branch locus is defined by $xy(z^2 - 4xy) = 0$ in $\mathbb{P}^2$, which is isomorphic to the ox. So Y is exactly X_2^∞, and it admits an explicit Kähler–Einstein metric.

Notice that $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ has no deformations, so their quotients by any finite group have no equisingular deformations. But clearly for

$\lambda \neq [1:0],[0:1]$, X_2^λ has non-trivial equisingular deformations, so it can not be a global quotient of $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.

5.2.1. Relation with moduli of curves. Naturally considering the associated branch locus for each double cover (i.e., the bi-anti-canonical map), we can regard our moduli M_2 as the GIT moduli of bicanonically embedded Hilbert polystable genus 3 curves, which is constructed in [**33**]. Indeed, by a direct comparison, the corresponding set of parametrized curves are the same. We have a one-dimensional family of tacnodal curves and a five-dimensional family of hyperelliptic curves. They intersect at one point corresponding to the curve $z^2 = x^4y^4$ in $\mathbb{P}(1,1,4)$. From this point of view, the proof that the moduli space is a blow up of the GIT moduli of plane quartics is given in [**6**] (which is due to David Hyeon). Our proof recovers this result, modulo the criterion of the Hilbert stability.

Thus a natural question would concern the corresponding "Del Pezzo surface modular interpretation" for the flipped contraction that contracts the tacnodal locus in [**33**]. In general, we can ask the following:

Question 5.15. *What are the modular interpretations via log Del Pezzo surfaces for each step of the Hassett–Keel program in* [**33**]*? In addition, are there also stability interpretations for them?*

5.3. Degree 1 case. From Section 2 we know that for any $X \in M_1^{GH}$, there are only three possible types for the non-canonical singularities. Moreover, we have the following:

Lemma 5.16. *The canonical singularities in $X \in M_1^{GH}$ are either $A_1, \cdots, A_8$ or D_4.*

Proof. This follows from Theorem 2.7 and the Noether formula for singular surfaces [**30**, Proposition 2.6],

$$\rho(X) + K_X^2 + \sum_{P \in Sing(X)} \mu_P = 12\chi(\mathcal{O}_X) - 2,$$

where $\rho(X)$ is the Picard rank of X and μ_P denotes the Milnor number. Notice that $\chi(\mathcal{O}_X) = 1$ by the Kodaira vanishing theorem and that the Milnor number of an A_k, D_k or E_k singularity is k. q.e.d.

We mention that, by using the Kähler–Ricci flow and calculating certain α-invariants, it has been proved in [**77**] and [**16**] that a degree 1 log Del Pezzo surface with only A_n singularities admits a Kähler–Einstein metric, if $n \leq 6$.

5.3.1. First step: GIT. By Corollary 5.4, a Gromov–Hausdorff limit in degree 1 is either a double cover of $\mathbb{P}(1,1,2)$ branched along a sextic or a double cover of $\mathbb{P}(1,2,9)$ branched along a degree 18 curve. As the first step, we will construct a moduli space of surfaces that are double cover of $\mathbb{P}(1,1,2)$ branched along a sextic that does not pass through $[0:0:1]$. These surfaces have equations $w^2 = F(x,y,z) \subset \mathbb{P}(1,1,2,3)$, where F contains a non-zero term z^3.

Although the automorphism group of $\mathbb{P}(1,1,2)$ in non-reductive, we can construct a compact moduli space of such sextics in $\mathbb{P}(1,1,2)$ that are polystable in an appropriate GIT sense, following [**64**]. Instead of the honest automorphism group $Aut(\mathbb{P}(1,1,2))$, we consider the action of $SL(2;\mathbb{C}) \ltimes H^0(\mathbb{P}^1, \mathcal{O}(2))$ that is a finite cover of $Aut(\mathbb{P}(1,1,2))$ and a subgroup of $Aut(\mathbb{P}(1,1,2), \mathcal{O}(2))$ (i.e., it also acts on the linearization). First, we fix the translation action of $H^0(\mathbb{P}^1, \mathcal{O}(2))$ by requiring the vanishing of the coefficient of z^2. Thus we only need to consider surfaces of the form

$$w^2 = z^3 + f_4(x,y)z + f_6(x,y).$$

Then, by dividing out by the natural $\mathbb{C}^*$-action on f_4 and f_6 with weights $4, 6$, repectively, we obtain a weighted projective space $\mathbb{P}_s := \mathbb{P}(2,2,2,2,2,3,3,3,3,3,3,3)$ as a parameter space. What is left is the action of $SL(2;\mathbb{C})$ in the two variables x, y. Thus we get a GIT quotient

$$M_1' := \mathbb{P}_s^{ss}//SL(2;\mathbb{C})$$

as a moduli space. This is similar to [**64**], where the GIT of degree 12 curves in $\mathbb{P}(1,1,4)$ was studied. We have the following classification of singularities for polystable locus (cf. [**64**, Theorem 4.3]):

Lemma 5.17. *With respect to the GIT stability of the above $SL(2;\mathbb{C})$-action, our surface $[w^2 = z^3 + zf_4(x,y) + f_6(x,y) \subset \mathbb{P}(1,1,2,3)]$ is:*

1) *stable if and only if it contains at worst A_k singularities;*
2) *strictly polystable if and only if it contains exactly two D_4 singularities or $SL(2;\mathbb{C})$-equivalent to $p_0 := [-\frac{1}{3}(x^2+y^2)^2 : \frac{2}{27}(x^2+y^2)^3]$ in $\mathbb{P}_s$ (in this case, it is non-normal).*

Proof. By the numerical criterion, a point $f = [f_4 : f_6]$ is unstable if and only if there is a point $u \in \mathbb{P}^1(x,y)$ such that f_4 and f_6 has multiplicity bigger than 2 and 3 at u, respectively. Without loss of generality, we may assume $u = [1:0]$, so that y^3 divides f_4 and y^4 divides f_6. Then it is easy to see that the corresponding sextic has a triple point at u, with unibranch (i.e. a unique tangent line). So the surface X_f has an E_k or worse singularity. Conversely, if X_f has a singularity of type E_k or worse, then by multiplying by an element in $SL(2;\mathbb{C})$ we may assume the singularity is of the form $[1:0:z_0] \in \mathbb{P}(1,1,2)$. In the affine chart where $x \neq 0$, the sextic is of the form $z^3 + zf_4(1,y) + f_6(1,y)$. It is easy to see that the only triple point must have $y = z = 0$. Then it follows that $[f_4 : f_6]$ is unstable. Similarly, it is easy to see that X_f is stable if and only if it contains at worst A_k singularities, i.e., the sextic contains at worst double points. If X_f is polystable, then $[f_4 : f_6]$ must be in the $SL(2;\mathbb{C})$ orbit of $[ax^2y^2 : bx^3y^3]$ for some non-zero $[a:b] \in \mathbb{P}(2,3)$. It is not hard to see that for $[a : b] \in \mathbb{P}(2,3)$ not equal to $[-1/3 : 2/27]$, X_f has exactly two D_4 singularities. q.e.d.

Remark 5.18. *We remark that, in the context of rational elliptic surfaces (which is the blow up of the base point of a complete anti-canonical system of degree* 1 *Del Pezzo surfaces), Miranda* [**51**] *also analyzed the equivalent GIT stability and constructed the corresponding compactified moduli variety that is isomorphic to our* M_1'.

5.3.2. Second step: Blow up. For the compatibility with later discussions, we replace characters x, y, z, w by x', y', z', w' for the homogeneous coordinates for $\mathbb{P}(1,1,2,3)$. Recall that in the statement of Theorem 5.2 when the Gromov–Hausdorff limit is the double cover of $\mathbb{P}(1,1,2)$, the branch locus could pass the vertex. This corresponds to the z'^3 term vanishing in the statement$F(x', y', z')$. By Lemma 5.6, it is easy to see that if we want the surface to have only quotient singularities, there must be a term of the form $z'^2 f_2(x', y')$ where f_2 must have rank at least 1. On the other hand, for these surfaces, there is no obvious reason that they do not appear as the Gromov–Hausdorff limit of Kähler–Einstein surfaces. Indeed, we have explicit examples of such surfaces that admit Kähler–Einstein metrics. The first is a one-dimensional family of degree 1 Kähler–Einstein log Del Pezzo surfaces that are Gorenstein except one whose f_2 is rank 2.

Example 5.19. *We consider a* $\mathbb{Z}/2\mathbb{Z}$ *action on the family of degree two surfaces* X_2^λ *as studied in the end of Section* 5.2. *The action is given by* $[x : y : z : w] \mapsto [x : y : -z : -w]$. *The fixed points are exactly the singularities of* X_2^λ. *One can check that for* $\lambda \neq [1 : 0], [0 : 1]$ *the quotient* X_1^λ *is a degree one log Del Pezzo surface with exactly two* D_4 *singularities. It is interesting that these surfaces admit a* $\mathbb{C}^*$ *action and correspond exactly to the polystable points in* M_1', *except* p_0. *From the discussion at the end of Section* 5.2 *we see they all admit Kähler–Einstein metrics.*

For the surface X_2^∞ *the action also fixes the* A_1 *singularity* $[0 : 0 : 1 : 0]$, *so the quotient* X_1^∞ *has two* D_4 *singularities and one* $\frac{1}{4}(1,1)$ *singularity. Denote the embedding* $\mathbb{P}(1,1,2) \hookrightarrow \mathbb{P}^3$ *by* $[x' : y' : z'] \mapsto [z' : x'^2 : x'y' : y'^2]$. *Then the bi-anti-canonical map realizes* X_1^∞ *as a double cover of* $\mathbb{P}(1,1,2) \subset \mathbb{P}^3$ *branched along the curve isomorphic to* $z'^2x'y' + x'^3y'^3$. *Indeed,* $|-2K_{X_1^\infty}| = |-2K_{X_2^\infty}|^{\mathbb{Z}/2\mathbb{Z}} = |\mathcal{O}_{\mathbb{P}(1,1,1,2)}(2)^{\mathbb{Z}/2\mathbb{Z}}|$, *which is spanned by* x^2, xy, y^2, z^2 *so the branch locus is* $xyz(z-xy)$. *The latter is isomorphic to the sextic described above.*

So X_1^∞ *corresponds to the case that* f_2 *has rank 2. Clearly,* X_1^∞ *admits a Kähler–Einstein metric, as a global quotient of* $\mathbb{P}^1 \times \mathbb{P}^1$.

The next example, which will be important in our further modification, is a degree 1 Kähler–Einstein log Del Pezzo surface that corresponds to f_2 being rank 1.

Example 5.20. *Consider the degree 2 surface $X_2^{\gamma_0}$ with $\gamma_0 = [1 : -1]$. It has two A_3 singularities, one at $[1 : 0 : 0 : 0]$ and one at $[0 : 1 : 0 : 0]$. Now consider the involution $\sigma : X_2^{\gamma_0} \to X_2^{\gamma_0}$ that sends $[x : y : z : w]$ to $[x : -y : -z : -w]$. Then σ has two fixed points exactly at the two singularities. It is straightforward to check that the quotient, which we will denote by X_1^e from now on, has one A_7 singularity and one $\frac{1}{8}(1, 3)$ singularity. $|-2K_{X_1^e}|$ is determined by the sections $\{x^2, y^2, yz, z^2\} \in H^0(\mathbb{P}(1, 1, 1, 2), \mathcal{O}(2))$, and this defines a double covering map from X_1^e to the quadric cone in $\mathbb{P}^3$. The corresponding involution σ maps $[x : y : z : w]$ to $[-x : -y : -z : -w] = [-x : y : z : w]$ (the identity holds on X_1^e). Then the fixed locus of σ consists of the curve $w = 0$ and the curve $x = 0$. Denote again the embedding $\mathbb{P}(1, 1, 2) \hookrightarrow \mathbb{P}^3$ by $[x' : y' : z'] \mapsto [z' : x'^2 : x'y' : y'^2]$. The branch locus in $\mathbb{P}(1, 1, 2)$ is isomorphic to the sextic $z'^2x'^2 - z'y'^4 = 0$. So X_1^e corresponds to f_2 having rank 1. Again X_1^e admits a Kähler–Einstein metric by the discussion in the end of Section* 5.2.

We have a more refined classification than Corollary 5.4.

Lemma 5.21. *Let X_∞ be the Gromov–Hausdorff limit of a sequence of degree one Kähler–Einstein Del Pezzo surfaces. If it is a hypersurface in $\mathbb{P}(1, 1, 2, 3)$ of the form $w^2 = F_6(x, y, z)$, then either F_6 has a term z^3 or F_6 is equivalent to $z^2(x^2 + y^2) + zg_4(x, y) + g_6(x, y)$ or X_0 is isomorphic to X_1^e.*

Proof. Consider the case when F_6 contains no z^3 term. Then we claim the term $z^2 f_2(x, y)$ must not vanish. Otherwise, $F_6 = zf_4(x, y)+f_6(x, y)$. Then in the affine chart $\{z \neq 0\}$ in $\mathbb{P}(1, 1, 2, 3)$ we have the equation $w^2 = f_4(x, y) + f_6(x, y)$ then by the Lemma 5.6, X_∞ has a non-quotient singularity, so it can not be a Gromov–Hausdorff limit by Theorem 2.1. So up to equivalence we may assume the z^2 term in F_6 is of the form $z^2(x^2 + y^2)$ or z^2x^2. In the former case we are done, so we assume the latter. Then we can write

$$F_6(x, y, z) = z^2x^2 + azy^4 + bzxf_3(x, y) + f_6(x, y).$$

Now if $a = 0$, then again in the affine chart $\{z \neq 0\}$ we have equation $w^2 = x^2 + bxf_3(x, y) + f_6(x, y)$. Then by a change of variable at $(0, 0, 0)$ we may assume it is locally equivalent to $w^2 = x^2 + a_1xy^3 + a_2xy^5 + a_3y^6$. It is easy to see this is either non-normal or has an A_i singularity $i \geq 5$ at the origin. The corresponding singularity on X_0 is a $(\mathbb{Z}/2\mathbb{Z})$-quotient by the action $(x, y, w) \mapsto (-x, -y, -w)$. So X_0 is either non-normal or has an orbifold point of order at least 12, and thus it cannot admit a Kähler–Einstein metric by Theorem 2.7.

So $a \neq 0$, and then by a change of variables $y \mapsto y + cx$ and $z \mapsto z + g_2(x, y)$ we may assume

$$F_6(x, y, z) = z^2x^2 + zy^4 + f_6(x, y). \tag{5.1}$$

X_1^e is isomorphic to the surface defined by $w^2 = z^2x^2 + zy^4$. The one-parameter subgroup $\lambda(t) = (t^2, t, 1, t^2)$ degenerates every surface defined by $(*)$ to X_1^e as t tends to zero. Since X_1^e admits a Kähler–Einstein metric, it has vanishing Futaki invariant. By Theorem 3.3 we see X_∞ must be isomorphic to X_1^e. q.e.d.

We first construct a moduli space for surfaces with f_2 being rank two, and we will show these surfaces are parametrized exactly by a weighted blow up of M_1' at p_0. The surfaces are defined by

$$w'^2 = z'^2(x'^2 + y'^2) + z'g_4(x', y') + g_6(x', y'). \tag{5.2}$$

Similarly as before, by considering the translation $z' \mapsto z' + a_2(x', y')$ for a certain quadric $a_2(x', y')$, we may assume g_4 lies in the space $T(x', y') := \mathbb{C}(x' + iy')^4 \oplus \mathbb{C}(x' - iy')^4$, which is the $SO(2; \mathbb{C})(\cong \mathbb{C}^*)$-invariant complement to the linear subspace of $Sym^4(\mathbb{C}x' \oplus \mathbb{C}y')$ that consists of those divisible by $(x'^2 + y'^2)$. In this way, we can obtain the GIT quotient $\mathbb{P}_e^{ss}//SO(2; \mathbb{C}) := \mathbb{P}(1, 1, 2, 2, 2, 2, 2, 2, 2)^{ss}//SO(2; \mathbb{C})$, which parametrizes surfaces of the form (5.2). Here we need to specify the weight of $SO(2; \mathbb{C}) \cong \mathbb{C}^*$ on the linearization, and we choose the natural one, so the action corresponding to $(x' + iy') \mapsto \mu(x' + iy')$, $(x' - iy') \mapsto \mu^{-1}(x' - iy')$ has weight

$$(4, -4, 6, 4, 2, 0, -2, -4, -6), \tag{5.3}$$

with respect to the basis consists of

$$(x' + iy')^4, (x' - iy')^4,$$

and

$$\begin{aligned}
&(x' + iy')^6, (x' + iy')^5(x' - iy'),\\
&(x' + iy')^4(x' - iy')^2, (x' + iy')^3(x' - iy')^3,\\
&(x' + iy')^2(x' - iy')^4, (x' + iy')(x' - iy')^5, (x' - iy')^6.
\end{aligned}$$

Then we have the following:

Lemma 5.22. *The GIT quotient $\mathbb{P}_e^{ss}//SO(2; \mathbb{C})$ with respect to the action with weight (5.3) above parametrizes log Del Pezzo surfaces; i.e., a polystable sextic defined by $[g_4 : g_6] \in P_e$ has only quotient singularities, or more precisely, the corresponding Del Pezzo surface has exactly one $\frac{1}{4}(1, 1)$ singularity besides canonical singularities.*

Proof. It is easy to check that if a sextic has the form $z'^2(x'^2 + y'^2) + z'(a(x' + iy')^4 + b(x' - iy')^4) + g_6(x', y')$ with $a, b \neq 0$, then it has only double points away from the vertex. If $a = b = 0$, then for it to be stable it has at most double points, and for it to be polystable it has exactly two D_4 singularities besides the vertex. If $a \neq 0$ and $b = 0$, then, if it is stable, the sextic has at most double points, and if it is semistable, then it degenerates to $z'^2(x'^2+y'^2)+a(x'+iy')^3(x'-iy')^3$, which has two D_4 singularities. q.e.d.

When we prove the moduli space we constructed in the end has property (KE), we need to show the following:

Lemma 5.23. *A surface of the form* (5.2) *that admits a Kähler–Einstein metric must be GIT polystable with respect to the chosen linearization as above.*

Proof. This does not follow directly from the general Theorem 3.4, as the group $SO(2;\mathbb{C})\cong\mathbb{C}^*$ has non-trivial characters. But in our case this can be done by explicit analysis as follows. Notice that since $\mathbb{P}_e$ contains a point parametrizing a K-polystable log Del Pezzo surface (e.g., X_1^∞), the CM line bundle must be isomorphic to $\mathcal{O}(k)$ for $k>0$. This follows from the proof of Theorem 3.4. X_1^∞ corresponds to the vector $v=[0:0:0:0:0:1:0:0:0]$ in $\mathbb{P}_e$ with respect to the quasi-homogeneous coordinates as above. So the weight of the action on the CM line bundle must also be the natural one as above, for otherwise it is easy to see that v is unstable. q.e.d.

The second step toward the construction of M_1 is to replace the point $[p_0]\in M_1'$ (which corresponds to a non-normal surface) by the above GIT quotient.

Theorem 5.24. *There is a blow up $M_1''\to M_1'$ at $[p_0]$ (with a non-reduced ideal) so that M_1'' is an analytic moduli space for degree 1 log Del Pezzo surfaces. The exceptional divisor E is isomorphic to $\mathbb{P}_e^{ss}//SO(2;\mathbb{C})$. Moreover, a point $s\in M_1''$ parametrizes the polystable sextic hypersurface X_s defined by it, and $s\in E$ if and only if the sextic passes through the vertex* $[0:0:1]$.

Proof. Let $\tilde{\mathbb{A}}\simeq Sym^4(\mathbb{C}x\oplus\mathbb{C}y)\oplus Sym^6(\mathbb{C}x\oplus\mathbb{C}y)$ be the cone over $\mathbb{P}_s$. In the tangent space at the point $p_0=(-\frac{1}{3}(x^2+y^2)^2,\frac{2}{27}(x^2+y^2)^3)$, we take an $SO(2;\mathbb{C})$-invariant Luna étale slice $\mathbb{A}_f:=p_0+\{T(x,y)\oplus Sym^6(\mathbb{C}x\oplus\mathbb{C}y)\}$ in $\tilde{\mathbb{A}}$. To include surfaces of the form (5.2), let $\mathbb{A}_g=T(x',y')\oplus Sym^6(\mathbb{C}x'\oplus\mathbb{C}y')$, and we consider the family of surfaces over $\mathbb{A}_g\times\mathbb{C}^*$ where we associate (g_4,g_6,t) with the sextic

$$tz'^3+z'^2(x'^2+y'^2)+z'g_4(x',y')+g_6(x',y'). \tag{5.4}$$

Making the change of variable

$$x':=tx,y':=ty,z':=z-\frac{t}{3}(x^2+y^2),$$

and

$$f_4(x,y)=-\frac{t^2}{3}(x^2+y^2)^2+t^3g_4(x,y);$$

$$f_6(x,y)=\frac{2t^3}{27}(x^2+y^2)^3-\frac{t^4}{3}(x^2+y^2)g_4(x,y)+t^5g_6(x,y),$$

the sextic in equation (5.4) is then transformed into the form

$$t[z^3 + f_4(x,y)z + f_6(x,y)].$$

Hence it corresponds to the point $[f_4(x,y) : f_6(x,y)] \in \mathbb{A}_f \subseteq \mathbb{P}_s$. If we keep g_4 and g_6 fixed, and let t tend to zero, this converges exactly to the point p_0.

The equation (5.4) defines a family of sextics over the trivial $\mathbb{P}_{x',y',z'}(1,1,2)$ bundle $\mathcal{P}'$ over $\mathbb{A}_g \times \mathbb{C}^*$, and it extends obviously over $\mathbb{A}_g \times \mathbb{C}$, which is the cone over the blow up $\mathbb{B}_g$ of $\mathbb{A}_g$ at 0. This family is invariant under $\mathbb{C}^*$ action $\lambda.(t, g_4, g_6) := (\lambda^{-1}t, \lambda g_4, \lambda^2 g_6)$, and thus descended to a family over $\mathbb{B}_g$. The above change of variables indeed defines an isomorphism Ψ between $\mathcal{P} = \mathbb{P}_{x,y,z}(1,1,2) \times (\mathbb{A}_f \times \mathbb{C}^*)$, and induces a $\mathbb{C}^*$ action on $\mathbb{A}_f$. We decompose $\mathbb{A}_f$ as $\mathbb{A}_f = p_0 + (L_1 \oplus L_2)$, where

$$L_1 := \{(f_4(x,y), -\frac{1}{3}(x^2+y^2)f_4(x,y))\} \mid f_4 \in T_{(x,y)}\},$$

and

$$L_2 := Sym^6(\mathbb{C}x \oplus \mathbb{C}y) \subset \mathbb{A}_f.$$

Denote the associated ideals of $L_i + p_0$ in $\mathbb{A}_f$ by $I_{(L_i+p_0)}$. Then we define $\mathbb{B}_f$ to be the blow up of $\mathbb{A}_f$ at $I^2_{(L_1+p_0)} + I_{(L_2+p_0)}$. The exceptional divisor is isomorphic to $\mathbb{P}_e$. Then by pulling back by Ψ we obtain a flat family of sextics over $\mathbb{B}_f$, and the exceptional divisor parametrizes sextics of the form (5.2).

Similarly to the degree 2 case, we consider the GIT of $\mathbb{B}_f$ with respect to the $SO(2;\mathbb{C})$-action and get a certain blow up $\mathbb{B}_f^{ss}//SO(2;\mathbb{C}) \to \mathbb{A}_f//SO(2;\mathbb{C})$. This induces a blow up of $\mathbb{P}_s//SL(2;\mathbb{C})$ at $[p_0]$, with exceptional divisor $E \cong \mathbb{P}_e^{ss}//SO(2;\mathbb{C})$. We denote this by $M_1'' \to M_1'$.

From the construction, as in the previous section, M_1'' is an analytic moduli space and a coarse moduli of an algebraic stack that is constructed by gluing

$$[\mathbb{B}_f^{ss}/SO(2;\mathbb{C})]$$

naturally with

$$[(\mathbb{P}_s^{ss} \setminus (PGL(2;\mathbb{C}).p_0))/PGL(2;\mathbb{C})]$$

in our context. q.e.d.

5.3.3. Construction of moduli: further modifications. We have a further refinement of Corollary 5.4, parallel to Lemma 5.21.

Lemma 5.25. *Let X_∞ be the Gromov–Hausdorff limit of a sequence of degree 1 Kähler–Einstein Del Pezzo surfaces. Then X_∞ is a sextic hypersurface in $\mathbb{P}(1,1,2,3)$ of the form $x_4^2 = f_6(x_1,x_2,x_3)$, or isomorphic to the toric surface X_1^T.*

Proof. By Theorem 5.2, we may assume X_∞ is a degree 18 hypersurface in $\mathbb{P}(1,2,9,9)$ of the form $x_4^2 = f_{18}(x_1,x_2,x_3)$ not passing through the point $[0:0:1]$. So we may assume $f_{18}(x_1, x_2, x_3) = x_3^2 + g_{18}(x_1, x_2)$. If the term x_2^9 appears in g_{18}, then the one-parameter subgroup Λ acting with weight $(0, 9, 2, 2)$ degenerates $x_4^2 - f_{18}$ to $x_4^2 - x_3^2 - ax_2^9$. This induces a test configuration for X_∞ with the central fiber isomorphic to X_1^T. Since X_1^T has vanishing Futaki invariant, and X_∞ is K-polystable, we conclude that X_∞ must be isomorphic to X_1^T. If x_2^9 does not appear in g_{18}, then the one-parameter subgroup Λ acting with weight $(0, 0, 1, 1)$ degenerates $x_4^2 - f_{18}(x_1, x_2, x_3)$ to $x_4^2 - x_3^2$. Again this induces a test configuration for X_∞ with central fiber the non-normal hypersurface Y defined by $x_4^2 - x_3^2 = 0$. We claim this has zero Futaki invariant, thus contradicting the fact that X_∞ is K-polystable. To see the claim, note that the Futaki invariant for a $\mathbb{C}^*$-action on a connected fixed component in the Hilbert scheme is constant. Since X_1^T obviously degenerates to Y and is fixed by the same Λ, we can compute the Futaki invariant on X_1^T, which is zero since it is Kähler–Einstein. q.e.d.

The analytic moduli space M_1'' constructed in the previous section does not have property (KE), since it does not parametrize the two examples X_1^e and X_1^T that we are unable to show that they cannot appear as a Gromov–Hausdorff limit. So we have to make a modification of M_1''. Now the only problem is to fit these two into M_1''. We first illustrate the phenomenon of modification of GIT by a simple example.

Example 5.26. *Let $\mathbb{C}^*$ act linearly on $\mathbb{C}^2$ by $t.(z_1, z_2) = (tz_1, z_2)$. Then the quotient is isomorphic to $\mathbb{C}$, and the polystable locus are points on the line $\{0\} \times \mathbb{C}$. If we remove the origin $(0, 0)$, then the quotient is again isomorphic to $\mathbb{C}$, but the polystable locus differs from the previous one in that the orbit of the origin is replaced by the punctured line $\mathbb{C}^* \times \{0\}$.*

Our situation is very similar to this. We first investigate the $\mathbb{Q}$-Gorenstein deformation of X_1^T studied in Section 3. Adopting the notation there, we have the following:

Lemma 5.27. *A point $v = (v_1, v_2, v_3) \in Def(X_1^T)$ is polystable under the action of $Aut^0(X_1^T)$ if and only if v_1, v_2 and v_3 are all non-zero or all zero and $(0, 0, 0)$ is the only strictly polystable point.*

Proof. If $v_1 = 0$, then we can destabilize v by the one-parameter subgroup $\lambda(t) = (t^{-1}, 1)$. If $v_2 = 0$, then we can destabilize v by the one-parameter subgroup $(1, t^{-1})$. If $v_3 = 0$, then we can destabilize v by the one-parameter subgroup (t^3, t^2). If all the v_i's are non-zero, then for $\lambda(t) = (t^a, t^b)$ to destabilize v we need $a - b \geq 0$, $-3a + 6b \geq 0$, and $-3a - 3b \geq 0$. It is easy to see that no non-trivial such pair (a, b) exists. q.e.d.

To fill X_1^T in our moduli, since we may locally identify $\mathrm{Kur}(X_1^T)$ with $\mathrm{Def}(X_1^T)$ and the $(\mathbb{C}^*)^2$-action on $\mathrm{Kur}(X_1^T)$ is compatible with the one on $\mathrm{Def}(X_1^T)$, it suffices to study the GIT on $\mathrm{Def}(X_1^T)$. By the above lemma, the stable points all represent canonical log Del Pezzo surfaces with at most a unique $A_k (k \leq 7)$ singularity, and the polystable point 0 represents X_1^T. The GIT quotient Q is then smooth at $[X_1^T]$. The semistable orbit $(0, v_2, v_3)$ (where $0 < |v_2|^2 + |v_3|^2 \ll 1$) represent a log Del Pezzo surface with a unique A_8 singularity. Since it is unique up to isomorphism by [**26**], we denote it by X_1^a. Due to Lemma 5.17, it has discrete automorphism group and it is parametrized by a point u_0 in $M_1'' \setminus E$.

Consider the analytic subset $\mathrm{Kur}'(X_1^T)$ of $\mathrm{Kur}(X_1^T)$ that represents only canonical log Del Pezzo surfaces, i.e., that consists of points with $v_2 \neq 0$ and $v_3 \neq 0$. Then the corresponding quotient Q' can be identified with the previous quotient Q, which identifies every stable orbit, except the orbit of X_1^a is replaced by X_1^T. Q' can be viewed as the universal deformation space X_1^a. There is an analytic neighborhood U of u_0, and an embedding $\iota : U \to Q' = Q$ such that $\iota(u_0) = 0$, and u and $\iota(u)$ parametrize equivalent surfaces. In terms of stack language, the open embedding of stacks $[(\mathrm{Kur}(X_1^T)\backslash \mathrm{Kur}(X_1^T)')/(\mathbb{C}^*)^2] \hookrightarrow [\mathrm{Kur}(X_1^T)/(\mathbb{C}^*)^2]$ induces an isomorphism of the categorical moduli. Now we can simply define $M_1''' = M_1''$ as a variety and only change the surface parametrized by u_0 from X_1^a to X_1^T. Then it is clear that M_1''' is again an analytic moduli space of degree 1 log Del Pezzo surfaces. So this modification takes care of the point X_1^T.

Now we treat X_1^e in a similar fashion. First notice that the linear system $|-2K_{X_1^e}|$ realizes X_1^e as the double cover of $\mathbb{P}(1,1,2)$, thus $\mathrm{Aut}^0(X_1^e)$ is induced from $\mathrm{Aut}(\mathbb{P}(1,1,2))$. Then one sees that $\mathrm{Aut}^0(X_1^e) \cong \mathbb{C}^*$ corresponds to the scaling $\lambda(t) = (t^2, t, 1, t^2)$. By Lemma 3.9 we have

$$\mathrm{Def}(X_1^e) = \mathrm{Def}' \oplus \mathrm{Def}_1 \oplus \mathrm{Def}_2,$$

where Def' corresponds to equisingular deformations, Def_1 corresponds to deformations of the local singularity at $[0:0:1:0]$, and Def_2 corresponds to deformations of the local singularity at $[1:0:0:0]$. By applying again the Main Theorem of [**47**], it follows that Def_1 is two dimensional and Def_2 is seven dimensional. Thus by dimension counting we must have $\mathrm{Def}' = 0$. We can write down a semi-universal deformation family:

$$w^2 = z^2x^2 + zy^4 + a_1z^3 + a_2z^2y^2 + \sum_{i=0}^{6} b_i x^i y^{6-j}. \tag{5.5}$$

In particular, note that we have $\mathrm{Aut}(X_1^e)$-invariant affine versal deformation space $\mathrm{Kur}(X_1^e)$ as claimed in the explanation after Lemma 3.9 and that in this case $\mathrm{Kur}(X_1^e)$ can be identified globally with the tangent space $\mathrm{Def}X_1^e$ so that $(a_1, a_2) \in \mathrm{Def}_1$ and $(b_0, \cdots, b_6) \in \mathrm{Def}_2$.

It is also easy to see the weights of the action are

$$\lambda(t).(a,b) = (t^{-4}, t^{-2}, t^{8}, t^{6}, \cdots, t^{2}).$$

So in the local GIT quotient by $\mathrm{Aut}(X_1^e)$ a point (a,b) is stable if and only if $a \neq 0$ and $b \neq 0$, in which case $X_{a,b}$ has either a unique $A_k (k \leq 6)$ singularity or a $\frac{1}{4}(1,1)$ plus $A_k (k \leq 6)$ singularity.

When we remove the subspace $\{0\} \oplus \mathrm{Def}_2$, every point becomes stable. In particular, the quotient of the subspace $(a,0)$ with $a \neq 0$ is exactly a $\mathbb{P}^1$, which parametrizes surfaces in M_1'' of the form

$$w^2 = a_1 z^3 + z^2 x^2 + z y^4 + a_2 z^2 y^2,$$

and intersects the exceptional divisor at one point corresponding to $a_1 = 0$. It is easy to see that $\lambda(t)$ degenerates all these surfaces to X_1^e as t tends to infinity, so they could not admit Kähler–Einstein metrics, and we need to remove them. Notice this family does not include the point corresponding to X_1^a, so we can make a further modification simultaneously with the previous one. When we add the the subspace $\{0\} \oplus \mathrm{Def}_2$, the point $(a,0)$ with $a \neq 0$ becomes semistable and in the GIT quotient this is contracted to the point 0. To be more precise, we take the neighborhood U in $\mathrm{Def}(X_1^e)$ consisting of points (a,b) with $||a|-1| \ll 1$ and $|b| \ll 1$, and the quotient V by $\mathbb{C}^*$ gives rise to a tubular neighborhood of the $\mathbb{P}^1$ in M_1''. When we add the subspace $\{0\} \oplus \mathrm{Def}_2$, we have that V gets mapped to a neighborhood of zero in the local GIT, with $\mathbb{P}^1$ contracted to 0.

As before, the GIT on $\mathrm{Kur}(X_1^e)$ and on $\mathrm{Def}(X_1^e)$ are equivalent so this allows us to perform the contraction in an analytic neighborhood of the $\mathbb{P}^1$ inside M_1'''. We obtain a new analytic moduli space M_1, which enjoys the Moishezon property. Thus it has a natural structure of an algebraic space as well.

Theorem 1.1 in the degree 1 case then follows from the theorem below.

Theorem 5.28. *M_1 has property (KE).*

Proof. The proof is very similar to Theorem 5.13. By Lemma 5.25 we only need to show that if a $X \in M_1^{GH}$ is a sextic hypersurface in $\mathbb{P}(1,1,2,3)$ defined by $w^2 = f_6(x,y,z)$, then it is parametrized by some element in M_1''. If f_6 contains a term az^3 with $a \neq 0$, then it is parametrized by a point u in $\mathbb{P}_s$. Then by Theorem 3.3 and Theorem 3.4, keeping in mind that $\mathbb{P}_s$ has Picard rank 1, we conclude that u is polystable under the $SL(2;\mathbb{C})$ action, and thus X is parametrized by a point p in M_1'. Then X cannot be isomorphic to X_1^T or the $\mathbb{P}^1$ family above. So X is parametrized by a point in M_1. If the term z^3 does not appear in f_6, then by Lemma 5.21 and Lemma 5.23 X is either isomorphic to X_1^e or is parametrized by a polystable point $u \in \mathbb{P}_e$. Again this point u can not be on the $\mathbb{P}^1$, and this means that u is in M_1. q.e.d.

We can construct a KE moduli stack $\mathcal{M}_1$ by gluing the previously constructed moduli stack with $[U/\mathrm{Aut}(X_1^e)]$, where U is some open $\mathrm{Aut}(X_1^e)$-invariant neighborhood of $0 \in \mathrm{Kur}(X_1^e)$ (along $[(U \setminus (\{0\} \oplus \mathrm{Def}_2))/\mathrm{Aut}(X_1^e)]$). Recall that in this case, we identified globally $\mathrm{Kur}(X_1^e)$ and $\mathrm{Def}(X_1^e)$. We can show that with a small enough $\mathrm{Aut}(X_1^e)$-invariant open neighborhood U of 0 in $\mathrm{Kur}(X_1^e)$, a stack $[(U \setminus (\{0\} \oplus \mathrm{Def}_2)))/\mathrm{Aut}(X_1^e)]$ has a natural étale morphism to the previously constructed moduli stack so that the glueing is possible. Indeed, the $\mathbb{Q}$-Gorestein deforming component (cf. [**39**, Section 5]) of a Luna étale slice in the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}(H^0(X_1^T, -K_{X_1^T}^{\otimes m})))$ at $[X_1^T]$ with respect to the standard SL action is an étale locally semi-universal deformation by the universality of the Hilbert scheme. Then the étale local uniqueness of semi-universal family tells us it is actually étale locally equivalent with U including the family on it. Then the assertion follows from the universality of Hilbert scheme again. Note especially that U includes the subspace $\mathrm{Def}_1 \oplus \{0\}$ so that the categorical moduli of the open immersion $[(U \setminus (\{0\} \oplus \mathrm{Def}_2)))/\mathrm{Aut}(X_1^e)] \hookrightarrow [U/\mathrm{Aut}(X_1^e)]$ represents the contraction of $\mathbb{P}^1$.

Then $\mathcal{M}_1$ is a KE moduli stack and M_1 constructed above is KE moduli space. This completes the proof of Theorem 1.1 for the degree 1 case as well. Note that our contraction of $\mathbb{P}^1$ on the coarse quotient is constructed just on an étale cover, not a priori an open substack. Indeed it is not, although we omit the lengthy proof for that. This is the reason our argument is not enough to show M_1 is a (projective) variety. Completely as before, there is a natural anti-holomophic involution on M_1 that gives rise to the complex conjugation.

5.3.4. A remark on a conjecture of Corti. In [**18**] Corti conjectured the following, motivated by the possibility of using birational geometry to get certain "nice" integral models over a discrete valuation ring:

Conjecture 5.29 ([**18**, Conjecture 1.16])**.** *For an arbitrary smooth punctured curve $C \setminus \{p\}$ and a smooth family of Del Pezzo surfaces $f\colon \mathcal{X} \to (C \setminus \{p\})$ over it, we can complete it to a flat family $\bar{f}\colon \bar{\mathcal{X}} \to C$ that satisfies:*

- *$\mathcal{X}$ is terminal.*
- *The $\mathbb{Q}$-Gorenstein index of $\bar{\mathcal{X}}_p$ is either $1, 2, 3,$ or 6 and $-6K_{\bar{\mathcal{X}}_p}$ is very ample.*

He called $\bar{\mathcal{X}}$ the *standard model.* We have the following partial solution to the above; it is rather weak, in the sense we permit base change, but, on the other hand, we even have a classification of the possible central fiber.

Proposition 5.30. *For an arbitrary smooth punctured curve $C \setminus \{p\}$ and a smooth family of Del Pezzo surfaces $f\colon \mathcal{X} \to (C \setminus \{p\})$ over it, by* *a possibly ramified base change* $p' \in \tilde{C} \to C$ *(with* $p' \mapsto p$*), we can fill the punctured family* $\mathcal{X} \times_{(C \setminus \{p\})} (\tilde{C} \setminus \{p'\})$ *to a flat family* $\bar{\mathcal{X}}' \to \tilde{C}$ *such that:*

- $\bar{\mathcal{X}}'$ *is terminal.*
- *The* $\mathbb{Q}$*-Gorenstein index of* $\bar{\mathcal{X}}'_{p'}$ *is either* $1, 2$ *and* $-6K_{\bar{\mathcal{X}}'_{p'}}$ *is very ample.*

Proof. We have constructed the moduli stack $\mathcal{M}''_1$ by gluing quotient stacks of a certain GIT semistable locus (Subsection 5.3.2). From the construction, it is a universally closed stack and it parametrizes log del Pezzo surfaces of $\mathbb{Q}$-Gorenstein index 1, 2. The $\mathbb{Q}$-Gorenstein property of $\mathcal{X}$ follows from our construction as well. q.e.d.

5.3.5. Relation with moduli of curves. We expect the KE moduli variety M_1 to be a divisor of one of the geometric compactifactions of moduli of curves with genus 4. In particular, we suspect that our moduli M_1 is the prime divisor of $\overline{M}_4(a)$ with $\frac{23}{44} < a < \frac{5}{9}$ in [**13**]. Note that it is the moduli of Hilbert polystable canonical curves.

6. Further discussion

6.1. Some remarks.

6.1.1. Lower bound of the Bergman function. The main technical part in the proof of Proposition 2.2 is a uniform lower bound of the Bergman function. Let (X, J, ω, L) be a polarized Kähler manifol;, then for any k there is an induced metric on $H^0(X, L^k)$. The Bergman function is defined by

$$\rho_{k,X}(x) = \sum |s_\alpha|^2(x),$$

where $\{s_\alpha\}$ is any orthonormal basis of $H^0(X, L^k)$. The Kodaira embedding theorem says that for fixed X, and for sufficiently large k the Bergman function is always positive. It is proved in [**23**] that for an n-dimensional Kähler–Einstein Fano manifold (X, J, ω), we always have $\rho_{k,X}(x) \geq \epsilon$ for some integer k (and thus every positive multiple of k) and $\epsilon > 0$ depending only on n. This was named the "partial C^0 estimate" in [**70**] and it is also proved there for two-dimensional case. It was explained in [**23**] that one may not take k to be all sufficiently large integers, and in our proof of the main theorem we have seen examples, see Remark 5.14. Indeed, we found explicitly all the integers k that we need to take in each degree in order to ensure a uniform positivity of the Bergman function for all Kähler–Einstein Del Pezzo surfaces. (Compare the strong partial C^0 estimate in [**70**, Theorem 2.2]):

- $d = 4, 3$: $k \geq 1$;
- $d = 2$: $k = 2l$, with $l \geq 1$;
- $d = 1$: $k = 6l$, with $l \geq 1$.

6.1.2. Kähler–Einstein metrics on Del Pezzo orbifolds. As a consequence of our main Theorem 1.1, we have a complete classification of Kähler–Einstein Del Pezzo surfaces with at worst canonical singularities in terms of K-polystability.

Corollary 6.1. *Let X be a Del Pezzo surface with at worst canonical singularities. Then*

$$X \text{ admits a Kähler–Einstein metric} \Longleftrightarrow X \text{ is K-polystable.}$$

Proof. The direction "$\Longrightarrow$" is known by Theorem 3.3. To prove the other direction, suppose that X is K-polystable and with at worst canonical singularities (in particular, it is automatically $\mathbb{Q}$-Gorenstein smoothable). Then by Theorem 3.4 X is also polystable with respect to the stability notions that we used in the construction of our moduli spaces, i.e. $[X] \in M_d$. Thus X admits a Kähler–Einstein metric as a consequence of Theorem 1.1. q.e.d.

The above result gives the answer to the conjecture of Cheltsov and Kosta ([**16**, Conjecture 1.19]) on the existence of Kähler–Einstein metrics on canonical Del Pezzo surfaces. In particular, we have the following exact list of possible singularities that can occur. Let (X, ω) be a degree $d \leq 4$ Del Pezzo surface with canonical singularities; then it admits a Kähler–Einstein metric if and only if X is smooth or:

- $d = 4$: $\mathrm{Sing}(X)$ consists of only two A_1 singularities and X is simultaneously diagonalizable, or exactly four singularities (in which case X is isomorphic to X_4^T).
- $d = 3$: $\mathrm{Sing}(X)$ consists of only points of type A_1, or of exactly three points of type A_2 (in which case X is isomorphic to X_3^T).
- $d = 2$: $\mathrm{Sing}(X)$ consists of only points of type A_1, A_2, or of exactly two A_3 singularities.
- $d = 1$: $\mathrm{Sing}(X)$ consists of only points of type A_k ($k \leq 7$), or of exactly two D_4 singularities, and X is not isomorphic to one the surfaces in the $\mathbb{P}^1$ family in the last section.

As we have seen, the class of log Del Pezzo surfaces with canonical singularities is not sufficient to construct a KE moduli variety. In particular, we have found some $\mathbb{Q}$-Gorenstein smoothable Kähler–Einstein log Del Pezzo surfaces, hence K-polystable, with non-canonical singularities. Thus it is natural to ask the following differential geometric/algebro-geometric question: Do there exist other $\mathbb{Q}$-Gorenstein smoothable Kähler–Einstein/K-polystable log Del Pezzo surfaces besides the ones that appear in our KE moduli varieties? If the answer to the above question is negative (as we conjecture), then the Yau–Tian–Donaldson conjecture for K-polystability also holds for the class of $\mathbb{Q}$-Gorenstein smoothable Del Pezzo surfaces. For this it is of course sufficient to prove the following: Let $\pi\colon \mathcal{X} \to \Delta$ be a $\mathbb{Q}$-Gorenstein deformation of a K-polystable

Del Pezzo surface X_0 over the disc Δ such that the generic fibers X_t are smooth (hence admit Kähler–Einstein metrics). Then X_0 admits a Kähler–Einstein metric ω_0, and (X_0, ω_0) is the Gromov–Hausdorff limit of a sequence of Kähler–Einstein metrics on the fibers (X_{t_i}, ω_{t_i}) for some sequence $t_i \to 0$.

6.2. On compact moduli spaces. In this final section, we would like to formulate a conjecture about the existence of certain compact moduli spaces of K-polystable/Kähler–Einstein Fano varieties. Before stating our conjecture, we recall some important steps in the history of the construction and compactifications of moduli spaces of varieties.

For complex curves of genus $g \geq 2$, the construction of the moduli spaces, and their "natural" compactifications, was completed during the seventies by Deligne, Mumford, Gieseker, and others *using* GIT. The degenerate curves appearing in the compactification are the so-called stable curves, i.e., curves with nodal singularities and discrete automorphisms group. Let us recall that these compact moduli spaces have also a "differential geometric" interpretation. It is classically well-known that every curve of genus g has a unique metric of constant Gauss curvature with fixed volume. As the curves move toward the boundary of the Deligne–Mumford compactification, the diameters, with respect to the constant curvature metrics, go to infinity and finally these metric spaces "converge" to a complete metric with constant curvature and hyperbolic cusps on the smooth part of a "stable curve."

The construction of compact moduli spaces of higher-dimensional polarized varieties turns out to be much more complicated than in the one-dimensional cases. Indeed, in the seminal paper [**39**] the authors discovered examples of surfaces with ample canonical class and semi-log-canonical singularities, which are the natural singularities to be considered for the compactification, which are *not* asymptotically GIT stable. The central point for this phenomenon is that there are semi-log-canonical singularities that have "too big" multiplicity compared to the one required to be asymptotically Chow stable ([**54**]). Nevertheless, proper separated moduli of canonical models of general type surfaces have been recently constructed using birational geometric techniques instead of classical GIT. These compactifications are sometimes known as *Kollár–Shepherd-Barron–Alexeev (KSBA) type moduli.* It is then natural to ask what is the "differential geometric" interpretation of these kind of moduli spaces.

In order to discuss this last point, we first recall that GIT theory became again a main theme for the following reason: the existence of a Kähler–Einstein, or more generally constant scalar curvature, metric on a polarized algebraic variety is found to be deeply linked to some GIT stability notions, e.g., asymptotic Chow, Hilbert stability, and in particular to the formally GIT-like notion of "K-stability" introduced in [**71**] and [**22**]. Similarly to the previous discussion, asymptotic Chow stability appears to not fully capture the existence of a Kähler–Einstein metric since there are examples of Kähler–Einstein varieties which are asymptotically Chow *un*stable ([**39**, **58**]). On the other hand, for $\mathbb{Q}$-Fano varieties it is indeed proved that the existence of a Kähler–Einstein metric implies K-polystability [**11**].

It turns out that the notion of K-stability is also closely related to the singularities allowed in the KSBA compactifications ([**57**, **58**]): for varieties with ample canonical class, the notion of K-stability *coincides* with the semi-log-canonicity property, and for Fano varieties K-(semi)stability *implies* log-terminalicity. This last condition on the singularities in the Fano case is also important for differential geometric reasons. As recently shown in [**23**], it is known that Gromov–Hausdorff limits of smooth Kähler–Einstein Fano manifolds (and more generally of polarized Kähler manifolds with control on the Ricci tensor, the injectivity radius and with bounded diameter) are indeed $\mathbb{Q}$-Fano varieties, i.e., they have at worst log-terminal singularities, and, moreover, they must be K-polystable, by [**11**].

Summing up, a central motivation of the present work was to investigate how Kähler–Einstein metrics and the compact moduli varieties are indeed related. Thus, motivated by our results on Del Pezzo surfaces and by the above discussion, we shall now try to state a conjecture on moduli of Kähler–Einstein/K-polystable Fano varieties.

Denote the category of algebraic schemes over $\mathbb{C}$ by $\mathrm{Sch}_{\mathbb{C}}$, and let

$$\mathcal{F}_h\colon \mathrm{Sch}^o_{\mathbb{C}} \to Set$$

be the contravariant moduli functor that sends an object $S \in Ob(\mathrm{Sch}_{\mathbb{C}})$ to isomorphic classes of $\mathbb{Q}$-Gorenstein flat families $\mathcal{X} \to S$ of K-semistable $\mathbb{Q}$-Fano varieties with Hilbert polynomial equal to h and sends a morphism to the pull-back of families making the corresponding squared diagram commuting. Moreover, adding isomorphism (or isotropy) structure on this functor, we should naturally get a stack $\mathcal{M}_h$ on which we conjecture, refining [**66**, Conjecture 1.3.1] and [**56**, Conjecture 5.2] in the $\mathbb{Q}$-Fano case, the following:

Conjecture 6.2. *$\mathcal{M}_h$ is a KE moduli stack (cf. Definition* 3.13*) that has a categorical moduli algebraic space*

$$\mathcal{M}_h \to M_h,$$

where M_h is a projective variety *(in general may not be irreducible) endowed with an ample CM line bundle. In particular, M_h is a KE moduli variety in the sense of Definition* 3.13*.*

Let M_h^{GH} be the Gromov–Hausdorff compactification of the moduli space of smooth Kähler–Einstein Fano manifolds with Hilbert polynomial *h. Then there is a natural homeomorphism*

$$\Phi\colon M_h^{GH} \to M_h,$$

where we use the analytic topology on M_h.

This paper explicitly settles the above conjecture for the ($\mathbb{Q}$-Gorenstein smoothable) log Del Pezzo surface case, except the issue in the previous subsection and the statement about the CM line bundle. A remark is that the CM line bundle [**60**] can be naturally regarded as a line bundle on $\mathcal{M}_h$ and so by "CM line bundle on M_h" we mean a $\mathbb{Q}$-line bundle descended from $\mathcal{M}_h$. The descent is possible for each $U_i \twoheadrightarrow U_i//G$ in the context of Definition 3.13 since for each K-semistable $x \in U_i$ the action of the identity component of the isotropy group of G on the CM line over x is trivial by the weight interpretation of the vanishing of Futaki invariant [**60**]. They canonically patch together due to the canonical uniqueness of the descended line bundle on each $U_i//G$.

From the point of view of the authors, one way toward establishing the above conjecture in higher dimensions is by combining the algebraic and differential geometric techniques, as we did in this article. In many concrete situations one can hope to construct the above KE moduli stack by glueing together quotient stacks from different GIT. This also fits into the general conjecture on the Artin stack [**1**, Conjecture 1].

Finally, we remark that the points in the boundary $M_h \setminus M_h^0$ should correspond to $\mathbb{Q}$-Fano varieties, admitting weak Kähler–Einstein metrics in the sense of pluripotential theory [**24**]. This is known for $M_h^{GH} \setminus M_h^0$, see [**23**].

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