# COMPACT RIEMANNIAN MANIFOLDS WITH HARMONIC CURVATURE <br> AND NON-PARALLEL RICCI TENSOR 

Andrzej Derdziñski<br>Wroclaw University, Mathematical Institute

For any (pseudo) Riemanian manifold, the divergence $\delta R$ of its curvature tensor $R$ satisfies the well-known identity

$$
\begin{equation*}
\delta R=d S \tag{1}
\end{equation*}
$$

where $S$ is the Ricci tensor (viewed as a vector-valued 1-form) and dS denotes its Riemannian exterior derivative (so that, using the metric, one may consider $d S$ as a 1-form with values in exterior 2 -forms). The local coordinate expression for (1) is

$$
\begin{equation*}
\nabla^{i_{R_{h i j k}}}=\nabla_{k} S_{h j}-\nabla_{j} S_{h k} \tag{2}
\end{equation*}
$$

While every manifold with parallel Ricci tensor has harmonic curvature (i.e. satisfies $\delta R=0$ ), there are examples ([3], Theorem 5.2) of open Riemannian manifolds with $\delta R=0$ and $\nabla S \neq 0$. In [1] J.P. Bourguignon has asked the question whether the Ricci tensor of a compact Riemannian manifold with harmonic curvature must be parallel.

The aim of this note is to describe an easy example answering this question in the negative. More precisely, metrics with $\delta \mathrm{R}=0$ and $\nabla S \neq 0$ are exhibited on $S^{1} \times N^{3}, N^{3}$ being e.g. the $3-s p h e r e$ or a lens space. By taking products of these manifolds with themselves or with arbitrary compact Einstein manifolds, one gets similar examples in all dimensions greater than three.
THEOREM. Let $\left(N^{3}, h\right)$ be a three-dimensional Riemannvan manifold with constant positive sectional curvature $K$. Define the Riemannian manifold $\left(M^{4}, g\right)$ by $M^{4}=S^{1} \times N^{3}$,

$$
g_{((\cos t, \sin t), x)}(u+X, v+Y)=\langle u, v\rangle+F(t) h_{x}(X, Y)
$$

for $u, v \in T(\cos t, \sin t) S^{1}$ and $X, Y \in T_{X} N^{3}$, where $<,>$ is the standard metric of $S^{1}=R / 2 \pi Z \quad$ and

$$
\begin{equation*}
F(t)=2 \mathrm{Km}^{-2}+A \cos m t+B \sin m t \tag{3}
\end{equation*}
$$

the positive integer $m$ and real numbers $A, B$ being chosen so that

$$
0<A^{2}+B^{2}<4 K_{m}^{-4}
$$

which implies that the function F is non-constant, positive and periodic.
Then ( $\mathrm{M}^{4}$,g) has harmonic curvature tensox, but its Ricci tensor fiela is not parallel.

Proof. It seems convenient to use local coordinates. In a product chart $x^{0}=t, x^{1}, x^{2}, x^{3}$ for $S^{1} \times N^{3}$ we have, setting $q(t)=\log F(t)$, $g_{00}=1, \quad g_{0 i}=0, \quad g_{i j}=e^{q_{h}}{ }_{i j}, \quad \Gamma_{00}^{0}=r_{0 i}^{0}=r_{00}^{i}=0, \quad r_{i j}^{0}=-\frac{1}{2} q^{\prime} e^{q_{h}}{ }_{i j}$, $\Gamma_{o j}^{i}=\frac{1}{2} q^{\prime} \delta_{j}^{i}, \quad r_{j k}^{i}=H_{j k}^{i}, \quad s_{o o}=-\frac{3}{4}\left(2 q^{\prime \prime}+\left(q^{\prime}\right)^{2}\right), \quad s_{o i}=0$, $S_{i j}=\left(2 k-\frac{1}{2} e^{q} q^{\prime \prime}-\frac{3}{4} e^{q}\left(q^{\prime}\right)^{2}\right)_{i j}{ }_{i j} \quad \nabla_{0} S_{o o}=-\frac{3}{2}\left(q^{\prime \prime \prime}+q^{\prime} q^{\prime \prime}\right)=-3 \frac{d}{d t}\left(E^{-\frac{1}{2}} \frac{d^{2}}{d t^{2}}\left(F^{\frac{1}{2}}\right)\right)$, $\nabla_{0} S_{i 0}=\nabla_{i} S_{o o}=0, \quad \nabla_{o} S_{i j}=-\left(2 K q^{i}+\frac{1}{2} e^{q} q^{\prime \prime \prime}+\frac{3}{2} e^{q^{q}} q^{\prime} q^{\prime \prime}\right) h_{i j}$, $\nabla_{i} S_{o j}=-\left(K q^{\prime}+\frac{1}{2} e^{q} q^{\prime} q^{\prime \prime}\right) h_{i j}, \quad \nabla_{k} S_{i j}=0$, where $i, j, k$ run through $\{1,2,3\}$ and the $\Gamma$ 's (resp. H's) are the Christoffel symbols of g (resp. of $h$ with respect to the chart $x^{1}, x^{2}, x^{3}$ of $N^{3}$ ). From (3) we obtain $q^{\prime \prime}+\left(q^{\prime}\right)^{2}-2 K e^{-q}+m^{2}=0$, whence $q^{\prime \prime}+2 q^{\prime} q^{\prime \prime}+2 K q^{\prime} e^{-q}=0$, i.e. $\nabla_{o} S_{i j}=\nabla_{i} S_{o j}$, and $\delta R=0$ is now immediate from (2). On the other hand, $\quad \nabla S \neq 0$, since $\nabla_{0} S_{00}=0$ would mean that the non-constant positive periodic function $f^{\frac{1}{2}}$ is an eigenfunction of $d^{2} / d t^{2}$. This completes the proof.
REMARK 1. It is easy to verify that the manifold ( $M^{4}$, g) defined above is conformally flat, that is, its Weyl tensor $W=0$. (conformal flatness together with constancy of the scalar curvature is well-known to imply harmonicity of the curvature tensor. On the other hand, the scalar curvature is constant whenever $\delta R=0$.)

REMARK 2. One can prove ([2], Theorem 3) that every four-dimensional compact analytic Riemannian manifold with harmonic curvature, whose Ricci tensor is not parallel and has less than three distinct eigenva-
lues at any point, is covered isometrically by $s^{1} \times s^{3}$ with a metric closely related to the one described above.

## REFERENCES

[1] J.P. Bourguignon, on hamonic forms of cunvature type (preprint).
[2] A. Dendzinski, Classification of certain compact Riemannian manifolds with harmonic curvature and non-parallel Ricci tensor, to appear in Mathematische Zeitschrift.
[3] A. Gray, Einstein-like manifolds which are not Einstein, Geometriae dedicata, $7(1978)$, 259-280.

