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COMPACTIFICATIONS OF $C \times C^*$ AND $(C^*)^2$

TETSUO UEDA

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0°. By a compactification of a complex manifold V we mean a compact complex manifold S together with an analytic set C in S such that $S - C$ is biholomorphic to V. It is known that every compactification of C^2 is a rational surface. (Kodaira [4], Morrow [5]) In this note we shall prove the following two theorems:

THEOREM 1. *Every compactification of $C \times C^*$ is a rational surface.*

THEOREM 2. *Every compactification of $(C^*)^2$ is of one of the following three types:*

(0) *a rational surface,*

(1) *a Hopf surface containing only one irreducible curve, or a manifold obtained from a Hopf surface of this type by blowing up at points of the curve* (*see* 4°, *case* (d)).

(2) *a unique P^1-bundle S over an elliptic curve admitting a unique global section C with $I(C, C) = 0$ such that $S - C$ is an analytically non-trivial principal C-bundle, i.e., the "Serre variety" according to Simha* [9], *or a manifold obtained from S by blowing up at points of C.*

As for Theorem 2, the existence of such three types has been pointed out by several authors (see for example the footnote of [10]), and recently Simha ([9]) has shown that every non-rational algebraic compactification with irreducible C is of type (2).

Our proof is based on (i) a result obtained by Kodaira from the value distribution theory, from which follows the vanishing of the geometric genus of S, (ii) the method used by Mumford to see the topology of the neighborhood of a compact curve in a surface, in particular his criterion for simplicity, by which we shall show that, in the non-rational case, C can be reduced to an irreducible elliptic curve, and (iii) Kodaira's classification theory of complex surfaces.

1°. Let S be a compact complex manifold of dimension 2. We denote by $b_k(S)$, $P_m(S)$, and $p_g(S)$ respectively, the k-th Betti number, the m-genus and the geometric genus of S. The intersection multiplicity $I(\Gamma_1, \Gamma_2)$ of $\Gamma_1, \Gamma_2 \in H_2(S, R)$ is a non-singular symmetric bilinear form on

$H_2(S, \boldsymbol{R})$. We denote by (b^+, b^-) the signature of this form.

By Kodaira [4, p. 45-46] we have

PROPOSITION 1. *If S is a compactification of $\boldsymbol{C} \times \boldsymbol{C}^*$ or $(\boldsymbol{C}^*)^2$, then $P_m(S) = 0$ for all m, in particular $p_g(S) = 0$.*

For $(\boldsymbol{C}^*)^2$ this is not stated in [4], but we can easily verify it in the same manner as for $\boldsymbol{C} \times \boldsymbol{C}^*$ calculating the order of the mean degree of the mapping $f: \boldsymbol{C}^2 \to (\boldsymbol{C}^*)^2$ defined by $f(z_1, z_2) = (e^{z_1}, e^{z_2})$.

COROLLARY. *S being as in the proposition,*

$$b^+ = 2p_g(S) + 1 = 1 \quad \textit{if} \quad b_1(S) \quad \textit{is even},$$
$$b^+ = 2p_g(S) = 0 \quad \textit{if} \quad b_1(S) \quad \textit{is odd}.$$

This follows from Theorem 3 in [2] and the proposition.

2°. **Proof of Theorem 1.** Let S be a compactification of $\boldsymbol{C} \times \boldsymbol{C}^*$ with an analytic set C, $S - C \cong \boldsymbol{C} \times \boldsymbol{C}^*$. C is connected, since there is no pair of open sets U_1 and U_2 in $\boldsymbol{C} \times \boldsymbol{C}^*$, which are not relatively compact such that $U_1 \cap U_2 = \emptyset$ and that $(\boldsymbol{C} \times \boldsymbol{C}^*) - (U_1 \cup U_2)$ is compact. C is of dimension 1. In fact, if C were a point, every holomorphic function on $\boldsymbol{C} \times \boldsymbol{C}^*$ would be extended to the whole S by Hartogs' theorem, and would be constant. Let us next consider the following exact homology sequence with real coefficients:

$$0 \to H_3(S) \to H_3(S, C) \to H_2(C) \xrightarrow{i_*} H_2(S) \to H_2(S, C).$$

We have $\dim H_3(S,C) = \dim H^1(S-C) = 1$ and $\dim H_2(S,C) = \dim H^2(S-C) = 0$ by the Poincaré-Lefschetz duality. Therefore $b_1(S) = b_3(S) = 0$ or 1. It is proved in [4] (using Proposition 1) that, if $b_1(S) = 0$, then S is rational. We shall show that the case $b_1(S) = 1$ does not occur. Suppose that $b_1(S) = b_3(S) = 1$. Then i_* is an isomorphism and induces on $H_2(C)$ a non-singular symmetric bilinear form, which is represented by the intersection matrix $(I(C_j, C_k))$ with respect to the irreducible components C_j of C. By the corollary to Proposition 1, this matrix is negative definite. Then by a theorem of Grauert ([1]), C is exceptional and by collapsing C to a point we obtain a normal complex space S/C. But this is a contradiction, since any holomorphic function on $S - C \cong \boldsymbol{C} \times \boldsymbol{C}^*$ would be extended to the whole S/C (see Narasimhan [7] p. 118, Proposition 4), and would be constant. q.e.d.

3°. Let S be a complex manifold of dimension 2 and consider a connected compact analytic set C of dimension 1 in S with n irreducible components, $C = \bigcup_{j=1}^n C_j$, satisfying the following conditions:

(♯) i) each irreducible component C_j is regular,

ii) if $C_j \cap C_k \neq \emptyset$ for $j \neq k$, C_j and C_k intersect transversally at only one point,

iii) $C_i \cap C_j \cap C_k = \emptyset$, if i, j and k are distinct.

Following Mumford [6] and Ramanujam [8], we construct the boundary of a tubular neighborhood of C, as follows: For every singular point p_ν of C we choose a coordinate neighborhood $U_\nu \cong \{(z_1, z_2); |z_1|, |z_2| < R\}$ $(R > 1)$, such that C is defined by the equation $z_1 z_2 = 0$, and that $U_\nu \cap U_\mu = \emptyset$ for $\nu \neq \mu$. Next we introduce a Riemannian metric ds in a neighborhood of C, which is of the form $ds^2 = |dz_1|^2 + |dz_2|^2$ in U_ν. Using this metric we define the exponential map $\exp_j: W_{j,\varepsilon} \to S$, where $W_{j,\varepsilon}$ is a closed ε-neighborhood of the zero section of the normal bundle of C_j. If we take ε sufficiently small, we may assume that $\exp_j$ is a homeomorphism from $W_{j,\varepsilon}$ onto V_j, where V_j is a closed neighborhood of C_j such that $V_j \cap V_k = \emptyset$ if $C_j \cap C_k = \emptyset$, and $V_j \cap V_k \subset U'_\nu = \{p \in U_\nu; |z_1(p)|, |z_2(p)| < 1\}$ if $C_j \cap C_k = p_\nu$. We define $M = \partial(\bigcup_{j=1}^n V_j)$. M is a topological manifold of dimension 3.

Let us next define a continuous map Φ of $[0, 1] \times M$ onto $\bigcup_{j=1}^n V_j$ such that $\Phi(0 \times M) = C$ and that $\Phi|(0, 1] \times M$ is a homeomorphism of $(0, 1] \times M$ onto $(\bigcup V_j) - C$, in the following manner (cf. [8]):

i) for (t, p) with $p \in M \cap \bar{U}'_\nu$. The coordinates (z_1, z_2) of p satisfy $\min\{|z_1|, |z_2|\} = \varepsilon$. Define Φ by

$$(t, z_1, z_2) \mapsto \left(tz_1 + (1-t)\frac{(|z_1| - \varepsilon)z_1}{(1-\varepsilon)|z_1|}, tz_2\right) \quad \text{if} \quad |z_1| \geqq |z_2| = \varepsilon\ ,$$

$$(t, z_1, z_2) \mapsto \left(tz_1, tz_2 + (1-t)\frac{(|z_2| - \varepsilon)z_2}{(1-\varepsilon)|z_2|}\right) \quad \text{if} \quad |z_2| \geqq |z_1| = \varepsilon\ .$$

ii) for (t, p) with $p \in M - (\bigcup U'_\nu)$. There is a unique j with $p \in \partial V_j$. Define Φ by $\Phi(t, p) = \exp_j(t \exp_j^{-1}(p))$, where t denotes the multiplication by t on the fiber of the normal bundle.

Let S/C be the topological space obtained by collapsing C, and let $\pi: S \to S/C$ denote the canonically defined continuous map. Set $\pi(C) = p_0$ and $\pi \circ \Phi = \Phi_0$. Then M has the following property: There exists a continuous map $\Phi_0: [0, 1] \times M \to S/C$ such that $\Phi_0(0 \times M) = p_0$ and that $\Phi_0|(0, 1] \times M$ is a homeomorphism. Let us see that any topological space with this property is of the same homotopy type. We take any topological space M' with a continuous map $\Phi': [0, 1] \times M' \to S/C$ such that $\Phi'(0 \times M') = p_0$ and that $\Phi'|(0, 1] \times M'$ is a homeomorphism, and show the homotopy equivalence of M and M'. Take $t, t' \in (0, 1]$ sufficiently small, such that $\Phi_0(t \times M) \subset \Phi'([0, 1] \times M')$ and $\Phi'(t' \times M') \subset \Phi_0([0, 1] \times M)$, respectively. Let $(\Phi_0|(0,1]\times M)^{-1}=(\tau,\sigma)$ and $(\Phi'|(0,1]\times M')^{-1}=(\tau',\sigma')$, and define $f\colon M\to M'$ by $f(p)=\sigma'\circ\Phi_0(t,p)$ and $g\colon M'\to M$ by $g(p')=\sigma\circ\Phi'(t',p')$. Then $g\circ f\colon M\to M$ is homotopic to the identity map of M, since we have the homotopy $\tilde{h}\colon[0,1]\times M\to M$ defined by

$$\tilde{h}(s,p)=\sigma\circ\Phi'(st'+(1-s)\tau'\circ\Phi_0(t,p),\sigma'\circ\Phi_0(t,p))\,.$$

Indeed, $\tilde{h}(0,p)=\sigma\circ\Phi'(\tau'\circ\Phi_0(t,p),\sigma'\circ\Phi_0(t,p))=\sigma\circ\Phi_0(t,p)=p$, and $\tilde{h}(1,p)=\sigma\circ\Phi'(t',\sigma'\circ\Phi_0(t,p))=g\circ f(p)$. Similarly we see that $f\circ g$ is homotopic to the identity map of M'. Thus M and M' are homotopically equivalent.

Now we consider the continuous map φ of M onto C defined by $\varphi(p)=\Phi(0,p)$. φ induces surjective homomorphisms $\pi_1(M)\to\pi_1(C)$ and $\varphi_*\colon H_1(M,\boldsymbol{Z})\to H_1(C,\boldsymbol{Z})$.

PROPOSITION 2. (Mumford [6, p. 10]). *The kernel K of the homomorphism φ_* is described as follows:*

i) *K is generated by $\alpha_1,\cdots,\alpha_n$, where α_j is a loop in M which goes around C_j with positive orientation.*

ii) *the fundamental relations of these generators are*

$$\sum_{k=1}^{n} I(C_j,C_k)\alpha_k=0\ ,\quad j=1,\cdots,n\ .$$

COROLLARY. $\operatorname{rank}(I(C_j,C_k))=n-\operatorname{rank}H_1(M,\boldsymbol{Z})+\operatorname{rank}H_1(C,\boldsymbol{Z})$.

In fact $\operatorname{rank}(I(C_j,C_k))=n-\operatorname{rank}K$, and $\operatorname{rank}K=\operatorname{rank}H_1(M,\boldsymbol{Z})-\operatorname{rank}H_1(C,\boldsymbol{Z})$.

C is said to be exceptional if there is a complex space $\check{S}$ and a holomorphic map $f\colon S\to\check{S}$ such that $f(C)$ is a point and that $f|S-C$ is a homeomorphism. By the theorem of Grauert referred to in 2°, C is exceptional if and only if the intersection matrix is negative definite. If $\check{S}$ is a manifold, C is said to be exceptional of the first kind.

THEOREM (Mumford [6]). *If C is exceptional, C is of the first kind if and only if M is simply connected.*

4°. **Proof of Theorem 2.** Let S be a compactification of $(\boldsymbol{C}^*)^2$ with C, $S-C\cong(\boldsymbol{C}^*)^2$. C is connected and of dimension 1. We may assume that $C=\bigcup_{j=1}^n C_j$ satisfies the conditions (#) by blowing up at points in C if necessary. M defined in 3° with respect to C is homotopically equivalent to a 3-dimensional torus. Indeed, $\boldsymbol{C}^*\cong S^1\times\boldsymbol{R}$, $(\boldsymbol{C}^*)^2\cong S^1\times S^1\times\boldsymbol{R}^2\cong S^1\times S^1\times S^2-S^1\times S^1\times p$ $(p\in S^2)$ as differentiable manifolds. The construction of M is also valid in this situation. So $M\cong S^1\times S^1\times S^1$. Hence $\pi_1(M)=H_1(M,\boldsymbol{Z})=\boldsymbol{Z}\oplus\boldsymbol{Z}\oplus\boldsymbol{Z}$. Since $\pi_1(M)\to\pi_1(C)$ is surjective, $\pi_1(C)$ is commutative.

In general $\pi_1(C) = \pi_1(C_1) * \cdots * \pi_1(C_n) * \pi_1(G)$, the free product of the fundamental groups of the irreducible components C_j and that of the graph G of $\bigcup C_j$. $\pi_1(G)$ is a free group with p generators when G is of the first Betti number p. In order that $\pi_1(C)$ is commutative, all but one of the components of the product must vanish, moreover if $\pi_1(C_j) \neq 1$, the genus of C_j is 1, and if $\pi_1(G) \neq 1$, G contains only one loop.

Thus we have the following three possibilities regarding the configuration of C:

i) $\pi_1(C) = H_1(C, \boldsymbol{Z}) = 0$, all C_j are rational curves and the graph is a tree,

ii) $\pi_1(C) = H_1(C, \boldsymbol{Z}) = \boldsymbol{Z}$, all C_j are rational curves and the graph contains only one loop.

iii) $\pi_1(C) = H_1(C, \boldsymbol{Z}) = \boldsymbol{Z} \oplus \boldsymbol{Z}$, there is only one elliptic curve, the others are, if they exist, all rational, and the graph is a tree.

Let us next consider the exact homology sequence with real coefficients:

$$0 \to H_3(S) \to H_3(S, C) \to H_2(C) \overset{i_*}{\to} H_2(S) \to H_2(S, C) \to H_1(C) \to H_1(S) \to 0 ,$$

where $b_1(S) = b_3(S)$, $\dim H_3(S, C) = \dim H^1(S - C) = 2$, and $\dim H_2(S, C) = \dim H^2(S - C) = 1$, by duality; $\dim H_2(C) = n$, the number of the irreducible components of C.

Combining these we have the following five cases:

(a) $b_1(C) = 0$, $b_1(S) = 0$, $b_2(S) = n - 1$, $b^+ = 1$,
(b) $b_1(C) = 1$, $b_1(S) = 0$, $b_2(S) = n - 2$, $b^+ = 1$,
(c) $b_1(C) = 1$, $b_1(S) = 1$, $b_2(S) = n$, $b^+ = 0$,
(d) $b_1(C) = 2$, $b_1(S) = 1$, $b_2(S) = n - 1$, $b^+ = 0$,
(e) $b_1(C) = 2$, $b_1(S) = 2$, $b_2(S) = n + 1$, $b^+ = 1$.

Before dealing with each case we prove two lemmas.

LEMMA 1. *Let E be an N-dimensional vector space over the field of real numbers and $I(x, y)$ a non-singular symmetric bilinear form defined on E with signature* $(1, N - 1)$. *If the restriction I' of I to an* $(N - 1)$*-dimensional subspace E' is of rank $N - 2$, then I' is negative semi-definite.*

PROOF. There is an orthogonal decomposition $E = E^+ \oplus E^-$, $\dim E^+ = 1$, $\dim E^- = N - 1$, $I|E^+$ is positive definite, $I|E^-$ is negative definite. Since $\dim E' \cap E^- \geqq N - 2$, $I' = I|E'$ has at least $N - 2$ negative eigenvalues. Then I' cannot have any positive eigenvalue.

q.e.d.

LEMMA 2. *Let $C = \bigcup_{j=1}^{n} C_j$ be as in* 3°, *and suppose that the intersection matrix* $(I(C_j, C_k))$ *is negative semi-definite, then*

i) *if* $I(\sum_{j=1}^{n} r_j C_j, \sum_{j=1}^{n} r_j C_j) = 0$ $(r_j \in \boldsymbol{R})$, *then* r_j *are all positive, negative or zero simultaneously.*

ii) $\mathrm{rank}\,(I(C_j, C_k)) \geqq n-1$,

iii) *if* $\{j(1), \cdots, j(n')\} \subsetneqq \{1, \cdots, n\}$, *then the* $n' \times n'$ *matrix* $(I(C_{j(\nu)}, C_{j(\mu)}))$ *is negative definite.*

PROOF. i) Set $J^+ = \{j; r_j > 0\}$ and $J^- = \{j; r_j < 0\}$. Then

$$I(\sum_{j \in J^+} r_j C_j, \sum_{j \in J^+} r_j C_j) + I(\sum_{k \in J^-} r_k C_k, \sum_{k \in J^-} r_k C_k) + 2 \sum_{j \in J^+, k \in J^-} r_j r_k I(C_j, C_k) = 0 .$$

Since all the terms are non-positive, each term is zero. If $J^+ \neq \emptyset$ or $\{1, \cdots, n\}$, there would exist, since C is connected, a $j_0 \notin J^+$ such that $I(C_{j_0}, \sum_{j \in J^+} r_j C_j) > 0$, and then

$$I(C_{j_0} + r \sum_{j \in J^+} r_j C_j, C_{j_0} + r \sum_{j \in J^+} r_j C_j) = I(C_{j_0}, C_{j_0}) + 2rI(C_{j_0}, \sum_{j \in J^+} r_j C_j)$$

would be positive for sufficiently large r. Therefore $J^+ = \emptyset$ or $\{1, \cdots, n\}$. Similarly $J^- = \emptyset$ or $\{1, \cdots, n\}$. ii) and iii) are easily derived from i).

We now return to the proof of the theorem.

Case (a). We shall show that this case does not occur. Let us suppose S and C satisfy the conditions in (a). By the corollary to Proposition 2, $\mathrm{rank}\,(I(C_j, C_k)) = n-3$. $\dim H_2(S) = n-1$, $\dim \mathrm{Im}\, i_* = n-2$ and $\mathrm{rank}\,(I|\mathrm{Im}\, i_*) = \mathrm{rank}\,(I(C_j, C_k)) = n-3$. Then by Lemma 1, $(I(C_j, C_k))$ is negative semi-definite. This contradicts Lemma 2 ii).

Case (b). S is rational by Proposition 1. Examples of this case are

(i) S: the complex projective plane $\boldsymbol{P}^2$, C: the union of three lines $L_j (j = 1, 2, 3)$ in general position,

(ii) S: a line bundle over a rational curve compactified with infinity section, i.e., $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1$ (Hirzebruch manifold Σ_n or Nagata's F_n), C: the union of zero section, infinity section and two fibers. It is difficult for the author to determine all the types of configuration of C. Some remarks on this case will be made at the end of this paper.

Case (c). This case does not occur. In fact, by the exact sequence $\dim \mathrm{Im}\, i_* = n-1$. Since I is negative definite, $\mathrm{rank}\,(I(C_j, C_k)) = \mathrm{rank}\,(I|\mathrm{Im}\, i_*) = n-1$. This contradicts the corollary to Proposition 2.

In cases (d) and (e), C is the union of an elliptic curve C_1 and a certain number of trees $D_1, D_2, \cdots$ composed of rational curves, $D_\nu \cap D_\mu = \emptyset$, $D_\nu \cap C_1 = p_\nu$. We shall prove that each D_ν is exceptional of the first kind. It suffices to show the following:

i) The intersection matrix with respect to the components of D_ν is negative definite.

In fact, in case (d), $(I(C_j, C_k))$ is negative semi-definite. We can apply Lemma 2 iii). In case (e), $\dim \operatorname{Im} i_* = n$, rank $(I(C_j, C_k)) = n-1$. Therefore by Lemma 1, $(I(C_j, C_k))$ is negative semi-definite, and we can apply Lemma 2 iii).

ii) M_ν, the boundary of a tubular neighborhood of D_ν, is simply connected.

We define everything as in 3°, then by definition $M_\nu = \partial(\bigcup_{C_j \subset D_\nu} V_j)$. In the coordinate neighborhood U_ν of p_ν, let $C_1 \cap U_\nu = \{z_1 = 0\}$ and $D_\nu \cap U_\nu = \{z_2 = 0\}$. Let Δ be a disc in C_1 with center p_ν defined by $\Delta = \{z_1 = 0, |z_2| < \varepsilon\}$. We form $M' = M_\nu \cup (C_1 - \Delta) \cup (\bigcup_{\mu \neq \nu} D_\mu)$ and define a continuous map $\psi\colon M \to M'$ by

$$\psi(z_1, z_2) = \left(\frac{(|z_1| - \varepsilon)z_1}{(1-\varepsilon)|z_1|}, z_2\right) \quad \text{for} \quad p \in M \cap U'_\nu \,,$$

$$\psi(p) = p \qquad \text{for} \quad p \in (M - U'_\nu) \cap M_\nu \,,$$

$$\psi(p) = \varphi(p) \qquad \text{for} \quad p \in (M - U'_\nu) - M_\nu \,.$$

The homomorphism $\pi_1(M) \to \pi_1(M')$ induced by ψ is surjective as is easily verified. Hence $\pi_1(M')$ is commutative. From this fact we shall derive $\pi_1(M_\nu) = 1$.

Let us recall here the notion of free product of groups with amalgamation. Let groups G and G' be represented by sets of generators and fundamental relations:

$$G = \langle X_1, \cdots, X_r \,|\, R_1 = 1, \cdots, R_s = 1\rangle \,,$$
$$G' = \langle X'_1, \cdots, X'_{r'} \,|\, R'_1 = 1, \cdots, R'_{s'} = 1\rangle \,.$$

If G and G' contain subgroups H and H' respectively with an isomorphism $i\colon H \to H'$, then the free product of G and G' with amalgamation along H and H' is defined by

$$\langle X_1, \cdots, X_r, X'_1, \cdots, X'_{r'} \,|\, R_1 = 1, \cdots, R_s = 1, \cdots, R'_{s'} = 1, i(x) = x \ (x \in H)\rangle \,.$$

It is known that this contains subgroups isomorphic to G and to G'.

We set $\pi_1(M_\nu) = \langle \alpha_1, \cdots, \alpha_r \,|\, R_1 = 1, \cdots, R_s = 1\rangle$, and $\pi_1((C_1 - \Delta) \cup (\bigcup_{\mu \neq \nu} D_\mu)) = \pi_1(C_1 - \Delta) = \langle \beta, \gamma\rangle$, the free group generated by β and γ. By van Kampen's theorem,

$$\pi_1(M') = \langle \alpha_1, \cdots, \alpha_r, \beta, \gamma \,|\, R_1 = 1, \cdots, R_s = 1, \alpha = \beta\gamma\beta^{-1}\gamma^{-1}\rangle \,,$$

where $\alpha \in \pi_1(M_\nu)$ is the element represented by the loop $\partial\Delta = M_\nu \cap ((C_1 - \Delta) \cup (\bigcup D_\mu))$. If the order of α in $\pi_1(M_\nu)$ is infinite, $\pi_1(M')$ is the free product of $\pi_1(M_\nu)$ and $\pi_1((C_1 - \Delta) \cup (\bigcup D_\mu))$ with amalgamation along $\langle\alpha\rangle$ and $\langle \beta\gamma\beta^{-1}\gamma^{-1}\rangle$, which contains a subgroup isomorphic to $\langle \beta, \gamma\rangle$, and is therefore not commutative. This is a contradiction. Hence α must be of a finite order m. It is easily seen that, in $\langle\beta, \gamma \mid (\beta\gamma\beta^{-1}\gamma^{-1})^m = 1\rangle$, $\beta\gamma\beta^{-1}\gamma^{-1}$ is of order m. Then $\pi_1(M')$ is the free product of $\pi_1(M_\nu)$ and $\langle\beta, \gamma \mid (\beta\gamma\beta^{-1}\gamma^{-1})^m = 1\rangle$ with amalgamation along the subgroups $\langle\alpha\rangle$ and $\langle\beta\gamma\beta^{-1}\gamma^{-1}\rangle$. $\langle\beta, \gamma \mid (\beta\gamma\beta^{-1}\gamma^{-1})^m = 1\rangle$ is commutative if and only if $m = 1$. Hence $\pi_1(M')$ is the free product of $\pi_1(M_\nu)$ and $\langle\beta\gamma \mid \beta\gamma\beta^{-1}\gamma^{-1} = 1\rangle$ and is not commutative when $\pi_1(M_\nu) \neq 1$. ii) is thus proved.

Case (d). By blowing down all the D_ν's, we obtain a compactification $\check{S}$ with an irreducible curve $\check{C}$, and $\check{S} - \check{C} \cong (\boldsymbol{C}^*)^2$. We have $b_2(\check{S}) = 0$. Hence any curve on S is homologous to zero and the intersection multiplicity of any two curves is zero. In particular there is no irreducible curve different from $\check{C}$ which meets $\check{C}$. On the other hand, there is no curve which does not meet $\check{C}$, since $\check{S} - \check{C} \cong (\boldsymbol{C}^*)^2$ does not contain any compact curve. Thus there is no curve on $\check{S}$ other than $\check{C}$, and consequently there is no non-constant meromorphic function on $\check{S}$. Now we apply a theorem of Kodaira ([3] Theorem 34): If $b_1(\check{S}) = 1$, $b_2(\check{S}) = 0$, and if $\check{S}$ contains at least one curve and admits no non-constant meromorphic function, then $\check{S}$ is a Hopf surface. Therefore $\check{S}$ is of the form $(\boldsymbol{C}^2 - 0)/G$, where G is a group of transformations generated by a transformation (A) or by transformations (A) and (B).

(A): $(z_1, z_2) \to (\alpha_1 z_1 + \lambda z_2^m, \alpha_2 z_2)$,

(B): $(z_1, z_2) \to (\varepsilon_1 z_1, \varepsilon_2 z_2)$

where m is an integer, α_1, α_2 and λ are complex numbers, ε_1 and ε_2 are primitive l-th roots of unity, with conditions $0 < |\alpha_1| \leqq |\alpha_2| < 1$, $(\alpha_1 - \alpha_2^m)\lambda = 0$, and $(\varepsilon_1 - \varepsilon_2^m)\lambda = 0$. ([3] Theorem 32).

In our case we have $\lambda \neq 0$, since otherwise $\check{S}$ contains two curves defined by $z_1 = 0$ and $z_2 = 0$. Conversely, if $\lambda \neq 0$, $\check{S}$ contains only one irreducible curve $\check{C}$ defined by $z_2 = 0$, and $\check{S} - \check{C} \cong (\boldsymbol{C}^*)^2$ as is easily verified. This is the type (1) of the theorem.

Case (e). By blowing down all the D_ν's we obtain a compactification $\check{S}$ with $\check{C}$. $\check{S}$ is algebraic ([2] Theorem 10), and $\check{C}$ is irreducible. This situation was investigated by Simha [9]: By the Albanese mapping, $\check{S}$ is mapped onto an elliptic curve T, and each fiber is regular and rational, i.e., $\check{S}$ is a $\boldsymbol{P}^1$-bundle over T. $\check{C}$ is a global section, and $\check{S} - \check{C}$ is a non-trivial principal $\boldsymbol{C}$-bundle. Conversely a unique $\boldsymbol{P}^1$-bundle of this type is a compactification of $(\boldsymbol{C}^*)^2$. This is the type (2) of the theorem.

REMARK. It is not known to the author whether, for every rational compactification S with C of $(\boldsymbol{C}^*)^2$, there exists a bimeromorphic map f

of S to $\boldsymbol{P}^2$ such that $f|S-C$ is a biholomorphic map of $S-C$ onto $\boldsymbol{P}^2 - \bigcup L_j$.

This problem can be, in the following special case, affirmatively answered.

PROPOSITION 3. *Assume that* $C = \bigcup_{j=1}^n C_j$ *satisfies the condition* (#), *and* $I(C_j, C_{j+1}) = 1$, $j = 1, \cdots, n$, *where we let* $C_{n+1} = C_1$, *and that* C_j*'s have no other intersection. Then, blowing down successively irreducible components* C_j *with* $I(C_j, C_j) = -1$, *we can reduce* S *with* C *to* (i) *or* (ii) *of the case* (b).

PROOF. The kernel K of the surjective homomorphism $H_1(M, \boldsymbol{Z}) \to H_1(C, \boldsymbol{Z})$ is isomorphic to $\boldsymbol{Z} \oplus \boldsymbol{Z}$. On the other hand, by Proposition 2, K is generated by $\alpha_1, \cdots, \alpha_n$, with the relations

$$\alpha_{j-1} + v_j\alpha_j + \alpha_{j+1} = 0 \ , \quad j = 1, \cdots, n \ ,$$

where $\alpha_0 = \alpha_n$, $\alpha_{n+1} = \alpha_1$, and $v_j = I(C_j, C_j)$.
These relations are written in the form

$$\begin{pmatrix} \alpha_{j-1} \\ \alpha_j \end{pmatrix} = A_j \begin{pmatrix} \alpha_j \\ \alpha_{j+1} \end{pmatrix} , \quad \text{where} \quad A_j = \begin{pmatrix} -v_j & -1 \\ 1 & 0 \end{pmatrix} , \quad j = 1, \cdots, n \ .$$

Hence we see, eliminating $\alpha_2, \cdots, \alpha_{n-1}$, that K is generated by α_0 and α_1 with the relation

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = A_1 \cdots A_n \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} .$$

In order that $K \cong \boldsymbol{Z} \oplus \boldsymbol{Z}$, this relation must be trivial, i.e., $A_1 \cdots A_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$.

By direct computation we have, in the case $n = 3$, $v_1 = v_2 = v_3 = 1$, and in the case $n = 4$, $v_1 = v_3 = v_2 + v_4 = 0$, or $v_2 = v_4 = v_1 + v_3 = 0$. These are cases (i) and (ii) as is readily seen.

We prove that, if $n \geqq 5$, there exists a C_j with $v_j = -1$. First we see that, if $n \geqq 6$, there is no pair C_j, C_k such that $v_j, v_k \geqq 0$ and $I(C_j, C_k) = 0$. Assume the contrary, and take $C_{j'}$ and $C_{k'}$ such that

$$I(C_j, C_{j'}) = I(C_k, C_{k'}) = 1 \ , \quad I(C_j, C_{k'}) = I(C_k, C_{j'}) = I(C_{j'}, C_{k'}) = 0 \ .$$

Then we have $I(r_jC_j + C_{j'}, r_kC_k + C_{k'}) = 0$, $I(r_jC_j + C_{j'}, r_jC_j + C_{j'}) > 0$ and $I(r_kC_k + C_{k'}, r_kC_k + C_{k'}) > 0$, for sufficiently large r_j, r_k. This contradicts the fact that $b^+ = 1$. We see similarly that, if $n = 5$, there is no pair C_j, C_k with $v_j > 0$, $v_k \geqq 0$ and $I(C_j, C_k) = 0$. It suffices to consider the following four cases: (a) $v_j < 0$, $j = 1, \cdots, n$. (b) $v_1 \geqq 0$, $v_j < 0$,

$j = 2, \cdots, n$. (c) $v_1, v_2 \geqq 0$, $v_j < 0$, $j = 3, \cdots, n$. (d) $n = 5$, $v_1 = v_3 = 0$. The case (d) is omitted by direct computation. In the first three cases, we have one of the equations, $I = A_1 \cdots A_n$, $A_1^{-1} = A_2 \cdots A_n$, or $A_2^{-1}A_1^{-1} = A_3 \cdots A_n$, where the left term is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & -v_1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} -1 & -v_1 \\ v_2 & v_1v_2 - 1 \end{pmatrix}.$$

If $v_j \neq -1$, setting the right term $= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $a > c > 0$, which is a contradiction. The above assertion is proved by induction using the fact:

In the equation $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} k & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if $k \geqq 2$ and if $a > c > 0$, then $a' > c' > 0$.

Thus one of the irreducible components of C can be blown down and n is diminished until we have $n = 4$, and the proof is completed.

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DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY
KYOTO, 606 JAPAN