C. Diemer, L. Katzarkov, G. Kerr, Compactifications of spaces of Landau–Ginzburg models, *Izv. RAN. Ser. Mat.*, 2013, Volume 77, Issue 3, 55–76

УДК 512.7
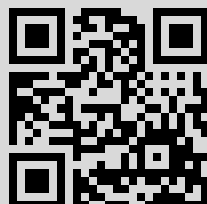
C. Diemer, L. Katzarkov, G. Kerr

Compactifications of spaces of Landau–Ginzburg models

This paper reviews the results and techniques from the authors' work on Landau–Ginzburg degenerations and applies them in basic examples. The main example is the A_n-category, where we observe a relationship to stability conditions and directed quiver representations. We conclude with a brief survey of applications to the birational geometry of del Pezzo surfaces.

Bibliography: 31 titles.

Dedicated to I. R. Shafarevich on the occasion of his 90*th birthday*

§ 1. Introduction

One case of homological mirror symmetry is an equivalence between the derived category of coherent sheaves on a Fano variety X and the Fukaya–Seidel category of its mirror Landau–Ginzburg, or LG, model $\mathbf{w} \colon X^{\mathrm{mir}} \to \mathbb{C}$. There are many constructions of the mirror [1], [2] but all depend on a choice of symplectic form on X. Moving within the complexified Kähler cone of X gives an open parameter space of mirror LG models. While the Fukaya–Seidel categories of any two mirror LG models from this space are equivalent, we may assign distinct exceptional collections and semi-orthogonal decompositions to certain regions. We observe that these decompositions should be related to the space of stability conditions of $D^b(X)$. For more on LG models from this vantage point see [3]–[6].

In [7], [8], the authors examine these phenomena in the toric context and compactify the space of LG models into a toric stack $\mathcal{M}_{A,A'}$ using methods from [9], [10]. The boundary of this stack gives degenerations of the LG models where both the fibres and the base of the model degenerate. Examining the fixed points of $\mathcal{M}_{A,A'}$, we see that an LG model decomposes into a chain of regenerated circuit LG models. In [7] we considered the symplectic topology of these degenerated pieces and observed that, under homological mirror symmetry, they correspond to semi-orthogonal components of $D^b(X)$ obtained by running the toric minimal model programme on the mirror toric Fano X. We expect that this type of correspondence between Mori-theoretic semi-orthogonal decompositions and degenerated LG models holds in much more generality, leading to a new approach to birational geometry.

In addition to reviewing the definitions and theorems in the approach outlined above, we give a detailed account of a basic example, which is still rich in structure. Here our LG models are simply single-variable polynomials $f \colon \mathbb{C} \to \mathbb{C}$ of degree $n+1$ and the Newton polytope is simply the interval $[0, n+1]$. Because the fibres

are finite sets, this allows us to set aside much of the technical symplectic topology in [7]. The secondary polytope of the interval was investigated in [9], where it was shown to be a cube whose lattice structure is strongly related to A_n representation theory. We investigate the universal family over the toric stack of this cube which was identified with a quotient of the Losev–Manin space in [11]. Finally, we analyze the monotone path polytope in this setting, as well as the combinatorial structure of the vanishing thimbles near the degenerated LG models. After this careful study, we observe connections with the classical representations of A_n quivers and an interpretation of reflection functors as wall crossing in the space of stability conditions of the A_n-category.

In the final section of the paper, we explore the homological mirror of the three-point blow-up X of $\mathbb{P}^2$. We build on the work of [12], which studied relations between Sarkisov links. In the usual set-up of Sarkisov links, earlier contractions do not play a prominent role. We explain how the toric compactification of the LG model mirror of X preserves these data and gives a more complete picture of the minimal model program for X.

§ 2. Toric Landau–Ginzburg models

In this section we review constructions from [7] which compactify the moduli of hypersurfaces in a toric stack, and a moduli space of LG models. This is followed by a detailed definition of radar screens, which are distinguished bases for the LG models designed to preserve categories in the degenerated models. The choices involved in defining these bases are condensed into a torsor over the monotone path stack.

2.1. Toric stacks and LG models. We start by introducing the toric machinery that we need for the rest of the paper. Letting M be a rank d lattice and A a finite subset of M, we take $\bar{A} \subset N$ to be the collection of primitive normal vectors to facets of $Q = \mathrm{Conv}(A)$. Here we use the usual notation for $N = \mathrm{Hom}(M, \mathbb{Z})$ and write $\mathrm{Conv}(A)$ for the convex hull of a set of points. The normal fan $\mathcal{F}_Q$ of Q has $\bar{A}$ as the set of generators for rays and defines an abstract simplicial complex structure on $\bar{A}$. We take the toric variety X_Q to be the variety associated with $\mathcal{F}_Q$.

To promote this to a toric stack, we follow the prescription given in [13] and [14]. For any set B, take $\{e_\alpha \colon \alpha \in B\}$ to be a basis for $\mathbb{R}^B$. Define $U_Q \subset \mathbb{C}^{\bar{A}}$ to be the open toric variety given by taking the fan in $\mathbb{R}^{\bar{A}}$ consisting of cones $\mathrm{Cone}\{e_\alpha \colon \alpha \in \sigma\}$, where σ is any cone in the normal fan $\mathcal{F}_Q$ of Q. The map $\beta_{\bar{A}} \colon \mathbb{Z}^{\bar{A}} \to N$ is defined to take e_α to α and we write its kernel and cokernel as $L_{\bar{A}}$ and $K_{\bar{A}}$. Define the group $\mathbb{L}_Q = (L_{\bar{A}} \otimes \mathbb{C}^*) \oplus \mathrm{Tor}(K_{\bar{A}}, \mathbb{C}^*)$ as a subgroup of $(\mathbb{C}^*)^{\bar{A}}$ using the inclusion and connecting homomorphism to obtain the stack

$$\mathcal{X}_Q = [U_Q / \mathbb{L}_Q].$$

The variety X_Q is the coarse space of $\mathcal{X}_Q$. As in the case of toric varieties defined from polytopes, the stack $\mathcal{X}_Q$ comes equipped with a line bundle $\mathcal{O}_Q(1)$. Letting $Q_{\mathbb{Z}} = Q \cap M$, the space of sections $H^0(\mathcal{X}_Q, \mathcal{O}_Q(1))$ has an equivariant basis $\{s_\alpha \colon \alpha \in Q_{\mathbb{Z}}\}$ and a linear system $\mathcal{L}_A = \mathrm{Span}\{s_\alpha \colon \alpha \in A\}$. We distinguish two open subsets of $\mathcal{L}_A$, the full sections

$$\mathcal{L}_A^f = \Big\{ s = \sum c_\alpha s_\alpha \colon c_\alpha \neq 0 \text{ for all vertices } \alpha \in Q \Big\},$$

and the very full sections

$$\mathcal{L}_A^{vf} = \Big\{ s = \sum c_\alpha s_\alpha \colon c_\alpha \neq 0 \text{ for every } \alpha \in A \Big\}.$$

As illustrated in Fig. 1, we take the sets $\mathbf{A}_1 = \{0, 1, 2, 3\} \subset \mathbb{Z}$ and $\mathbf{A}_2 = \{(0,0), (-1,-1), (-1,0), (0,1), (1,1), (1,0), (0,-1)\} \subset \mathbb{Z}^2$. The first example gives $X_{\mathbf{Q}_1} = \mathbb{P}^1$, with line bundle $\mathcal{O}_{\mathbf{A}_1}(1) = \mathcal{O}(3)$ and the linear system $\mathcal{L}_{\mathbf{A}_1}$ consisting of all sections. The full sections $\mathcal{L}_{\mathbf{A}_1}^f$ are those that do not vanish at the torus fixed points 0 and ∞. The second example $X_{\mathbf{Q}_2}$ is a 3-point blow-up of $\mathbb{P}^2$. The bundle $\mathcal{O}_{\mathbf{Q}_2}(1)$ in this case is the anti-canonical bundle and the linear system again consists of all sections.

FIGURE 1. The sets $\mathbf{A}_1$ and $\mathbf{A}_2$

Many of the LG models arising in homological mirror symmetry are obtained from pencils on $\mathcal{X}_Q$ contained in $\mathcal{L}_A$. It is common for the behaviour of these pencils at infinity and zero to be prescribed. We now give a concise definition of this constraint.

DEFINITION 2.1. Let A' be a proper subset of A. An A'-*sharpened pencil* on $\mathcal{X}_Q$ is a pencil $W \subset \mathcal{L}_A$ which has a basis $\{s_1, s_\infty\}$ for which $s_1 = \sum_{\alpha \in A} c_\alpha s_\alpha \in \mathcal{L}_A^{vf}$ and $s_\infty = \sum_{\alpha \in A'} c_\alpha s_\alpha$.

Let $U_{A,A'}$ be the open subset of A'-sharpened pencils in the Grassmannian $\mathrm{Gr}_2(H^0(\mathcal{X}_Q, \mathcal{O}_Q(1)))$. We now examine two other equivalent ways of defining an A'-sharpened pencil. Given $s_1 \in W$ as in the definition above, take $s_0 = \sum_{\alpha \notin A'} c_\alpha s_\alpha$. Letting $(A')^\circ$ be the complement of A', the pair $(s_0, s_\infty) \in \mathbb{C}^{(A')^\circ} \times \mathbb{C}^{A'}$ gives another basis for W which is unique up to a multiple $(\lambda s_0, \lambda s_\infty)$ for some $\lambda \in \mathbb{C}^*$. We define

$$\mathbf{w} = [s_0 : s_\infty] \colon \mathcal{X}_Q - \{s_\infty = 0\} \to \mathbb{C}$$

to be the LG model of the A'-sharpened pencil W.

Alternatively, we may write $W \in U_{A,A'}$ as the closure of an equivariant map, or orbit, $i_W \colon \mathbb{C}^* \to \mathcal{L}_A^{vf}$. Taking the one-parameter subgroup $G_{A'} \subset (\mathbb{C}^*)^A$ given by

the cocharacter $\gamma_{A'} = \sum_{\alpha \in A'} e_\alpha^\vee \in (\mathbb{Z}^A)^\vee$ and any very full section $s \in W$, observe that $W = \overline{\{\lambda \cdot s \colon \lambda \in G_{A'}\}}$. When referring to an A'-sharpened pencil, we may utilize any of these three equivalent viewpoints. We will observe examples of these pencils in the next section and see that the orbit perspective turns out to be quite useful.

In general, the fibres of $\mathbf{w}$ over 0 and ∞ have bad behaviour which is corrected by judicious blow-ups. We explain this bad behaviour from a global perspective. Let $\mathcal{D}_Q = \sum \mathcal{D}_i$, where the sum is over the facets of Q, be the toric boundary of $\mathcal{X}_Q$ and for any subset J of facets, let $\mathcal{Z}_J = \bigcap_{j \in J} \mathcal{D}_j$. If $s \in \mathcal{L}_A$, write $\mathcal{Y}_s$ for the hypersurface defined by s and $\mathcal{Y}_{s,J} = \mathcal{Y}_s \cap \mathcal{Z}_J$. For any subset $U \subset \mathcal{L}_A$, we have the incidence stacks $\mathcal{I}(U) \subset U \times \mathcal{X}_Q$ and $\mathcal{I}_J \subset U \times \mathcal{X}_Q$, whose points are given by pairs $\{(s,y) \colon s \in U, y \in \mathcal{Y}_s\}$ and $\{(s,y) \colon s \in U, y \in \mathcal{Y}_{s,J}\}$ respectively.

PROPOSITION 2.2. *The set* $U = \mathcal{L}_A^f$ *is the maximal open subset of* $\mathcal{L}_A$ *for which the projection* $\pi_{\mathcal{L}_A} \colon \mathcal{I}_J(U) \to U$ *is flat for all subsets* J.

PROOF. This follows from the observation that the sections which are not full are equivalent to sections that contain fixed points of the toric action. Thus they contain zero-dimensional intersections $\mathcal{Z}_J$.

For our purposes, a reasonable moduli space of sections should not exhibit this behaviour. In the next subsection, we modify these sections along with their fibres in order to obtain a proper flat family.

2.2. The secondary stack. To remedy the fact that the incidence varieties give a poorly behaved parameter space for the hypersurface, we review the constructions of the secondary and Lafforgue stacks given in [7], where more details can be found. We assume that the reader is familiar with material found in [13], [14] and [9]. Given A as above, the *secondary polytope* $\Sigma(A) \subset \mathbb{R}^A$ is an $(|A| - d - 1)$-dimensional polytope whose faces correspond to regular subdivisions $S = \{(Q_i, A_i) \colon i \in I\}$ of A. Here I is some indexing set of marked polytopes (Q_i, A_i) which subdivide (Q, A). The normal fan $\mathcal{F}_{\Sigma(A)}$ of $\Sigma(A)$ can be refined to a fan $\mathcal{F}_{\Theta(A)}$ as in [15] and [10] by considering pairs (S, Q') of a subdivision S along with a set Q' which is a face of a subdivided polytope $(Q_i, A_i) \in S$. Then a cone $\sigma_{(S,Q')}$ in $\mathcal{F}_{\Theta(A)}$ is defined as all functions η on A whose lower convex hull gives the marked subdivision S and whose minimum is achieved on $Q' \cap A$.

PROPOSITION 2.3 [7]. *If* $\Delta^A \subset \mathbb{R}^A$ *is the unit simplex, then* $\mathcal{F}_{\Theta(A)}$ *is the normal fan of the Minkowski sum* $\Theta(A) := \Sigma(A) + \Delta^A \subset \mathbb{R}^A$.

As the secondary polytope $\Sigma(A)$ lies in a translation of the kernel $L_A \otimes \mathbb{R} := \ker(\mathbb{R}^A \to M_\mathbb{R} \oplus \mathbb{R})$, it enjoys $(d + 1)$ constraints. Similarly, the polytope $\Theta(A)$, which we call the *Lafforgue polytope*, lies in a translation of the hyperplane $H = \{\sum_\alpha c_\alpha e_\alpha \colon \sum_\alpha c_\alpha = 0\}$ and therefore has $(|A| - 1)$ dimensions. This drastically limits the number of examples of Lafforgue polytopes that one can visualize, but our case $\mathbf{A}_1$ rendered in Fig. 2 gives an indication of the relationship between the original marked polytope (Q, A), the secondary polytope $\Sigma(A)$ and Lafforgue polytope $\Theta(A)$. Observe that corresponding to each vertex of the secondary polytope, there is a regular triangulation of (Q, A) which can be seen as a unique subset
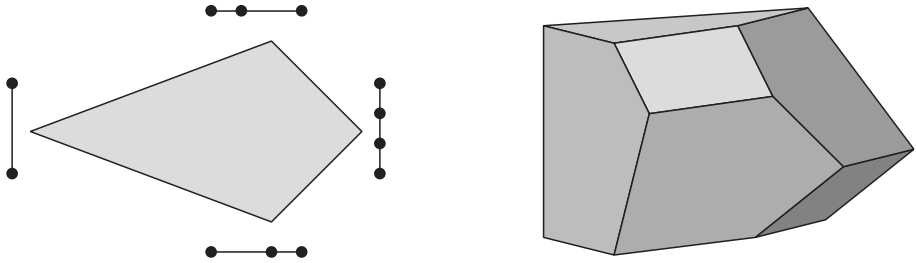
FIGURE 2. The secondary polytope and Lafforgue polytope for $\mathbf{A}_1$

of the facets of the Lafforgue polytope. The second example, $\mathbf{A}_2$ has a 4-dimensional secondary polytope with 32 vertices and a 7-dimensional Lafforgue polytope. Nevertheless, we will be able to use a polytope derived from these data to analyze the minimal model runs of the homological mirror of $\mathcal{X}_{\mathbf{Q}_2}$ in the last section.

Since $\Theta(A)$ is a Minkowski sum, we have maps $\widetilde{\pi}_A \colon X_{\Theta(A)} \to X_{\Sigma(A)}$ and $\widetilde{\pi}_Q \colon X_{\Theta(A)} \to \mathbb{P}^{|A|-1}$. If $i \colon p \hookrightarrow X_{\Sigma(A)}$ sends a point to the orbit $\operatorname{orb} S$ associated with a subdivision $S = \{(Q_i, A_i) \colon i \in I\}$, then we may define X_S as the pullback in the fibre square

$$
\begin{array}{ccc}
X_S & \xrightarrow{\ j\ } X_{\Theta(A)} & \xrightarrow{\ \widetilde{\pi}_Q\ } \mathbb{P}^{|A|-1} \\
\rho_S \downarrow & \ \ \downarrow \widetilde{\pi}_A & \\
p & \xrightarrow{\ i\ } X_{\Sigma(A)} &
\end{array}
\tag{2.1}
$$

One only needs to trace through the definitions to see that $\widetilde{\pi}_Q \circ j$ maps X_S into the union of the images of the toric varieties X_{Q_i} via their $\mathcal{L}_{A_i}$ maps. At the level of varieties, this gives us a simultaneous degeneration of X_Q and $\mathcal{O}_Q(1)$. Taking a global section of $\widetilde{\pi}_Q^*(\mathcal{O}(1))$ yields a degeneration of hypersurfaces. In this way, we have a universal space for performing the degenerations along the lines of the Mumford construction.

We would like to promote this set-up to a morphism of stacks $\pi \colon \mathcal{X}_{\Theta(A)} \to \mathcal{X}_{\Sigma(A)}$ in such a way that $\mathcal{I}(\mathcal{L}_A^f)$ has an étale map to $\mathcal{X}_{\Theta(A)}$ and the quotient $[\mathcal{I}(\mathcal{L}_A^f)/(\mathbb{C}^*)^{d+1}]$ is naturally an open substack of $\mathcal{X}_{\Sigma(A)}$. This was carefully done in [7] and we review the procedure here.

Both the secondary and Lafforgue polytopes have vertices in a hyperplane parallel to $H_{\mathbb{Z}} = \{\sum_\alpha c_\alpha e_\alpha \colon \sum_\alpha c_\alpha = 0\} \subset \mathbb{Z}^A$. We consider both polytopes to live in $H \subset \mathbb{R}^A$ and write $i_H \colon H_{\mathbb{Z}} \to \mathbb{Z}^A$ for the inclusion. As in the case of $\bar{A}$, there is an exact sequence

$$
0 \longrightarrow L_A \xrightarrow{\ \delta_A\ } \mathbb{Z}^A \xrightarrow{\ \beta_A\ } M \longrightarrow K_A \longrightarrow 0
\tag{2.2}
$$

The hyperplanes supporting $\Theta(A)$ can be partitioned into horizontal and vertical hyperplane sections $\overline{\Theta(A)} = \overline{\Theta(A)}^h \cup \overline{\Theta(A)}^v$, and the vertical hyperplanes are all scalar multiples of $(\beta_A \circ i_H)^\vee(\bar{A})$. To define a stacky fan, one must choose

generators for the one-cones. As opposed to taking primitives for the one-cones in $\overline{\Theta(A)}^v$, we take the generators in the image $(\beta_A \circ i_H)^\vee(\bar{A})$, while for the horizontal hyperplanes in $\overline{\Theta(A)}^h$, we choose the primitives of the hyperplanes in $\overline{\Theta(A)}^h$. This defines the stacky fan which gives Lafforgue stack $\mathcal{X}_{\Theta(A)}$ associated with A.

To give the definition of the secondary stack, we rely on a universal colimit construction for toric stacks. It is not hard to show that, if $\widetilde{\mathcal{X}}_{\Sigma(A)}$ is the stack given by $\Sigma(A)$, then $\delta_A^\vee$ induces a map

$$g\colon \mathcal{X}_{\Theta(A)} \to \widetilde{\mathcal{X}}_{\Sigma(A)}. \tag{2.3}$$

The colimit stack $\mathcal{X}_{\Sigma(A)}$ of g comes equipped with a map $\pi\colon \mathcal{X}_{\Theta(A)} \to \mathcal{X}_{\Sigma(A)}$. Both $\mathcal{X}_{\Sigma(A)}$ and π can be described by the universal property that if g can be factored into two flat, equivariant morphisms $h_1 \circ h_2$, where $h_1\colon \mathcal{X}_{\Theta(A)} \to \mathcal{X}$ and $\mathcal{X}$ is a (good) toric stack, then there is a map $k\colon \mathcal{X}_{\Sigma(A)} \to \mathcal{X}$ with $h_1 = k \circ \pi$. This property makes $\mathcal{X}_{\Sigma(A)}$ the best toric candidate for the moduli stack of hypersurfaces in $\mathcal{X}_Q$.

THEOREM 2.4 [7]. *There is a hypersurface* $\mathcal{Y}_A \subset \mathcal{X}_{\Theta(A)}$ *for which the map* $\pi\colon$ $\mathcal{Y}_A \cap (\partial\mathcal{X}_{\Theta(A)})_J \to \mathcal{X}_{\Sigma(A)}$ *is flat for all horizontal boundary strata* $J \subset \overline{\Theta(A)}^h$. *The stack* $\mathcal{X}_{\Sigma(A)}$ *contains a dense open substack* $V \approx [\mathcal{I}(\mathcal{L}_A^f)/(\mathbb{C}^*)^{d+1}]$ *for which* $\pi\colon \mathcal{Y}_A(V) \to V$ *is equivalent to the quotient map of* $[\mathcal{I}(\mathcal{L}_A^f)/\mathbb{C}^*]$.

This theorem shows that $\pi\colon \mathcal{Y}_A \to \mathcal{X}_{\Sigma(A)}$ is a reasonable compactification of the universal hypersurface over the moduli stack of toric hypersurfaces. It is this compactification that allows us to degenerate LG models and understand their components.

It will be useful to identify the hypersurface of sections in $V \subset \mathcal{X}_{\Sigma(A)}$ that do not transversely intersect the toric boundary of $\mathcal{X}_Q$. Recall that the principal A-determinant from [9] does precisely this and has the secondary polytope as its Newton polytope. Thus it can naturally be written as a section of $\mathcal{O}_{\mathcal{X}_{\Sigma(A)}}(1)$, and we write $\mathcal{E}_A$ for its zero loci.

2.3. The stack of LG models. In this subsection we will review a toric compactification of the space of LG models arising from A'-sharpened pencils. Near the fixed points of this compactification, we give a procedure for obtaining a semi-orthogonal decomposition of the directed Fukaya category of the model.

The geometry of a fibre polytope has already been proven useful in the case of a secondary polytope. As it turns out, this more general notion works well in describing several moduli problems in the toric setting [16]. In particular, given two toric varieties, or stacks, $\mathcal{X}_{Q_1}$ and $\mathcal{X}_{Q_2}$ arising from marked polytopes, one may define a space of equivariant morphisms $\psi\colon \mathcal{X}_{Q_1} \to \mathcal{X}_{Q_2}$ for which $\psi^*(\mathcal{O}_{Q_2}(1)) = \mathcal{O}_{Q_1}(1)$ up to toric equivalence. This space has a reasonable compactification to a toric stack whose moment polytope is the fibre polytope $\Sigma(Q_2, Q_1)$.

In the previous section, we examined the case where Q_2 was the simplex and $Q = Q_1$. This gave the secondary polytope $\Sigma(A) = \Sigma(Q_2, Q_1)$ as the moment polytope of the stack $\mathcal{X}_{\Sigma(A)}$, which was regarded as a compactification of the moduli stack of toric hypersurfaces of $\mathcal{O}_Q(1)$ in $\mathcal{X}_Q$. Prior to this construction, we considered LG models on Q to be A'-sharpened pencils W which were given as equivariant maps $i_W\colon \mathbb{C}^* \to \mathcal{L}_A^f$. Two A'-sharpened pencils $W, \widetilde{W}$ are equivalent if $W = \lambda\widetilde{W}$

for some $\lambda \in (\mathbb{C}^*)^{d+1}$. Thus, from the perspective of equivariant maps, an LG model is an equivariant map $\iota_W \colon \mathbb{C}^* \to [\mathcal{L}_A^f/(\mathbb{C}^*)^{d+1}]$ up to toric equivalence. By equivariant, we mean with respect to the torus embedding $\mathbb{C}^* \to L_A^\vee \otimes \mathbb{C}^*$ given by the cocharacter $\gamma := \delta_A^\vee(\gamma_{A'})$, where δ_A is defined in equation (2.2) and $\gamma_{A'} \in (\mathbb{Z}^A)^\vee$ in § 2.1. Note that the codomain of ι_W is an open chart for the stack $\mathcal{X}_{\Sigma(A)}$, implying that the collection of such maps is an open chart of equivariant maps from $\mathbb{P}^1$ to the toric stack $\mathcal{X}_{\Sigma(A)}$ with respect to the character map $\gamma \colon L_A \to \mathbb{Z}$.

This map $\gamma \colon L_A \to \mathbb{Z}$ induces a map $\Sigma(A) \to [0, N]$ on polytopes for some N determined by A'. The fibre polytope $\Sigma(\Sigma(A), [0, N])$ is known as the monotone path polytope (an example of an iterated polytope; see [17]) and is denoted by $\Sigma_\gamma(\Sigma(A))$. The associated fibre stack $\mathcal{M}_{A,A'}$ defined in [7] then serves as a compactification of the open set of $G_{\gamma_{A'}}$-orbits (where $G_{\gamma_{A'}} := \mathbb{Z}\langle\gamma\rangle \otimes \mathbb{C}^*$) contained in the dense subset of $\mathcal{X}_{\Sigma(A)}$. Its coarse space is simply the toric variety associated with $\Sigma_\gamma(\Sigma(A))$. We codify these notions in the following proposition.

PROPOSITION 2.5 [7]. *The quotient stack* $\mathcal{U}_{A,A'} = [U_{A,A'}/(\mathbb{C}^*)^{d+1}]$ *of* A'-*sharpened pencils forms an open dense subset of the proper toric stack* $\mathcal{M}_{A,A'}$. *The fixed points of* $\mathcal{M}_{A,A'}$ *are in one-to-one correspondence with parametric simplex paths of* $\gamma \colon \Sigma(A) \to [0, N]$ *and will be called maximal degenerations of* $\mathbf{w}$.

Recall from [18] that a parametric simplex path of a linear function on a polytope is an edge path which increases relative to the linear function. One consequence of the above construction is that, over any point ψ in $\mathcal{M}_{A,A'}$, there is a chain $\langle\psi_1, \ldots, \psi_t\rangle$ of projective lines which has a flat family of degenerated toric varieties (or stacks) lying over it. In the dense orbit, there is one such line, and the toric variety is irreducible, so we obtain an LG model. As we approach the toric boundary, we bubble ψ into a stable map $\{\psi_i\}$ on several components $\bigcup_1^t \mathbb{P}^1$, and simultaneously degenerate the fibres of the LG model. In a maximally degenerated LG model, we have a chain $\langle C_1, \ldots, C_k\rangle$ of maps to one-dimensional orbits of $\mathcal{X}_{\Sigma(A)}$. Such strata correspond to the edges of the secondary polytope $\Sigma(A)$, which in turn correspond to circuit modifications or bistellar flips of the triangulations at the vertices. In [7], components of the fibres over each stable component were examined and found to reproduce well-known relations in the mapping class group. They were also conjectured to represent homological mirrors to birational maps of the minimal model on $\mathcal{X}_Q^{\mathrm{mir}}$.

2.4. Semi-orthogonal decompositions. Our next goal is to stratify our space of LG models so that for every stratum, we obtain a semi-orthogonal decomposition of the Fukaya–Seidel category of the associated model. The decomposition we obtain will bear a direct relationship to the monotone paths corresponding to the maximal degenerations. To do this, we start by recalling the notion of a *radar screen*, which will yield the class of a distinguished basis of paths for the LG model [19]. The definition given here differs from that in [7], but generalizes it and has the advantage of being valid for a generic LG model. Before we start the definition, it is worth keeping in mind that radar screens are auxiliary concepts depending only on configurations of points in $\mathbb{C}^*$ and do not depend on any of the toric stack definitions given earlier. In fact, one can consider their definition to be a logarithmic

variant of the more conventional procedure which chooses a distinguished basis of paths to be those with constant imaginary value in the positive real direction [2].

Let $\mathfrak{G}_r$ be the symmetric group on r letters, $E_r = (\mathbb{C}^*)^r/\mathfrak{G}_r$ the parameter space of r unmarked points in $\mathbb{C}^*$ and $P = \{z_1, \ldots, z_r\} \in E_r$. We order the points in such a way that $|z_i| \geqslant |z_{i+1}|$ for $1 \leqslant i \leqslant r$ and choose a lift $\widetilde{P} = \{w_1, \ldots, w_r\}$ such that $e^{w_i} = z_i$. Inductively define paths $p_i \colon [0, \infty) \to \mathbb{C}$ starting at w_i as follows. For $i = 1$, we take the path $p_1(t) = w_1 + t$. Assuming that p_i has been defined, we take p_{i+1} to be the concatenation $p_i * \ell_i * \ell'_i$, where $\ell'_i(t) = w_i + t \operatorname{Im}(w_{i+1} - w_i)$ for $t \in [0, 1]$ and $\ell_i(t) = \operatorname{Re}(w_i) + \operatorname{Im}(w_{i+1}) - t \operatorname{Re}(w_i - w_{i+1})$. While the p_i form a set of overlapping paths, it is clear that for any ε, we can perturb p_i to $\tilde{p}_i$ in such a way that $\|p_i - \tilde{p}_i\|_{L^2} < \varepsilon$ and $\{\tilde{p}_i \colon 1 \leqslant i \leqslant r\}$ forms a distinguished basis for $\widetilde{P}$. Furthermore, if the values of $|z_i|$ are distinct, the distinguished basis defined in this way is unique up to an isotopy for $\varepsilon \ll 1$.

DEFINITION 2.6. With the notation above, we say that $\mathcal{B}_{\widetilde{P}} = \{e^{\tilde{p}_i} \colon 1 \leqslant i \leqslant r\}$ is a radar screen distinguished basis and take $\mathcal{R}_P$ to be the collection of all such bases. If $\widetilde{P} \subset \{w \in \mathbb{C} \colon 0 \leqslant \operatorname{Im}(w) < 2\pi\}$, we write $\mathcal{B}_P$ and call any such basis a fundamental radar screen.

Our main application of this definition is when P is the collection of critical values of an LG model $\mathbf{w} \in U_{A,A'}$. Let $\Delta_{A,A'}$ be the variety of all A'-sharpened pencils that do not transversely intersect the principal A-determinant $\mathcal{E}_A$ regarded as a subvariety of $U_{A,A'}$, $\mathcal{U}_{A,A'}$ or $\mathcal{M}_{A,A'}$. We denote its complements in $U_{A,A'}$ and $\mathcal{U}_{A,A'}$ by $V_{A,A'}$ and $\mathcal{V}_{A,A'}$ respectively. Take $\widetilde{E}_r = E_r/\mathbb{C}^*$ to be the quotient, where $\mathbb{C}^*$ acts by multiplication.

PROPOSITION 2.7. *Suppose that $r = |i_W^{-1}(\mathcal{E}_A)|$ for some $W \in V_{A,A'}$. The map $\mathbf{c} \colon V_{A,A'} \to E_r$ given by $\mathbf{c}(W) = i_W^{-1}(\mathcal{E}_A)$ can be completed to a commutative diagram*

$$
\begin{array}{ccc}
V_{A,A'} & \xrightarrow{\ \mathbf{c}\ } & E_r \\
\downarrow & & \downarrow \\
\mathcal{V}_{A,A'} & \xrightarrow{\ \tilde{\mathbf{c}}\ } & \widetilde{E}_r
\end{array}
$$

PROOF. This follows from the quasi-homogeneous property of the principal A-determinant with respect to the $(\mathbb{C}^*)^{d+1}$ action on $\mathbb{C}^A$ (see [9]). Indeed, if $i_W, i_{\widetilde{W}} \in U_{A,A'}$ are equivalent, then there exists $(\lambda, \eta) \in (\mathbb{C}^*)^{d+1} = \mathbb{C}^* \times (\mathbb{C}^* \otimes N)$ such that $\lambda(1 \otimes \beta_A)^\vee(\eta) i_W(z) = i_{\widetilde{W}}(z)$ for all $z \in \mathbb{C}^*$. But then $E_A(i_{\widetilde{W}}(z)) = 0$ if and only if $\lambda^v \beta_A(p)(\eta) E_A(i_W(z)) = 0$, where $v = (d+1)\operatorname{Vol}(Q)$, $p \in \Sigma(A)$ and M is identified with $\operatorname{Hom}(\mathbb{C}^* \otimes N, \mathbb{C}^*)$.

This proposition shows that a choice of radar screen for the critical values of an A'-sharpened pencil can be consistently made on the quotient space $\mathcal{V}_{A,A'}$. Now, the discriminant $\Delta \colon E_r \to \mathbb{C}$ given by $\Delta(z_1, \ldots, z_r) = \prod_{i<j}(z_i - z_j)^2$ is homogeneous and thus its zero locus is pulled back from $\widetilde{E}_r$. The associated braid group $\widetilde{B}_r = \pi_1(\widetilde{E}_r - \{\Delta = 0\})$ is in fact a quotient of the subgroup of the braid group B_{r+1} which is pure on the strand at the origin. It is clear that the map $\tilde{\mathbf{c}}$ induces

a representation of the fundamental group of $\mathcal{V}_{A,A'}$ into $\widetilde{B}_r$. More generally, we have a representation of fundamental groupoids,

$$\mathbf{r}\colon \Pi(\mathcal{V}_{A,A'}) \to \Pi(\widetilde{E}_r - \Delta).$$

Define $\Delta_{\mathbb{R}}(z_1,\ldots,z_r) = \prod_{i<j}(|z_i| - |z_j|)^2$ to obtain a real stratification $\widetilde{\mathcal{S}} = \{R_\rho\colon \rho \in \mathcal{P}\}$ of $\widetilde{E}_r$. Here $\mathcal{P}$ denotes the set of partitions of $\{1,\ldots,r\}$ and $R_\rho = \{\{z_1,\ldots,z_r\}\colon |z_i| = |z_j|$ for $i \sim_\rho j$, and $|z_i| \leqslant |z_{i+1}|\}$.

DEFINITION 2.8. The pullback stratification

$$\mathcal{S} = \{R \text{ a component of } \tilde{\mathbf{c}}^{-1}(R_\rho)\colon R_\rho \in \widetilde{\mathcal{S}}\}$$

on $\mathcal{U}_{A,A'}$ will be called *the norm stratification*.

From the description of the toric boundary strata of $\mathcal{M}_{A,A'}$, we may extend the norm stratification on $\mathcal{V}_{A,A'}$ to the boundary. We will suppress the details of this extension, which are evident from the fact that the orbits degenerate to sequences of orbits, and refer to the resulting stratification on $\mathcal{M}_{A,A'}$ as the extended norm stratification.

To use the definitions given above, we need a result that gives us a well-defined category on which to work. If the sharpening set A' is chosen carefully, the associated LG model $\mathbf{w}$ has a sensible definition of a Fukaya–Seidel category $\mathcal{F}^{\to}(\mathbf{w})$. For example, we have the following proposition.

PROPOSITION 2.9 [7]. *Assume that $A' \subset A$ is contained in the interior of Q. If W transversely intersects the principal A-determinant, then its LG model $\mathbf{w}$ is a Lefschetz pencil.*

In particular, the singularities are isolated and Morse, and parallel transport is well defined along the base so that the usual notion of Fukaya–Seidel categories applies [19]. Moreover, the collection of distinguished bases is acted on by the full braid group B_r, and this extends to an action on exceptional collections via mutations [20].

From this proposition and the results in the above references, it is not difficult to obtain our main theorem for this section.

THEOREM 2.10. *The space $\mathcal{V}_{A,A'}$ has a stratification $\mathcal{S}$ such that, for any component $R \in \mathcal{S}$, there exists a $\mathbb{Z}^r$ torsor $\mathcal{C}_R$ of semi-orthogonal decompositions of $\mathcal{F}^{\to}(\mathbf{w})$ with the following properties.*

(i) If $R_1 < R_2 \in \mathcal{S}$, then there is a bijective map $\tau\colon \mathcal{C}_2 \to \mathcal{C}_1$ such that the semi-orthogonal decomposition $S \in \mathcal{C}_2$ refines that given by $\tau(S)$.

(ii) If $\gamma\colon [0,1] \to \mathcal{V}_{A,A'}$ is any morphism in $\Pi(\mathcal{V}_{A,A'})$ with $\gamma(0), \gamma(1) \in R_0$ giving exceptional collections, then $\mathbf{r}(\gamma)$ acts by mutation to map $\mathcal{C}_{R_0}$ to itself.

(iii) Assume that $R \in \mathcal{S}$ and $\mathbf{w} \in \overline{R}$ is in the boundary of $\mathcal{M}_{A,A'}$ and corresponds to a sequence $\langle \mathbf{w}_1,\ldots,\mathbf{w}_t \rangle$ of partial Lefschetz fibrations. Then every semi-orthogonal decomposition in $\mathcal{C}_R$ refines the decomposition $\langle \mathcal{F}^{\to}(\mathbf{w}_1),\ldots,\mathcal{F}^{\to}(\mathbf{w}_t) \rangle$.

PROOF. The first two parts follow directly from the definition. The third follows from the fact that we may choose a path $\mathbf{f}_s$ of LG models in $\mathcal{C}_R$ converging to $\mathbf{w}$ without effecting the semi-orthogonal decomposition. At $s = 0$, the partition ρ becomes coarser and the semi-orthogonal categories of $\mathbf{w}$ are generated by unions of those in the decomposition of $\mathbf{f}_s$. From the description of the boundary of $\pi \colon \mathcal{X}_{\Theta(A)} \to \sec A$, we observe that the k-category $\mathcal{T}_k$ in the decomposition of $\mathbf{w}$ is generated by the union of the vanishing thimbles of $\mathbf{f}_s$, whose critical value converges to the critical set of $\mathbf{w}_k$. From the definition of radar screens, the set of distinguished paths with this property follow a single path in the base of $\mathbf{f}_s$ outside any annulus for which $\mathbf{f}_s$ converges to $\mathbf{w}_k$. Thus, the category $\mathcal{T}_k$ is equivalent to $\mathcal{F}^{\rightharpoonup}(\mathbf{w}_k)$.

Note that this gives a direct relationship between the combinatorics of maximal degenerations, or parametric simplex paths, and decompositions of Fukaya–Seidel categories of LG models near such degenerations.

§ 3. A_n-categories

In this section we consider the most basic possible case, the directed A_n-category. We give a detailed construction of the secondary, Lafforgue and monotone path stacks in this case. In particular, we describe the combinatorics of the monodromy maps around the discriminant and toric boundary. Symplectic geometry in these cases is completely absent, as the Fukaya categories are more of a combinatorial nature. Nevertheless, the structure and geometry of the decompositions and representations of this category is surprisingly rich and illustrates some of the techniques that are applied in higher dimensions.

3.1. The Lafforgue stack of an interval. The derived A_n category can be given by the Fukaya–Seidel category of a single polynomial,

$$\mathbf{w}(x) = c_{n+1}x^{n+1} + \cdots + c_1 x + c_0 \in \mathbb{C}[x]$$

with $c_{n+1} \neq 0$, whose marked Newton polytope (Q, A) is clearly $Q = [0, n+1]$, $A = \{0, 1, \ldots, n+1\}$.

A first step towards understanding this example is to characterize the stacks $\mathcal{X}_{\Sigma(A)}$ and $\mathcal{X}_{\Theta(A)}$. We will derive $\mathcal{X}_{\Theta(A)}$ by obtaining its stacky fan. The secondary polytope of A was examined in [9] and was seen to be affinely equivalent to the representation-theoretic polytope $P(2\rho)$ which is the convex hull of the dominant weights $\{\omega\}$ of A_n such that $\omega \leqslant 2\rho$, where 2ρ is the sum of the positive roots. We begin by reviewing the observations there and establishing notation.

Take $\{e_0, \ldots, e_{n+1}\}$ as a basis for $\mathbb{Z}^A$ and $\{\alpha_1, \ldots, \alpha_n\}$ as a basis for $\Lambda_r \approx \mathbb{Z}^n$ and examine the fundamental exact sequence for A

$$0 \longrightarrow \Lambda_r \xrightarrow{\delta_A} \mathbb{Z}^A \xrightarrow{\beta_A} \mathbb{Z}^2 \longrightarrow 0,$$

where $\beta_A(e_i) = (1, i)$ and $\delta_A(\alpha_i) = -e_{i-1} + 2e_i - e_{i+1}$. Write C_n for the A_n Cartan matrix and recall that this serves as a transformation matrix from the simple roots to the fundamental weights. We will view Λ_r as the A_n root lattice with simple

roots $\{\alpha_i\}$, fundamental weights $\{\lambda_1, \ldots, \lambda_n\} \subset \Lambda := \Lambda_r^\vee$ and $\rho = \sum_{i=1}^n \lambda_i = \frac{1}{2}\sum_{i=1}^n i(n+1-i)\alpha_i$ the Weyl element. Using the Cartan form, we embed Λ_r in Λ and view the α_i as elements of Λ as well. We also write $[n]$ for the set $\{1, \ldots, n\}$.

PROPOSITION 3.1 [9]. *The normal fan $\mathcal{F}$ of $\Sigma(A)$ in $\Lambda \otimes \mathbb{R}$ has 1-cone generators $\{\alpha_1, \ldots, \alpha_n, -\lambda_1, \ldots, -\lambda_n\}$ in the weight lattice Λ and cones*

$$\sigma_{I,J} = \mathrm{Span}_{\mathbb{R}_{\geqslant 0}}\{-\lambda_i, \alpha_j : i \in I, j \in J\}$$

for any pair of disjoint sets $I, J \subset [n]$.

The vertices of $\Sigma(A)$ are easily seen to be in one-to-one correspondence with subsets $K = \{k_0 < \cdots < k_m\} \subset [n]$ representing the triangulations $T_K = \{([k_i, k_{i+1}], \{k_i, k_{i+1}\})\}$ and corresponding to the vertex $\varphi_K = \sum_{i=0}^m (k_{i+1} - k_{i-1})e_{k_i}$ of $\Sigma(A)$.

In order to obtain the secondary stack, we need to write out the stacky fan for the Lafforgue stack and find the colimit stack. The facets of the Lafforgue polytope were shown in [7] to correspond to pointed coarse subdivisions of A. Since $\Theta(A)$ is an $(|A|-1)$-dimensional polytope, we define the supporting hyperplane functions $\overline{\Theta(A)}$ as elements in $(\mathbb{Z}^A)^\vee$ but restrict them to linear functions on $\Gamma = \{\sum_i c_i e_i : \sum_i c_i = 0\}$. Letting $f_i = \delta_A(\alpha_i)$ and $f_0 = e_0 - e_1$, we take $\{f_0, f_1, \ldots, f_n\}$ as a basis for Γ so that $\delta_A \colon \Lambda \to \mathbb{Z}^A$ lifts to $\tilde{\delta}_A \colon \Lambda \to \Gamma$. As we will show in a moment, there are $3n+2$ facets of $\Theta(A)$, so the stacky fan is obtained by a fan in $\mathbb{R}^{3n+2}$ along with a map $\xi_A \colon \mathbb{Z}^{3n+2} \to \Gamma^\vee$ which gives the group $\mathbb{G}_{\Theta(A)} \simeq \ker(\xi_A) \otimes \mathbb{C}^*$. We write $\{g_i \colon 1 \leqslant i \leqslant 3n+2\}$ for the standard basis of $\mathbb{Z}^{3n+2}$.

The pointed coarse subdivisions (S, B) of A can be classified into three types. For each i, $1 \leqslant i \leqslant n$, there is a pointed subdivision $((Q, A - \{i\}), A - \{i\})$ whose supporting hyperplane function g_i is given by $e_i^\vee \in (\mathbb{Z}^A)^\vee$, so that $\xi_A(g_i) = -f_{i-1} + 2f_i - f_{i+1}$. Also, for every i, $1 \leqslant i \leqslant n$, there are two pointed subdivisions corresponding to $Q = [0, i] \cup [i, n+1]$ with pointing sets $\{0, \ldots, i\}$ and $\{i, \ldots, n+1\}$ respectively. The supporting primitives are easily seen to be $g_{2i} = \sum_{j=i+1}^{m+1}(j-i)e_j^\vee$ and $g_{3i} = \sum_{j=0}^i (i-j)e_j^\vee$, implying that $\xi_A(g_{2i}) = -f_i$ and $\xi_A(g_{3i}) = -f_i - f_0$. Finally, there are two vertical pointed subdivisions $((Q, A), \{0\})$ and $((Q, A), \{n+1\})$ corresponding to the one-cones for $\mathcal{X}_Q$. The linear functions corresponding to these two subdivisions are g_{3n+1}, g_{3n+2}, which map to $\xi_A(g_{3n+i}) = (-1)^{i+1}f_0^\vee$. We can write the map ξ_A as the matrix

$$\xi_A = \left[\begin{array}{c|c|c|cc} -1\,0\,\ldots\,0 & 0\,\ldots\,0 & 1\,\ldots\,1 & 1 & -1 \\ \hline C_n & -I & -I & 0 & 0 \end{array}\right]$$

where I is the $n \times n$ identity matrix.

The maximal cones of the Lafforgue fan $\mathcal{F}_{\Theta(A)}$ are indexed by pointed triangulations $\{(K, k) \colon K \subset [n], k \in K \cup \{0, n+1\}\}$. For example, if $k \neq 0, n+1$, the cone in $\mathcal{F}_{\Theta(A)}$ associated with (K, k) is

$$\sigma_{(K,k)} = \mathrm{Cone}\big(\{g_j : j \notin K\} \cup \{g_{2j} : j \in K, j \leqslant k\} \cup \{g_{3j} : j \in K, j \geqslant k\}\big).$$

If $k \in \{0, n+1\}$, we add g_{3n+1}, g_{3n+2} respectively to the generating set above. This in particular implies that $\mathcal{F}_{\Theta(A)}$ is a simplicial fan.

There is a map $g_{\mathcal{F}} \colon \mathcal{F}_{\Theta(A)} \to \mathcal{F}_{\Sigma(A)}$ of fans which can be promoted to a map $\overline{g}_{\mathcal{F}}$ of their canonical stacky fans, inducing the map $g \colon \mathcal{X}_{\Theta(A)} \to \widetilde{\mathcal{X}}_{\Sigma(A)}$ of toric stacks from equation (2.3).

PROPOSITION 3.2. *The Lafforgue stack* $\mathcal{X}_{\Theta(A)}$ *for* $A = \{0, \ldots, n+1\}$ *is smooth with covering* $\{U_{(K,k)}\}$. *The secondary stack* $\mathcal{X}_{\Sigma(A)}$ *is smooth and is given by the stacky fan described in Proposition* 3.1.

PROOF. The first sentence follows from the observation above that $\mathcal{F}_{\Theta(A)}$ is simplicial and therefore its canonical stack is smooth. For the second, recall from [21] and [7] that the map $\overline{g}_{\mathcal{F}}$ of stacky fans consists of a pair of lattice maps $((\overline{g}_{\mathcal{F}})_1, (\overline{g}_{\mathcal{F}})_2) \colon (\mathbb{Z}^{\overline{\Theta(A)}}, \Gamma) \to (\mathbb{Z}^{\overline{\Sigma(A)}}, L_A^{\vee})$. An elementary computation shows that $(\overline{g}_{\mathcal{F}})_1 \colon \mathbb{Z}^{\overline{\Theta(A)}} \to \mathbb{Z}^{\overline{\Sigma(A)}}$ is onto if $n > 1$, implying that the colimit stack $\mathcal{X}_{\Sigma(A)}$ of g equals $\widetilde{\mathcal{X}}_{\Sigma(A)}$. The case $n = 1$ was covered in [7] and gives $\mathbb{P}(1, 2)$. Also, the fan described in Proposition 3.1 is simplicial, implying that $\mathcal{X}_{\Sigma(A)}$ is a smooth stack.

This implies that for $n > 1$, the secondary stack is given by taking the canonical stack of the normal fan $\mathcal{F}_{\Sigma(A)}$ while for $n = 1$, we have $\mathcal{X}_{\Sigma(A)} = \mathbb{P}(1, 2)$. This reproduces the stacks studied in [11] which are quotients of the Losev–Manin stack. In all cases, there is a covering of $\mathcal{X}_{\Sigma(A)}$ by $\{U_K\}_{K \subset [n]}$, where U_K is the chart associated with the cone $\sigma_{K,[n]-K}$. Let us describe an open chart U_K in the covering given above. For $K = \{k_1, \ldots, k_m\} \subset [n]$, assume that $k_1 < \cdots < k_m$ and write $k_0 = 0$, $k_{m+1} = n+1$ and $r_i = k_i - k_{i-1}$ for $i = 1, \ldots, m+1$. Write μ_r for the group of rth roots of unity and let μ_r act on $\mathbb{C}^r$ via $\zeta(z_1, \ldots, z_r) = (\zeta z_1, \zeta^2 z_2, \ldots, \zeta^{r-1} z_{r-1}, z_r)$. We also take this as an action on the first $(r-1)$ coordinates. Then, using the basis of the open cones in the weight lattice given in Proposition 3.1, the open stack U_K is easily seen to be the quotient stack

$$U_K \approx [\mathbb{C}^{r_1} \times \cdots \times \mathbb{C}^{r_{m+1}-1}/\mu_{r_1} \times \cdots \times \mu_{r_{k+1}}].$$

This local description extends over the Lafforgue stack and the universal hypersurface. Indeed, writing G_K for the group $\mu_{r_1} \times \mu_{r_{m+1}}$, it is not hard to show that there is a polydisc neighbourhood $V_K = [D_1 \times \cdots \times D_m/G_K]$ near the origin of U_K with

$$\pi_{\mathcal{H}}^{-1}(V_K) \approx \left[\left(\bigcup_{j=1}^{m+1} \mu_{r_j} \right) \times D_1 \times \cdots \times D_m/G_K \right].$$

Here G_K acts in the obvious way on the set $\bigcup \mu_{r_j}$.

3.2. Vanishing trees of maximal degenerations. Having described the secondary stack and Lafforgue stack for A_n, we would like to consider our space of LG models which define the directed A_n-category. For this purpose, we choose $A' = \{0\}$ and consider all A'-sharpened pencils on $\mathcal{X}_Q = \mathbb{P}^1$. Since A' is not in the interior of A, we cannot apply Proposition 2.9. However, in this case we can state the following proposition, whose proof is evident from the definitions in §2.

PROPOSITION 3.3. *Let* $A = \{0, \ldots, n+1\}$ *and* $A' = \{0\}$. *The LG model of an* A'-*sharpened pencil is a polynomial* $\mathbf{w}(z) = c_{n+1} z^{n+1} + \cdots + c_0$ *of degree* $(n+1)$ *such that* $c_i \neq 0$ *for* $0 \leqslant i \leqslant n+1$.

Note that the Fukaya–Seidel category is extremely sensitive to the choice of the sharpening point (or set). For example, if $A = \{0, 1, 2\}$ and we chose $A' = \{1\}$ instead of $\{0\}$, we would obtain the homological mirror of $\mathbb{P}^1$ instead of the category of vector spaces (or the A_1-category).

Recall that the moment polytope of the stack of $\mathcal{M}_{A,A'}$ is the monotone path polytope of $\Sigma(A)$ relative to the function $\gamma_{A'} = e_0^\vee$ and that maximal degenerated LG models correspond to monotone edge paths of $\Sigma(A)$. We describe this polytope in the following proposition.

PROPOSITION 3.4. *If $A = \{0, \ldots, n+1\}$ and $A' = \{0\}$, then the monotone path polytope $\Sigma_{\gamma_{A'}}(\Sigma(A))$ is combinatorially equivalent to an $(n-1)$-dimensional cube.*

PROOF. By the results of [18], the vertices of $\Sigma_{\gamma_{A'}}(\Sigma(A))$ correspond to parametric simplex paths on $\Sigma(A)$. We recall that the vertices of $\Sigma(A)$ are labeled by subsets $K = \{k_0, \ldots, k_m\} \subset \{1, \ldots, n\}$ with associated triangulation $T_K = \{[k_i, k_{i+1}]\}$. The image of $\gamma_{A'}$ is easily seen to be $[1, n+1]$, where the set of vertices of $\Sigma(A)$ sent to 1 are all subdivisions $\{1, k_1, \ldots, k_m\}$. Omitting the element 1, we identify these with subsets $J = \{k_1, \ldots, k_m\} \subset \{2, \ldots, n\}$. Now observe that to any such vertex, there is a unique parametric simplex path on $\Sigma(A)$ relative to $\gamma_{A'}$ which has J as its minimum. Indeed, if $P = (J = K_0, K_1, \ldots, K_r)$ is a sequence of vertices in a parametric simplex path, then $\{K_i, K_{i+1}\}$ is an edge of $\Sigma(A)$ and $\gamma_{A'}(K_i) < \gamma_{A'}(K_{i+1})$. It is not hard to see that $K_i = \{k_{i+1}, \ldots, k_m\}$ gives such a path, establishing the existence claim. To see that it is unique, suppose that $P' = (K_0, \ldots, K_i, K'_{i+1}, \ldots, K_r)$ is any other parametric simplex path. Since $\{K_i, K'_{i+1}\}$ is an edge of $\Sigma(A)$, K'_i is obtained from K_i by inserting or deleting an element. As $\gamma_{A'}(K'_{i+1}) > \gamma_{A'}(K_i)$, we cannot insert a point, and deleting any element besides k_{i+1} does not affect the value of $\gamma_{A'}$. Therefore $K'_{i+1} = K_{i+1}$ and the path is unique.

By the Minkowski integral description of fibre polytopes, one easily observes that any face of $\gamma_{A'}^{-1}(1)$ gives a face of $\Sigma_{\gamma_{A'}}(\Sigma(A))$. Since the vertices are in bijection, this implies that the face lattices are equal and yields the proposition.

From the proof of Proposition 3.4 we obtain a combinatorial description of the sequence of circuits associated with maximal degenerations. Our next goal is to give a complete description of the semi-orthogonal decompositions connected with such sequences. We first recall the degeneration and regeneration procedure from [7]. Consider a monotone path specified by $J = \{0, 1 = k_0, k_1, \ldots, k_m = n+1\}$ and a function $\eta \colon A \to \mathbb{Z}$ which defines the triangulation given by J (see [22] or [9]). Briefly recall that, for $a < b \in A$, if we write $\eta'_{a,b} = \eta(b) - \eta(a)/(b-a)$, then this means that $\eta'_{k_i, k_{i+1}}$ is increasing relative to i and that $\eta(a)$ lies above the undergraph of η for $a \notin J$. To simplify the treatment, we also assume that $\eta'_{k_i, k_{i+1}} \in \mathbb{Z}$. Fix any $\mathbf{c} = (c_{n+1}, \ldots, c_0) \in \mathbb{C}^{n+2}$ such that $c_i = 1$ for $i \in J$, and define the family of polynomials

$$\psi(\mathbf{c}, s, t)(z) = \left(\sum_{i=1}^{n+1} c_i s^{\eta(i)} z^i \right) + s^{\eta(0)} t$$

which for $s \neq 0$ give very full sections $\psi(\mathbf{c}, s, t) \in \mathcal{L}_A^{vf}$. Notice that this gives an s-parameterized family of A'-sharpened pencils $\psi(\mathbf{c}, s, \_)$. After quotienting by the appropriate group, we can think of ψ as a function from $\mathbb{C}^*$ to $\mathcal{M}_{A,A'}$, or as a function from $(\mathbb{C}^*)^2$ to $\mathcal{X}_{\Sigma(A)}$. We will shift between these perspectives in what follows.

3*

As was seen in the proof of Proposition 3.4, the circuit modifications $\langle C_1, \ldots, C_m \rangle$ associated with J are supported on $C_i = \{0, k_{i-1}, k_i\}$ and correspond to edges of $\Sigma(A)$ with vertices $T_{K_{i-1}}$ and T_{K_i} where $T_i = \{0, k_i, k_{i+1}, \ldots, k_m\}$. For any i, $1 \leqslant i \leqslant m$, we may reparameterize ψ so as to obtain a regeneration of C_i. First recall that such a regeneration of C_i is a map $\tilde{\psi}_i$ completing the diagram

$$
\begin{array}{ccc}
\mathbb{C}^* & \lhook\joinrel\longrightarrow & \mathbb{C} \times \mathbb{C}^* \\
\downarrow{\scriptstyle \rho_i} & & \downarrow{\scriptstyle \tilde{\psi}_i} \\
\mathcal{X}_{\Sigma(C_i)} & \lhook\joinrel\longrightarrow & \mathcal{X}_{\Sigma(A)}
\end{array}
\tag{3.1}
$$

where ρ_i is étale onto the complement of $\{0, \infty\} \subset \mathcal{X}_{\Sigma(C_i)}$, the top arrow is the inclusion into $\{0\} \times \mathbb{C}^*$, and $\tilde{\psi}_i$ is a finite map. Explicitly, this is given by reparameterizing $\tilde{\psi}_i(s,t) = \psi(\mathbf{c}, s, s^{\eta(k_{i+1}) - \eta(0) - k_{i+1} \eta'_{k_{i+1}, k_i} t})$ and completing to $s = 0$. Indeed, letting $z = s^{\eta'_{k_i, k_{i+1}}} u$, we have that $\lim_{s \to 0} \tilde{\psi}_i(s,t)$ converges to the circuit pencil which can be written in the coordinate u as

$$
\mathbf{w}_i(u) = u^{k_{i+1}} + u^{k_i} + t.
\tag{3.2}
$$

The functions $\mathbf{w}_i$ are precisely those yielding the subcategories in the semi-orthogonal decomposition from Theorem 2.10. One can think of the reparameterization as giving an asymptotic prescription for the ith bubble in the stable map limit of ψ as s tends to 0. Moreover, it is important to remember that the fibres of $\pi \colon \mathcal{X}_{\Theta(A)} \to \mathcal{X}_{\Sigma(A)}$ over $\tilde{\psi}$ themselves degenerate into reducible chains of projective lines $\bigcup_{j=i+1}^{m} \mathbb{P}^1$ as in Fig. 3.
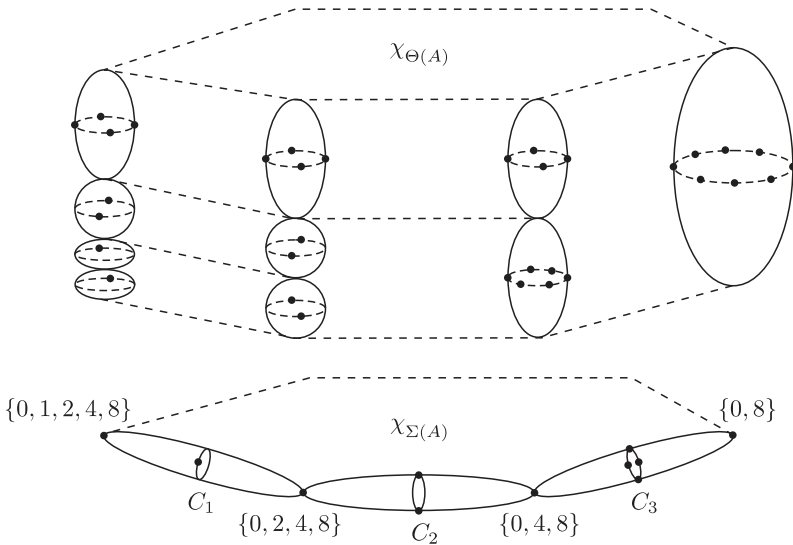

FIGURE 3. The pullback of $\mathcal{X}_{\Sigma(A)}$ and $\mathcal{X}_{\Theta(A)}$ for the monotone path $J = \{0, 1, 2, 4, 8\}$

Letting s and t tend to 0 in $\tilde{\psi}_{i+1}(s,t)$, one approaches the fixed point of $\mathcal{X}_{\Sigma(A)}$ associated with the triangulation T_{K_i} in the monotone path. The u roots $\mathcal{Z}_A(q) := \pi_{\mathcal{H}}^{-1}(q)$ of $q = \tilde{\psi}_{i+1}(s,t) \in \mathcal{X}_{\Sigma(A)}$ converge to the degenerated hyperplane section. As described at the end of the previous subsection, this hypersurface degeneration results in a partition of the fibre $\mathcal{Z}_A(q)$ into $(m-i)$ subsets $F_{i,i+1}(q), \ldots, F_{i,m}(q)$. For every $j > i$, the set $F_{i,j}(q)$ converges to $(k_j - k_{j-1})$ roots of unity of the jth component of the degeneration of the fibre, while the component $F_{i,i+1}(q)$ converges to the k_i roots of unity. Thus all the $F_{i,j}(q)$ are cyclically ordered sets. For the example illustrated in Fig. 3, the sets $F_{0,1}(q), F_{0,2}(q), F_{0,3}(q), F_{0,4}(q)$ are clearly seen in the leftmost fibre of $\pi\colon \mathcal{X}_{\Theta(A)} \to \mathcal{X}_{\Sigma(A)}$ as the points in the lowest to highest components.

For $j > i+2$, the subsets $F_{i,j}(q)$ experience no monodromy as q varies near the component C_i, regardless of the path. Thus for $j > i+2$, there is a collection of unique monodromy isomorphisms $\tau_{i,j}\colon F_{i,j}(q) \to F_{i+1,j}(q')$ where q and q' approach 0 and ∞ respectively of C_i. To identify the remaining sets, we must choose a radar screen $\mathcal{B}$ for $\tilde{\psi}_{i+1}(s,t)$ to obtain the isomorphism

$$\tau_{i,i+2}\colon F_{i,i+1}(q) \cup F_{i,i+2}(q) \to F_{i+1,i+2}(q').$$

To define $\tau_{i,i+2}$, we first take a path $p_j \in \mathcal{B}$ which does not end at a point in C_i, degenerate, and reparameterize the component γ_i of p_j in such a way that it is a path from 0 to ∞ in C_i (if $i=1$, take the last path and concatenate it to extend it to 0). Then $\tau_{i,i+2}$ is defined as the monodromy along $\gamma_i\colon [0,1] \to C_i$.

Note that in the 1-dimensional case, vanishing thimbles of a polynomial $\mathbf{w}$ are simply paths in $\mathbb{C}$ with endpoints on a fibre $\mathbf{w}^{-1}(q)$. Labelling them according to which path in the radar screen they are defined by, we obtain an edge-labelled tree which we refer to as the vanishing tree of $\mathbf{w}$ with respect to $\mathcal{B}$. If we omit the grading, this tree encodes all the data necessary to compute the algebra of the morphisms between exceptional objects in $\mathcal{F}^{\to}(\mathbf{w})$. We would like to give a concrete combinatorial formulation of this vanishing tree.

Towards this end, suppose that S_1, S_2 and S_3 are finite sets such that S_1 and S_3 have a cyclic order and $|S_3| = |S_1| + |S_2|$. We call a bijection $\sigma\colon S_1 \cup S_2 \to S_3$ a cyclic $|S_2|$-insertion if $\sigma|_{S_1}$ preserves the cyclic order. Now, assume that S_2 comes equipped with a total order $<$ and label $S_2 = \{s_1, \ldots, s_{|S_2|}\}$. Extend this to a partial order on $S_1 \cup S_2$ by taking $s < s'$ if $s \in S_1$ and $s' \in S_2$. If σ is a cyclic $|S_2|$-insertion and $s_k \in S_2$, define $m_\sigma(s_k) = s' \in S_1 \cup S_2$ to be the unique element less than s_k such that every element $s \in S_3$ in the cyclic interval between $\sigma(m_\sigma(s_k))$ and $\sigma(s_k)$ satisfies $s_k < \sigma^{-1}(s)$. We define the incidence graph of this function:

$$I_{\sigma,<} = \{(\sigma(s), \sigma(m_\sigma(s)))\colon s \in S_2\} \subset S_3 \times S_3.$$

Now, let $\widetilde{P}(s)$ be a choice of logarithms of the critical values of $\psi(\mathbf{c}, s, \_)$.

THEOREM 3.5. *For $s \ll 1$ and a radar screen distinguished basis $\{p_1, \ldots, p_n\} = \mathcal{B}_{\widetilde{P}(s)}$, the map $\tau_{i,i+2}$ is a cyclic $(k_{i+1} - k_i)$-insertion. There is a unique total order $<$ on $F_{i,i+2}$ such that the vanishing graph associated with $\{p_{k_i}, \ldots, p_{k_{i+1}}\}$ is $I_{\tau_{i,i+2},<}$. Furthermore, every cyclic $(k_{i+1} - k_i)$-insertion σ and total order $<$ arises as a monodromy map for some radar screen and regeneration of $\mathbf{w}_i$.*

The dictionary to use for the input structures of this theorem is as follows. The sets $F_{i,i+1}(q)$, $F_{i,i+2}(q)$, $F_{i+1,i+2}(q')$ give S_1, S_2, S_3 respectively, the perturbation coefficient $\mathbf{c}$ gives a total order on S_2, and the radar screen gives $\sigma = \tau_{i,i+2}$.

PROOF OF THEOREM 3.5. We sketch a proof. Let $a = k_{i+1} - k_i$, $b = k_i$ and recall that the LG model $\tilde{\psi}_i(s,t)$ corresponding to C_i converges to $\mathbf{w}_i(u) = u^{a+b} + u^b + t$. As an A'-sharpened pencil on $\mathbb{C}^*$, this is $[f(u) : 1] := [u^{a+b} + u^b : 1] = [-t : 1]$. We take a moment to understand the geometry of this elementary polynomial f. First observe that $\mathbf{w}_i$ has a critical values at scaled roots of unity $d\zeta$ for $R = |b/(a+b)|^{1/a}$ and $\zeta \in \mu_a$, as well as a $(b-1)$ ramified critical value at 0. Let S_r^1 be the circle of radius r and examine the contour $S_r := f^{-1}(S_r^1) \subset \mathbb{C}$ as we vary r. It is not hard to see that for $r > R$, S_r is a circle which is an $(a+b)$-fold cover of S_r^1, while for $r < R$ it is a union of $a+1$ circles, a of which cover S_r^1 once and the remaining circle covers it b times. For $r = R$, S_r is a circle with a pinched pairs of points. This is illustrated in Fig. 4.
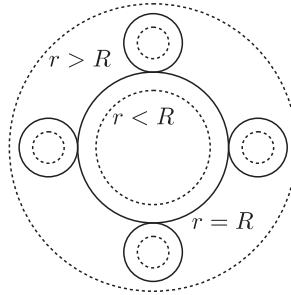

FIGURE 4. Contours for $|f(u)| = r$

Note that $\mathbf{w}_i$ is degenerate in the sense that it lies in the closure of the discriminant $\Delta_{A,A'}$. Nevertheless, the sets $F_{i,i+1}(q)$ and $F_{i,i+2}(q)$ converge, up to a phase, to the b roots of the inner circle of Fig. 4 and points contained in one of each of the a outer circles, respectively. Were we to regenerate a straight line path from q to q', it is not hard to see that the monodromy would then give a cyclic b-insertion $F_{i,i+1}(q) \cup F_{i,i+2}(q) \to F_{i+1,i+2}(q')$.

To obtain the actual monodromy map $\tau_{i,i+2}$, we need to define a radar screen $\mathcal{B}$ for the regeneration of $\mathbf{w}_i/\mu_a$ along $\tilde{\psi}_i(s,t)$. Recall from §2 that a radar screen is a distinguished basis of paths $\mathcal{B} = \{p_1, \ldots, p_a, p_{a+1}\}$ ending at the $(a+1)$ critical values $\{q_1, \ldots, q_a, 0\}$ of $\tilde{\psi}_i(s, \_)$, ordered in such a way that $|q_i| > |q_{i+1}|$. It is uniquely determined by the regeneration $\tilde{\psi}_i$ and a choice of logarithmic lifts $\{\tilde{q}_1, \ldots, \tilde{q}_a\}$ of the critical values. We note that, up to first order, only the projection $(c_{a+b-1}, \ldots, c_{b+1})$ of the coefficient $\mathbf{c} = (c_{n+1}, \ldots, c_0)$ matters in determining the norm ordering of the critical values for $\tilde{\psi}_i(s, \_)$. It is easy to see that one may prescribe any ordering with a judicious choice of such coefficients.

The key point in the proof is that for $s \ll 1$, Fig. 4 is only mildly modified, so that instead of pinching off a circles at once when $r = R$, we pinch them off one by one, each time $r = |q_i|$. Through the identification of $F_{i,i+2}$ with the outer contours

given above, we see that the ordering of critical values gives a total ordering $<$ of $F_{i,i+2}$. Between the pinches, the choice of logarithmic branch $\tilde{q}_i = \log(q_i)$ for the radar screen $\mathcal{B}$ has the effect of rotating the circle S_r as one performs monodromy along p_{i+1}. Note that this monodromy preserves the cyclic ordering of the b points that survive the pinching, so that the total monodromy of $F_{i,i+1}(q)$ along δ_{a+1}^{-1} also preserves the cyclic order. These observations show that $\tau_{i,i+2}$ is a cyclic a-insertion and $F_{i,i+2}(q)$ is totally ordered.

To establish the claim about the vanishing tree, simply observe that, for each pinch, we add a vanishing cycle connecting two points of the fibre $f^{-1}(p_j(z))$ on the central component of S_r. One of the points will always be the point pinched off, and the other will be one of its cyclic neighbours. The fact that this is always the clockwise neighbour corresponds to our choice of anticlockwise orientation for the radar screen distinguished basis. It is left as an exercise to see that the resulting collection of pairs of points is $I_{\tau_{i,i+2},<}$.
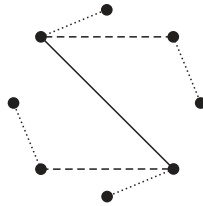

FIGURE 5. A vanishing tree for $J = \{0, 1, 2, 4, 8\}$

Applying this to the example $J = \{0, 1, 2, 4, 8\}$ illustrated in Fig. 3 with fundamental radar screen gives the vanishing tree in Fig. 5. One starts with the unique cyclic $(1, 1)$-insertion yielding the solid vanishing thimble, proceeds to a $(2, 2)$-insertion which gives two dashed vanishing thimbles, and completes the tree with a $(4, 4)$-insertion producing the 4 dotted vanishing thimbles. In general, the proof above shows that the insertions can be chosen arbitrarily. However, if we choose the fundamental radar screen distinguished basis for an exponential sequence $J = \{0\} \cup \{2^i : 0 \leqslant i \leqslant m\}$, it can be shown that each insertion is a perfect shuffle [23]. This reflects the general phenomenon that the fundamental radar screen gives insertions $\tau_{i,i+1}$ that maximally separate the points in $F_{i,i+1}(q)$ and $F_{i,i+2}(q)$.

3.3. Interpretations of A_n-degenerations. We conclude this section with some observations and corollaries of Theorem 3.5. First note that, were we to find the actual exceptional collection associated with the vanishing tree, we would need to include gradings on each of the edges. For ease of exposition, we neglect these gradings, commenting only that choosing different logarithmic branches in a radar screen that yield equivalent vanishing trees will generally alter the graded version.

Now, recall that Gabriel's theorem classifies quivers of finite type as directed Dynkin diagrams [24]. The set of quivers $\mathcal{Q}$ whose underlying graph is A_n can be identified with the power set of $\{2, \ldots, n\}$. To obtain a precise correspondence, order the vertices of the quiver by $\{v_1, \ldots, v_n\}$ and edges $\{e_2, \ldots, e_n\}$, where $e_i = \{v_{i-1}, v_i\}$. We say that $\mathfrak{o}(e_i) = \pm 1$ if e_i is directed towards i or $(i-1)$ respectively.

Then map $J = \{k_1, \ldots, k_{m-1}\} \subset \{2, \ldots, n\}$ to the unique quiver Γ_J which satisfies $\mathfrak{o}(e_i) = -1$ if and only if $i \in J$.

Given such a quiver, we assert that there is a natural T-structure on the derived category $\mathcal{D}$ of right modules over the path algebra $\mathcal{P}(\Gamma_J)$ (recall that, up to equivalence, $\mathcal{D}$ is independent of J). First, write P_i for the projective module of all paths with target v_i. Let $p_J \colon \{1, \ldots, n\} \to \mathbb{Z}$ be the function $p_J(j) = \frac{j}{2} - \frac{1}{2}\sum_{i=1}^{j} \mathfrak{o}(e_{i+1})$. We view p_J as a perversity function and define $(\mathcal{D}_J^{\geq 0}, \mathcal{D}_J^{\leq 0})$ as subcategories for which a bounded chain complex C^* of right $\mathcal{P}(\Gamma_J)$-modules is in $\mathcal{D}_J^{\geq 0}$ if $H^k(\mathrm{Hom}(P_i, C^*)) = 0$ for all $k < p_J(i)$ and likewise for $\mathcal{D}_J^{\leq 0}$.

Let us now construct a complete exceptional collection $\mathbf{E}_J = \langle E_1, \ldots, E_n \rangle$ in the heart $\mathcal{D}_J^{\geq 0} \cap \mathcal{D}_J^{\leq 0}$ for a given J. Define $E_i = P_i[p_J(i)]$ if $\mathfrak{o}(e_{i+1}) = 1$ and

$$E_i = \big(0 \leftarrow P_i[p_J(i)] \xleftarrow{e_{i+1}} P_{i+1}[p_J(i+1)] \leftarrow 0\big)$$

otherwise.

Given an exceptional collection $\mathbf{E} = \langle E_1, \ldots, E_n \rangle$, write $R_{\mathbf{E}} = \mathrm{Ext}^*(\bigoplus_{i=1}^{n} E_i, \bigoplus_{i=1}^{n} E_i)$ for its Yoneda algebra.

PROPOSITION 3.6. *The collection $\mathbf{E}_J$ is a complete, strong, exceptional collection for $\mathcal{D}$. The heart of $(\mathcal{D}_J^{\geq 0}, \mathcal{D}_J^{\leq 0})$ is equivalent to the category of finitely generated right modules over $R_{\mathbf{E}_J}$.*

PROOF. The first sentence follows from an elementary computation of the Ext groups between the objects $\{E_1, \ldots, E_n\}$. The second follows from the facts that $E_i \in \mathcal{D}_J^{\geq 0} \cap \mathcal{D}_J^{\leq 0}$ for all $1 \leq i \leq n$ and the algebra $R_{\mathbf{E}}$ is intrinsically formal (as the collection is strong).

To connect this collection with maximal degenerations of LG models, we approach the fixed point $\psi_J \in \mathcal{M}_{A,A'}$ associated with J via a degeneration path $\psi(\mathbf{c}, s, \_)$. Using Theorem 3.5, we can describe the vanishing tree of a radar screen $\mathcal{B}$ through a sequence of totally ordered, cyclic $(k_{i+1} - k_i)$-insertions. Partitioning $\{1, \ldots, n+1\}$ into the ordered sets $S_i = \{k_{i-1} + 1, \ldots, k_i\}$ for $1 \leq i \leq m$, we identify

$$F_{i,i+1} = \{1, k_1, k_1 - 1, \ldots, k_0 + 1, k_2, \ldots, k_1 + 1, \ldots, k_{i-1}, \ldots, k_{i-2} + 1\},$$

where the cyclic order is as written. We define the cyclic insertions $\sigma_i \colon F_{i,i+1} \cup S_{i+1} \to F_{i+1,i+2}$ as the inclusion. Let $R(J)$ be the radar screen which yields these data and write the resulting exceptional collection as $\mathbf{E}_{R(J)}$.

PROPOSITION 3.7. *For every $J \subset \{2, \ldots, n\}$, $R_{\mathbf{E}_{R(J)}} \approx R_{\mathbf{E}_J}$.*

This proposition suggests that the space $\mathcal{M}_{A,A'}$ is connected with the space of stability conditions for $\mathcal{D}$. In the A_n case, near maximal degeneration points, we obtain T-structures for the triangulated category of the LG model which are directly related to Abelian categories of directed A_n-quivers. In other words, we have categorified the bijection between fixed points of the monotone path stack and directed A_n-quivers to an equivalence between an exceptional collection of a degeneration near the fixed point and an exceptional collection naturally associated with the

directed quiver. Moving from one fixed point to another along certain edges of the monotone path polytope crosses a wall in the norm stratification which results in Coxeter functors, or tiltings, of the ambient triangulated category.

We end the section on the A_n case by a brief comment on homological mirror symmetry. It is known that the homological mirror category for A_n is the graded derived category of singularities for $z^n \colon \mathbb{C} \to \mathbb{C}$ [2]. One can view this category as a weighted divisor blow-up with weight n of the origin in $\mathbb{C}$. The monotone path associated with J may then be viewed as mirror to a sequence of m blow-ups with weights $(k_{i+1} - k_i)$. This perspective accords well with birational mirror symmetry landscape discussed in [7] and [25].

§ 4. Three-point blow-up of $\mathbb{P}^2$

We conclude this paper with an example of a different flavour from those in the previous sections. Throughout, let X_3 denote a smooth del Pezzo surface of degree 6, that is, a blow-up of $\mathbb{P}^2$ at three distinct non-collinear points. This space is the Batyrev mirror to $X_{\mathbf{Q}_2}$ as given in § 2. The case of a del Pezzo surface of degree 7 was considered in [7, § 5]. Recall that $\mathrm{Pic}(X_3) \otimes \mathbb{R}$ is spanned by the pull-back of the hyperplane class and the exceptional divisors E_1, E_2, E_3 corresponding to the blown-up points, and that the effective cone $\mathrm{Eff}(X_3)$ is generated by E_1, E_2, E_3, along with the pullbacks of the lines through the pairs of points, E_{12}, E_{13}, E_{23}. The effective cone admits a decomposition into Zariski chambers, with each maximal chamber corresponding to a birational model obtained from X_3 by birational contractions; moreover, the external walls of codimension 1 of $\mathrm{Eff}(X_3)$, equipped with this decomposition, correspond to Mori fibrations obtained from X_3, and the external walls of codimension 2 correspond to Sarkisov links between the fibrations. We refer to [26] for a general discussion of Mori fibrations and Sarkisov links from the perspective of chambers.

The structure of the external walls of $\mathrm{Eff}(X_3)$ was considered in particular by Kaloghiros in [12, Example 4.7], as a special case of a substantially more general result concerning external walls of codimension 3 and relations amongst Sarkisov links. It is convenient to consider a dual graph Γ_3, with vertices corresponding to the external walls of codimension 1 (that is, the Mori fibrations) and edges corresponding to the external walls of codimension 2 (that is, Sarkisov links). A picture of this graph appears in [12, Fig. 6]. By inspection, this graph is observed to be the edge graph of the 3-dimensional associahedron. This is consistent with toric mirror symmetry and the results of [7, § 5] because the associahedron appears as a facet of the secondary polytope of the point configuration $\mathbf{A}_2 \subset \mathbb{Z}^2$ which is the Batyrev mirror of X_3.

As noted in [12], the graph Γ_3 has 14 vertices, which correspond to Mori fibrations as follows:

(i) 2 vertices correspond to the trivial fibration $\mathbb{P}^2 \to \{\mathrm{pt}\}$, where $\mathbb{P}^2$ is obtained from X_3 by blowing down E_1, E_2, E_3 and E_{12}, E_{13}, $E_{2,3}$ respectively;

(ii) 6 vertices correspond to the fibration $\mathbb{F}_1 \to \mathbb{P}^1$, where the map $X_3 \to \mathbb{F}_1$ factors through the blow-down of one of E_1, E_2, E_3, E_{12}, E_{13}, E_{23};
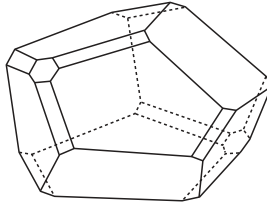
FIGURE 6. The monotone path polytope for $(\mathbf{Q}_2, \mathbf{A}_2)$

(iii) 6 vertices correspond to the fibration $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, where the map $X_3 \to \mathbb{P}^1 \times \mathbb{P}^1$ factors through a blow-down of one of E_1, E_2, E_3, and one of the two projections $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is fixed.

On the other hand, the monotone path polytope of the secondary polytope of $\mathbf{A}_2$ with respect to the $\{0\}$-sharpening is of small enough complexity to be constructed via software. A picture of the resulting truncated associahedron appears in Fig. 6. We observe that it has 36 vertices. Qualitatively, they correspond to the possible choices in the above description of Γ_3.

(i) 12 vertices correspond to the trivial fibration $\mathbb{P}^2 \to \{\text{pt}\}$, where $X_3 \to \mathbb{P}^2$ is one of the six ordered blow-downs of E_1, E_2, E_3, or one of the six ordered blow-downs of E_{12}, E_{13}, E_{23}.

(ii) 12 vertices correspond to the fibration $\mathbb{F}_1 \to \mathbb{P}^1$, where the map $X_3 \to \mathbb{F}_1$ is given by an ordered blow-down of two of E_1, E_2, E_3, or an ordered blow-down of two of E_{12}, E_{13}, E_{23}.

(iii) 12 vertices correspond to the fibration $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, where the map $X_3 \to \mathbb{P}^1 \times \mathbb{P}^1$ factors through a blow-down of one of E_1, E_2, E_3, and a blow-down of one of E_{12}, E_{13}, E_{23} not disjoint from E_i, and one of the two projections $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is fixed.

We observe that a vertex of the dual graph representing a Mori fibration is replaced by the collection of full runs of the minimal model program on X_3 whose last birational map is that Mori fibration. It was conjectured in [7] that the semi-orthogonal decompositions of $D^b(X_3)$ arising from such runs yield subcategories equivalent to those arising from the maximal degenerations of the mirror LG model.

We conclude with a brief discussion of prospects for extending beyond the toric case, and in particular to del Pezzo surfaces X_k of degrees from 1 to 6. The birational geometry of these surfaces is classical, though intricately structured [27]. Motivated by the Hori–Vafa Ansatz, [28] posited the mirror in each case to be an LG model $f_k : Y_k \to \mathbb{P}^1$ of a rational elliptic surface Y_k with prescribed fibre at ∞. That paper verified homological mirror symmetry in the form $D^b X_k \cong \mathcal{F}^{\rightharpoonup}(f_k)$; however, the identification of the Kähler moduli of X_k with the complex moduli of Y_k was not pursued. This identification was completed in an unpublished work of Pantev.[1]

In general, if $f : Y \to \mathbb{P}^1$ is a compactified LG model, results from [5] show that the complex $T_{Y,Y_\infty} \to f^*(T_{\mathbb{P}^1,\infty})$ defining perturbations of f that fix the fibre at

---

[1]T. Pantev, "Notes on homological mirror symmetry for del Pezzo surfaces", private communication.

infinity can be integrated to produce a smooth moduli stack $\mathcal{M}$ of LG models. It is not hard to see that, when f arises as a sharpened pencil, the quotient of $\mathcal{M}$ by the action of $\mathbb{C}^* \times \mathbb{C}$ naturally embeds as a substack of $\mathcal{M}_{A,A'}$. In the toric cases, Y_k can be obtained from the Batyrev mirror family by explicit blow-ups, and we have seen that $\mathcal{M}_{A,A'}$ is a natural geometric compactification of the complex moduli of the Batyrev mirror. While we suspect that a similar nested compactification exists in the non-toric cases, they do not appear to have been studied from this vantage point. However, see [29] for a thorough study of compact moduli of rational elliptic surfaces. The recent investigations of Donaldson [30] regarding K-stability and b-stability of Fano manifolds will be relevant, replacing the role that classical geometric invariant theory plays in constructing the chamber decomposition on the effective cone.

The above considerations provide a convenient way of studying surfaces whose derived category is close to being generated by an exceptional collection. In particular our analysis suggests the following conjecture.

CONJECTURE 4.1. *Given any Barlow surface S, the derived category $D^b(S)$ has an exceptional collection with* 11 *objects, but has no full exceptional collection.*

It is known that some Barlow surfaces have rank 11 rational K-theory. By taking a left orthogonal of the collection for such a surface, a proof of this conjecture will lead to examples of non-trivial categories with trivial K-theory.

Bibliography

1. D. Auroux, "Mirror symmetry and T-duality in the complement of an anticanonical divisor", *J. Gökova Geom. Topol. GGT*, **1** (2007), 51–91.
2. K. Hori and C. Vafa, *Mirror symmetry*, 2000, arXiv: hep-th/0002222.
3. F. Haiden, L. Katzarkov, and M. Kontsevich, "Mod of stability conditions for FS categories", 2013 (to appear).
4. A. Iliev, L. Katzarkov, and E. Scheidegger, "Automorphic forms and gaps", 2013 (to appear).
5. L. Katzarkov, M. Kontsevich, T. Pantev, and Y. Soibelman, "Stability Hodge structures", 2013 (to appear).
6. V. Przyjalkowski, "Birational geometry and LG models", 2013 (to appear).
7. C. Diemer, L. Katzarkov, and G. Kerr, *Symplectomorphism group relations and degenerations of Landau–Ginzburg models*, 2012, arXiv: 1204.2233.
8. G. Kerr, "Weighted blowups and mirror symmetry for toric surfaces", *Adv. Math.*, **219**:1 (2008), 199–250.
9. I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Birkhäuser, Boston, MA, 2008.
10. L. Lafforgue, "Chirurgie des grassmanniennes", CRM Monogr. Ser., **19**, Amer. Math. Soc., Providence, RI, 2003.
11. M. Blume, *Toric orbifolds associated to Cartan matrices*, arXiv: 1110.2761.
12. A.-S. Kaloghiros, *Relations in the Sarkisov program*, arXiv: 1203.3767.
13. L. A. Borisov, L. Chen, and G. G. Smith, "The orbifold Chow ring of toric Deligne–Mumford stacks", *J. Amer. Math. Soc.*, **18**:1 (2005), 193–215.
14. D. A. Cox, "The homogeneous coordinate ring of a toric variety", *J. Algebraic Geom.*, **4**:1 (1995), 17–50.

15. P. Hacking, *Compact moduli of hyperplane arrangements*, arXiv: math/0310479.

16. Q. Chen and S. Matthew, *Chow quotients of toric varieties as moduli of stable log maps*, arXiv: 1201.3406.

17. L. J. Billera and B. Sturmfels, "Iterated fiber polytopes", *Mathematika*, **41**:2 (1994), 348–363.

18. L. J. Billera and B. Sturmfels, "Fiber polytopes", *Ann. of Math.* (2), **135**:3 (1992), 527–549.

19. P. Seidel, *Fukaya categories and Picard–Lefschetz theory*, Zur. Lect. Adv. Math., European Math. Soc., Zürich, 2008.

20. P. Seidel, "Vanishing cycles and mutation", 3*rd European congress of mathematics*, vol. II (Barcelona, Spain 2000), Progr. Math., **202**, Birkhäuser, Basel, 2001, 65–85.

21. A. Geraschenko and M. Satriano, *Toric stacks* I: *The theory of stacky fans*, arXiv: 1107.1906.

22. L. J. Billera, P. Filliman, and B. Sturmfels, "Constructions and complexity of secondary polytopes", *Adv. Math.*, **83**:2 (1990), 155–179.

23. P. Diaconis, R. L. Graham, and W. M. Kantor, "The mathematics of perfect shuffles", *Adv. in Appl. Math.*, **4**:2 (1983), 175–196.

24. P. Gabriel, "Unzerlegbare Darstellungen. I", *Manuscripta Math.*, **6**:1 (1972), 71–103.

25. L. Katzarkov, "Birational geometry and homological mirror symmetry", *Real and complex singularities* (University of Sydney, Australia 2005), World Scientific, Hackensack, NJ, 2007, 176–206.

26. C. D. Hacon and J. McKernan, *The Sarkisov program*, arXiv: 0905.0946.

27. Yu. I. Manin, "Cubic forms. Algebra, geometry, arithmetic", North-Holland Math. Library, **4**, North-Holland, Amsterdam, 1986.

28. D. Auroux, L. Katzarkov, and D. Orlov, "Mirror symmetry for Del Pezzo surfaces: Vanishing cycles and coherent sheaves", *Invent. Math.*, **166**:3 (2006), 537–582.

29. G. Heckman and E. Looijenga, "The moduli space of rational elliptic surfaces", *Algebraic geometry* (Nagano, Japan 2000), Adv. Stud. Pure Math., **36**, Math. Soc. Japan, Tokyo, 2002, 185-248.

30. S. Donaldson, *Stability, birational transformations and the Kähler–Einstein problem*, arXiv: 1007.4220.

COLIN DIEMER   Поступило в редакцию
Department of Mathematics, University of Miami,   28.06.2012
Coral Gables, FL, USA   19.07.2012
E-mail: diemer@math.miami.edu

LUDMIL KATZARKOV
Department of Mathematics, University of Miami,
Coral Gables, FL, USA;
Fakultät für Mathematik, Universität Wien, Austria
E-mail: l.katzarkov@math.miami.edu

GABRIEL KERR
Department of Mathematics, University of Miami,
Coral Gables, FL, USA
E-mail: gdkerr@math.miami.edu