Compactness along the branch of semi-stable and unstable solutions for an elliptic problem with a singular nonlinearity

PIERPAOLO ESPOSITO* NASSIF GHOUSSOUB† YUJIN GUO ‡

September 5, 2018

Abstract

We study the branch of semi-stable and unstable solutions (i.e., those whose Morse index is at most one) of the Dirichlet boundary value problem $-\Delta u = \frac{\lambda f(x)}{(1-u)^2}$ on a bounded domain $\Omega \subset \mathbb{R}^N$, which models –among other things– a simple electrostatic Micro-Electromechanical System (MEMS) device. We extend the results of [11] relating to the minimal branch, by obtaining compactness along unstable branches for $1 \leq N \leq 7$ on any domain Ω and for a large class of "permittivity profiles" f . We also show the remarkable fact that power-like profiles $f(x) \simeq |x|^\alpha$ can push back the critical dimension $N = 7$ of this problem, by establishing compactness for the semi-stable branch on the unit ball, also for $N \geq 8$ and as long as $\alpha > \alpha_N = \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$. As a byproduct, we are able to follow the second branch of the bifurcation diagram and prove the existence of a second solution for λ in a natural range. In all these results, the conditions on the space-dimension and on the power of the profile are essentially sharp.

Keywords: Compactness, Electrostatic MEMS, Semi-stable Branch, Unstable branch, Critical Parameter, Extremal Solution.

AMS subject classification: 35J60, 35B40, 35J20.

Contents

*Dipartimento di Matematica, Università degli Studi "Roma Tre", Roma, Italy & Pacific Institute for the Mathematical Sciences, University of British Columbia, Vancouver, B.C. Canada V6T 1Z2. E-mail: pesposito@pims.math.ca. Author supported by M.U.R.S.T., project "Variational methods and nonlinear differential equations", and by a PIMS Postdoctoral Fellowship

†Department of Mathematics, University of British Columbia, Vancouver, B.C. Canada V6T 1Z2. E-mail: nassif@math.ubc.ca. Research partially supported by the Natural Science and Engineering Research Council of Canada

‡Department of Mathematics, University of British Columbia, Vancouver, B.C. Canada V6T 1Z2. E-mail: yjguo@math.ubc.ca. Author partially supported by the Natural Science Foundation of P. R. China (10171036) and by a U.B.C. Graduate Fellowship.

1 Introduction

We continue the analysis of [11] for the problem:

$$\begin{cases} -\Delta u = \dfrac{\lambda f(x)}{(1-u)^2} & \text{in } \Omega, \\ 0 < u < 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (S)_\lambda$$

where $\lambda > 0$, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $f \in C(\bar{\Omega})$ is a nonnegative function. This equation models a simple electrostatic Micro-Electromechanical System (MEMS) device consisting of a thin dielectric elastic membrane with boundary supported at 0 below a rigid plate located at +1. When a voltage – represented here by λ– is applied, the membrane deflects towards the ceiling plate and a snap-through may occur when it exceeds a certain critical value λ^* (pull-in voltage). This creates a so-called "pull-in instability" which greatly affects the design of many devices (see [10, 19] for a detailed discussion on MEMS devices). The mathematical model lends to a nonlinear parabolic problem for the dynamic deflection of the elastic membrane which has been considered by the second and third-named authors in [12, 13]. Concerning the stationary equation, in [11] the challenge was to estimate λ^* in terms of material properties of the membrane, which can be fabricated with a spatially varying dielectric permittivity profile $f(x)$. In particular, lower bounds for λ^* were proved completing in this way the upper bounds of [14, 18]. In all the above-mentioned papers, one can recognize a clear distinction –in techniques and in the available results– between the case where the permittivity profile f is bounded away from zero, from where it is allowed to vanish somewhere. A test case for the latter situation –that has generated much interest among both mathematicians and engineers– is when we have a power-law permittivity profile $f(x) = |x|^\alpha$ ($\alpha \geq 0$) on a ball.

There already exist in the litterature many interesting results concerning the properties of the branch of semi-stable solutions for Dirichlet boundary value problems of the form $-\Delta u = \lambda h(u)$ where h is a regular nonlinearity (for example of the form e^u or $(1+u)^p$ for $p > 1$). See for example the seminal papers [9, 15, 16] and also [7] for a survey on the subject and an exhaustive list of related references. The singular situation was considered in a very general context in [17], and this analysis was completed in [11] to allow for a general continuous permittivity profile $f(x) \geq 0$. Fine properties of steady states –such as regularity, stability, uniqueness, multiplicity, energy estimates and comparison results– were shown there to depend on the dimension of the ambient space and on the permittivity profile.

Let us fix some notations and terminology. The *minimal solutions* of the equation are those classical solutions u_λ of $(S)_\lambda$ that satisfy $u_\lambda(x) \leq u(x)$ in Ω for any solution u of $(S)_\lambda$. Throughout and unless otherwise specified, solutions for $(S)_\lambda$ are considered to be in the classical sense. Now for any solution u of $(S)_\lambda$, one can introduce the linearized operator at u defined by:

$$L_{u,\lambda} = -\Delta - \frac{2\lambda f(x)}{(1-u)^3},$$

and its corresponding eigenvalues $\{\mu_{k,\lambda}(u); k = 1, 2, ...\}$. Note that the first eigenvalue is simple and is given by:

$$\mu_{1,\lambda}(u) = \inf\left\{ \langle L_{u,\lambda}\phi, \phi\rangle_{H_0^1(\Omega)} ; \phi \in C_0^\infty(\Omega), \int_\Omega |\phi(x)|^2 dx = 1 \right\}$$

with the infimum being attained at a first eigenfunction ϕ_1, while the second eigenvalue is given by the formula:

$$\mu_{2,\lambda}(u) = \inf \left\{ \langle L_{u,\lambda}\phi, \phi\rangle_{H_0^1(\Omega)} \,;\, \phi \in C_0^\infty(\Omega), \int_\Omega |\phi(x)|^2 dx = 1 \text{ and } \int_\Omega \phi(x)\phi_1(x) dx = 0 \right\}.$$

This construction can then be iterated to obtain the k-th eigenvalue $\mu_{k,\lambda}(u)$ with the convention that eigenvalues are repeated according to their multiplicities.

The usual analysis of the minimal branch (composed of semi-stable solutions) was extended in [11] by Ghoussoub and Guo to cover the singular situation $(S)_\lambda$ above and the subsequent result – best illustrated by the following bifurcation diagram– was obtained.

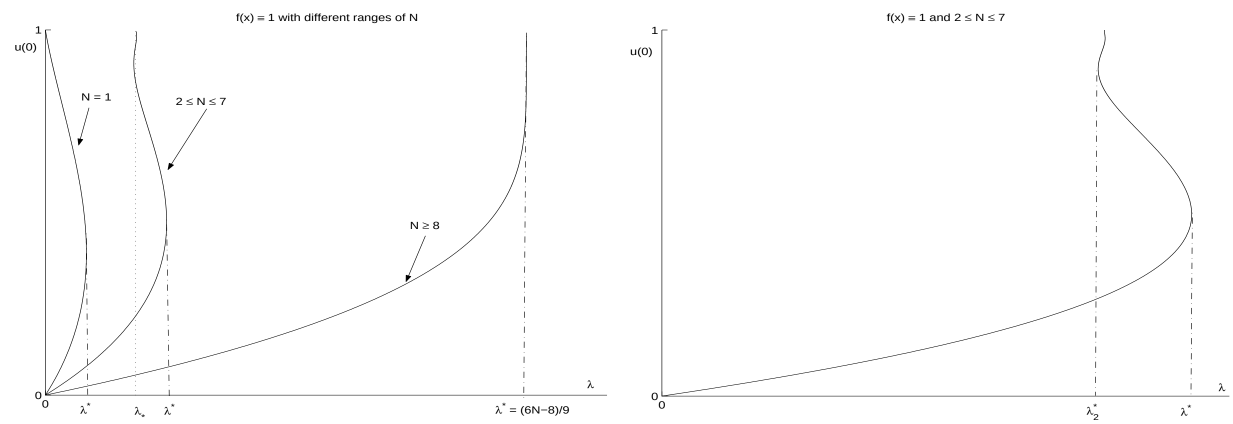

Figure 1: *Left figure: plots of $u(0)$ versus λ for the case where $f(x) \equiv 1$ is defined in the unit ball $B_1(0) \subset \mathbb{R}^N$ with different ranges of dimension N, where we have $\lambda^* = (6N-8)/9$ for dimension $N \geq 8$. Right figure: plots of $u(0)$ versus λ for the case where $f(x) \equiv 1$ is defined in the unit ball $B_1(0) \subset \mathbb{R}^N$ with dimension $2 \leq N \leq 7$, where λ^* (resp. λ_2^*) is the first (resp. second) turning point.*

Theorem A (Theorem 1.1-1.3 in [11]): *Suppose $f \in C(\bar{\Omega})$ is a nonnegative function on Ω. Then, there exists a finite $\lambda^* > 0$ such that*

1. *If $0 \leq \lambda < \lambda^*$, there exists a unique minimal solution u_λ of $(S)_\lambda$ such that $\mu_{1,\lambda}(u_\lambda) > 0$;*
2. *If $\lambda > \lambda^*$, there is no solution for $(S)_\lambda$.*
3. *Moreover, if $1 \leq N \leq 7$ then –by means of energy estimates– one has*

$$\sup_{\lambda \in (0,\lambda^*)} \| u_\lambda \|_\infty < 1 \tag{1.1}$$

and consequently, $u^* = \lim\limits_{\lambda \uparrow \lambda^*} u_\lambda$ is a solution for $(S)_{\lambda^*}$ such that

$$\mu_{1,\lambda^*}(u^*) = 0. \tag{1.2}$$

In particular, u^* –often referred to as the extremal solution of problem $(S)_\lambda$– is unique.

4. *On the other hand, if $f(x) = |x|^\alpha$ and Ω is the unit ball, then the extremal solution is necessarily $u^*(x) = 1 - |x|^{\frac{2+\alpha}{3}}$ and $\lambda^* = \frac{(2+\alpha)(3N+\alpha-4)}{9}$, provided $N \geq 8$ and $0 \leq \alpha \leq \alpha_N = \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$.*

We note that in general, the function u^* exists in any dimension, does solve $(S)_{\lambda^*}$ in an appropriate weak sense and is the unique solution in some suitable class (see the Appendix).

Our first goal is the study of the effect of power-like permittivity profiles $f(x) \simeq |x|^\alpha$ for the problem $(S)_\lambda$ on the unit ball $B = B_1(0)$. We extend the previous result in higher dimensions:

Theorem 1.1. *Assume $N \geq 8$ and $\alpha > \alpha_N = \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$. Let $f \in C(\bar{B})$ be such that:*

$$f(x) = |x|^\alpha g(x)\,, \quad g(x) \geq C > 0 \text{ in } B. \tag{1.3}$$

Let $(\lambda_n)_n$ be such that $\lambda_n \to \lambda \in [0, \lambda^*]$ and u_n be a solution of $(S)_{\lambda_n}$ so that $\mu_{1,n} := \mu_{1,\lambda_n}(u_n) \geq 0$. Then,

$$\sup_{n\in\mathbb{N}} \| u_n \|_\infty < 1.$$

In particular, the extremal solution $u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$ is a solution of $(S)_{\lambda^*}$ such that (1.2) holds.

As to non-minimal solutions, it is also shown in [11] –following ideas of Crandall-Rabinowitz [9]– that, for $1 \leq N \leq 7$, and for λ close enough to λ^*, there exists a unique second branch U_λ of solutions for $(S)_\lambda$, bifurcating from u^*, such that

$$\mu_{1,\lambda}(U_\lambda) < 0 \quad \text{while} \quad \mu_{2,\lambda}(U_\lambda) > 0. \tag{1.4}$$

For $N \geq 8$ and $\alpha > \alpha_N$, the same remains true for problem $(S)_\lambda$ on the unit ball with $f(x)$ as in (1.3) and U_λ is a radial function.

In the sequel, we try to provide a rigorous analysis for other features of the bifurcation diagram, in particular the second branch of unstable solutions, as well as the second bifurcation point. But first, and for the sake of completeness, we shall give a variational characterization for the unstable solution U_λ in the following sense:

Theorem 1.2. *Assume f is a non-negative function in $C(\bar{\Omega})$ where Ω is a bounded domain in $\mathbb{R}^N$. If $1 \leq N \leq 7$, then there exists $\delta > 0$ such that for any $\lambda \in (\lambda^* - \delta, \lambda^*)$, the second solution U_λ is a mountain pass solution for some regularized energy functional $J_{\varepsilon,\lambda}$ on the space $H_0^1(\Omega)$.*

Moreover, the same result is still true for $N \geq 8$ provided Ω is a ball, and $f(x)$ is as in (1.3) with $\alpha > \alpha_N$.

We are now interested in continuing the second branch till the second bifurcation point, by means of the Implicit Function Theorem. For that, we have the following compactness result:

Theorem 1.3. *Assume $2 \leq N \leq 7$. Let $f \in C(\bar{\Omega})$ be such that:*

$$f(x) = \left(\prod_{i=1}^k |x - p_i|^{\alpha_i}\right) g(x)\,, \quad g(x) \geq C > 0 \text{ in } \Omega, \tag{1.5}$$

for some points $p_i \in \Omega$ and exponents $\alpha_i \geq 0$. Let $(\lambda_n)_n$ be a sequence such that $\lambda_n \to \lambda \in [0, \lambda^*]$ and let u_n be an associated solution such that

$$\mu_{2,n} := \mu_{2,\lambda_n}(u_n) \geq 0. \tag{1.6}$$

Then, $\sup_{n\in\mathbb{N}} \| u_n \|_\infty < 1$. Moreover, if in addition $\mu_{1,n} := \mu_{1,\lambda_n}(u_n) < 0$, then necessarily $\lambda > 0$.

Let us mention that Theorem 1.3 yields another proof –based on a blow-up argument– of the compactness result for minimal solutions (1.1) established in [11] by means of some energy estimates, though under the more stringent assumption (1.5) on $f(x)$. We expect that the same result should be true for radial solutions on the unit ball for $N \geq 8$, $\alpha > \alpha_N$, and $f \in C(\bar{\Omega})$ as in (1.3).

As far as we know, there are no compactness results of this type in the case of regular nonlinearities, marking a substantial difference with the singular situation. Theorem 1.3 is based on a blow up argument and the knowledge of linear instability for solutions of a limit problem on $\mathbb{R}^N$, a result which is interesting in itself (see for example [8]) and which somehow explains the special role of dimension 7 and $\alpha = \alpha_N$ for this problem.

Theorem 1.4. *Assume that either $1 \le N \le 7$ and $\alpha \ge 0$ or that $N \ge 8$ and $\alpha > \alpha_N$. Let U be a solution of*

$$\begin{cases} \Delta U = \dfrac{|y|^\alpha}{U^2} & \text{in } \mathbb{R}^N, \\ U(y) \ge C > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{1.7}$$

Then, U is linearly unstable in the following sense:

$$\mu_1(U) = \inf\left\{\int_{\mathbb{R}^N} \big(|\nabla\phi|^2 - \frac{2|y|^\alpha}{U^3}\phi^2\big)dx;\ \phi \in C_0^\infty(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} \phi^2 = 1\right\} < 0\,. \tag{1.8}$$

Moreover, if $N \ge 8$ and $0 \le \alpha \le \alpha_N$, then there exists at least a solution U of (1.7) such that $\mu_1(U) \ge 0$.

Theorem 1.4 is the main tool to control the blow up behavior of a possible non compact sequence of solutions. The usual asymptotic analysis for equations with Sobolev critical nonlinearity, based on some energy bounds (usually $L^{\frac{2N}{N-2}}(\Omega)$-bounds), does not work in our context. In view of [11], a possible loss of compactness can be related to the $L^{\frac{3N}{2}}(\Omega)$-norm along the sequence. Essentially, the blow up associated to a sequence u_n (in the sense of the blowing up of $(1-u_n)^{-1}$) corresponds exactly to the blow up of the $L^{\frac{3N}{2}}(\Omega)$-norm. We replace these energy bounds by some spectral information and, based on Theorem 1.4, we provide an estimate of the number of blow up points (counted with their "multiplicities") in terms of the Morse index along the sequence.

We now define the second bifurcation point in the following way for $(S)_\lambda$:

$$\lambda_2^* = \inf\{\beta > 0 : \exists \text{ a curve } V_\lambda \in C([\beta,\lambda^*]; C^2(\Omega)) \text{ of solutions for } (S)_\lambda \text{ s.t. } \mu_{2,\lambda}(V_\lambda) \ge 0, V_\lambda \equiv U_\lambda\, \forall \lambda \in (\lambda^*-\delta,\lambda^*)\}.$$

We then have the following multiplicity result:

Theorem 1.5. *Assume $f \in C(\bar\Omega)$ to be of the form (1.5). Then, for $2 \le N \le 7$ we have that $\lambda_2^* \in (0,\lambda^*)$ and for any $\lambda \in (\lambda_2^*, \lambda^*)$ there exist at least two solutions u_λ and V_λ for $(S)_\lambda$, so that*

$$\mu_{1,\lambda}(V_\lambda) < 0 \quad \text{while} \quad \mu_{2,\lambda}(V_\lambda) \ge 0.$$

In particular, for $\lambda = \lambda_2^*$, there exists a second solution, namely $V^* := \lim_{\lambda \downarrow \lambda_2^*} V_\lambda$ so that

$$\mu_{1,\lambda_2^*}(V^*) < 0 \quad \text{and} \quad \mu_{2,\lambda_2^*}(V^*) = 0.$$

One can compare Theorem 1.5 with the multiplicity result of [1] for nonlinearities of the form $\lambda u^q + u^p$ $(0 < q < 1 < p)$, where the authors show that for p subcritical, there exists a second –mountain pass– solution for any $\lambda \in [0,\lambda^*)$. On the other hand, when p is critical, the second branch blows up as $\lambda \to 0$ (see also [3] for a related problem). We note that in our situation, the second branch cannot approach the value $\lambda = 0$ as illustrated by the bifurcation diagram above.

Let now V_λ, $\lambda \in (\beta,\lambda^*)$ be one of the curves appearing in the definition of λ_2^*. By (1.4), we have that $L_{V_\lambda,\lambda}$ is invertible for $\lambda \in (\lambda^*-\delta,\lambda^*)$ and, as long as it remains invertible, we can use the Implicit Function Theorem to find V_λ as the unique smooth extension of the curve U_λ (in principle U_λ exists only for λ close to λ^*). Let now λ^{**} be defined in the following way:

$$\lambda^{**} = \inf\{\beta > 0 : \forall \lambda \in (\beta, \lambda^*) \,\exists\, V_\lambda \text{ solution of } (S)_\lambda \text{ so that } \mu_{2,\lambda}(V_\lambda) > 0,\ V_\lambda \equiv U_\lambda \text{ for } \lambda \in (\lambda^* - \delta, \lambda^*)\}.$$

Then, $\lambda_2^* \le \lambda^{**}$ and there exists a smooth curve V_λ for $\lambda \in (\lambda^{**}, \lambda^*)$ so that V_λ is the unique maximal extension of the curve U_λ. This is what the second branch is supposed to be. If now $\lambda_2^* < \lambda^{**}$, then for $\lambda \in (\lambda_2^*, \lambda^{**})$ there is no longer uniqueness for the extension and the "second branch" is defined only as one of potentially many continuous extensions of U_λ.

It remains open the problem whether λ_2^* is the second turning point for the solution diagram of $(S)_\lambda$ or if the "second branch" simply disappears at $\lambda = \lambda_2^*$. Note that if the "second branch" does not disappear, then it can continue for λ less than λ_2^* but only along solutions whose first two eigenvalues are negative.

In dimension 1, we have a stronger but somewhat different compactness result. Recall that $\mu_{k,\lambda_n}(u_n)$ is the $k-$th eigenvalue of L_{u_n,λ_n} counted with their multiplicity.

Theorem 1.6. *Let I be a bounded interval in $\mathbb{R}$ and $f \in C^1(\bar{I})$ be such that $f \ge C > 0$ in I. Let $(u_n)_n$ be a solution sequence for $(S)_{\lambda_n}$ on I, where $\lambda_n \to \lambda \in [0, \lambda^*]$. Assume that for any $n \in \mathbb{N}$ and k large enough, we have:*

$$\mu_{k,n} := \mu_{k,\lambda_n}(u_n) \ge 0. \tag{1.9}$$

If $\lambda > 0$, then again $\sup\limits_{n\in\mathbb{N}} \| u_n \|_\infty < 1$ and compactness holds.

Even in dimension 1, we can still define λ_2^* but we don't know when $\lambda_2^* = 0$ (this is indeed the case when $f(x) = 1$, see [20]) or when $\lambda_2^* > 0$. In the latter situation, there would exist a solution V^* for $(S)_{\lambda_2^*}$ which could be –in some cases– the second turning point. Let us remark that the multiplicity result of Theorem 1.5 holds also in dimension 1 for any $\lambda \in (\lambda_2^*, \lambda^*)$.

The paper is organized as follows. In Section 2 we provide the mountain pass variational characterization of U_λ for λ close to λ^* as stated in Theorem 1.2. The compactness result of Theorem 1.1 on the unit ball is proved in Section 3. Section 4 is concerned with the compactness of the second branch of $(S)_\lambda$ as stated in Theorem 1.3. Section 5 deals with the dimension 1 of Theorem 1.6. In Section 6 we give the proof of the multiplicity result in Theorem 1.5. Finally, the linear instability property of Theorem 1.4 and the details of the above mentioned counterexample to the C^2-regularity of u^* in dimension $N \ge 8$, $0 \le \alpha \le \alpha_N$, are given in the Appendix.

2 Mountain Pass solutions

This Section is devoted to the variational characterization of the second solution U_λ of $(S)_\lambda$ for $\lambda \uparrow \lambda^*$ and in dimension $1 \le N \le 7$. Let us stress that the argument works also for problem $(S)_\lambda$ on the unit ball with $f(x)$ in the form (1.3) provided $N \ge 8$, $\alpha > \alpha_N$.

Since the nonlinearity $g(u) = \frac{1}{(1-u)^2}$ is singular at $u = 1$, we need to consider a regularized C^1 nonlinearity $g_\varepsilon(u)$, $0 < \varepsilon < 1$, of the following form:

$$g_\varepsilon(u) = \begin{cases} \frac{1}{(1-u)^2} & u \le 1 - \varepsilon\,, \\ \frac{1}{\varepsilon^2} - \frac{2(1-\varepsilon)}{p\varepsilon^3} + \frac{2}{p\varepsilon^3(1-\varepsilon)^{p-1}} u^p & u \ge 1 - \varepsilon\,, \end{cases} \tag{2.1}$$

where $p > 1$ if $N = 1, 2$ and $1 < p < \frac{N+2}{N-2}$ if $3 \le N \le 7$. For $\lambda \in (0, \lambda^*)$, we study the regularized semilinear elliptic problem:

$$\begin{cases} -\Delta u = \lambda f(x) g_\varepsilon(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

From a variational viewpoint, the action functional associated to (2.2) is

$$J_{\varepsilon,\lambda}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega f(x) G_\varepsilon(u) dx\,, \quad u \in H_0^1(\Omega)\,, \tag{2.3}$$

where $G_\varepsilon(u) = \int_{-\infty}^u g_\varepsilon(s)ds$.

Fix now $0 < \varepsilon < \frac{1-\|u^*\|_\infty}{2}$. For $\lambda \uparrow \lambda^*$, the minimal solution u_λ of $(S)_\lambda$ is still a solution of (2.2) so that $\mu_1\left(-\Delta - \lambda f(x) g_\varepsilon'(u_\lambda)\right) > 0$. In order to motivate the choice of $g_\varepsilon(u)$, we briefly sketch the proof of Theorem 1.2. First, we prove that u_λ is a local minimum for $J_{\varepsilon,\lambda}(u)$ for $\lambda \uparrow \lambda^*$. Then, by the well known Mountain Pass Theorem [2], we show the existence of a second solution $U_{\varepsilon,\lambda}$ for (2.2). Since $U_{\varepsilon,\lambda} \to u^*$ in $C(\bar\Omega)$ as $\lambda \uparrow \lambda^*$, we have that $U_{\varepsilon,\lambda} \le 1-\varepsilon$ and $U_{\varepsilon,\lambda}$ is then a second solution for $(S)_\lambda$ bifurcating from u^*. But since $U_{\varepsilon,\lambda}$ is a MP solution and $(S)_\lambda$ has exactly two solutions u_λ, U_λ for $\lambda \uparrow \lambda^*$, we get that $U_{\varepsilon,\lambda} = U_\lambda$.
The subcritical growth:

$$0 \le g_\varepsilon(u) \le C_\varepsilon(1+|u|^p) \tag{2.4}$$

and the inequality:

$$\theta G_\varepsilon(u) \le u g_\varepsilon(u) \quad \text{for } u \ge M_\varepsilon, \tag{2.5}$$

for some C_ε, $M_\varepsilon > 0$ large and $\theta = \frac{p+3}{2} > 2$, will yield that $J_{\varepsilon,\lambda}$ satisfies the Palais-Smale condition and, by means of a bootstrap argument, we get the uniform convergence of $U_{\varepsilon,\lambda}$. On the other hand, the convexity of $g_\varepsilon(u)$ ensures that problem (2.2) has the unique solution u^* at $\lambda = \lambda^*$, which then allows us to identify the limit of $U_{\varepsilon,\lambda}$ as $\lambda \uparrow \lambda^*$.

In order to complete the details for the proof of Theorem 1.2, we first need to show the following:

Lemma 2.1. *For $\lambda \uparrow \lambda^*$, the minimal solution u_λ of $(S)_\lambda$ is a local minimum of $J_{\varepsilon,\lambda}$ on $H_0^1(\Omega)$.*

Proof: First, we show that u_λ is a local minimum of $J_{\varepsilon,\lambda}$ in $C^1(\bar\Omega)$. Indeed, since

$$\mu_{1,\lambda} := \mu_1\left(-\Delta - \lambda f(x) g_\varepsilon'(u_\lambda)\right) > 0,$$

we have the following inequality:

$$\int_\Omega |\nabla\phi|^2 dx - 2\lambda \int_\Omega \frac{f(x)}{(1-u_\lambda)^3}\phi^2 dx \ge \mu_{1,\lambda}\int_\Omega \phi^2 \tag{2.6}$$

for any $\phi \in H_0^1(\Omega)$, since $u_\lambda \le 1-\varepsilon$. Now, take any $\phi \in H_0^1(\Omega) \cap C^1(\bar\Omega)$ such that $\|\phi\|_{C^1} \le \delta_\lambda$. Since $u_\lambda \le 1 - \frac{3}{2}\varepsilon$, if $\delta_\lambda \le \frac{\varepsilon}{2}$, then $u_\lambda + \phi \le 1-\varepsilon$ and we have that:

$$\begin{aligned} J_{\varepsilon,\lambda}(u_\lambda+\phi) - J_{\varepsilon,\lambda}(u_\lambda) &= \frac{1}{2}\int_\Omega |\nabla\phi|^2 dx + \int_\Omega \nabla u_\lambda \cdot \nabla\phi dx - \lambda\int_\Omega f(x)\left(\frac{1}{1-u_\lambda-\phi} - \frac{1}{1-u_\lambda}\right) \\ &\ge \tfrac{\mu_{1,\lambda}}{2}\int_\Omega \phi^2 - \lambda\int_\Omega f(x)\left(\frac{1}{1-u_\lambda-\phi} - \frac{1}{1-u_\lambda} - \frac{\phi}{(1-u_\lambda)^2} - \frac{\phi^2}{(1-u_\lambda)^3}\right)\,, \end{aligned} \tag{2.7}$$

where we have applied (2.6). Since now

$$\left|\frac{1}{1-u_\lambda-\phi} - \frac{1}{1-u_\lambda} - \frac{\phi}{(1-u_\lambda)^2} - \frac{\phi^2}{(1-u_\lambda)^3}\right| \le C|\phi|^3$$

for some $C > 0$, (2.7) gives that

$$J_{\varepsilon,\lambda}(u_\lambda+\phi) - J_{\varepsilon,\lambda}(u_\lambda) \ge \left(\frac{\mu_{1,\lambda}}{2} - C\lambda \parallel f \parallel_\infty \delta_\lambda\right)\int_\Omega \phi^2 > 0$$

provided δ_λ is small enough. This proves that u_λ is a local minimum of $J_{\varepsilon,\lambda}$ in the $\mathcal{C}^1$ topology. Since (2.4) is satisfied, we can then directly apply Theorem 1 in [5] to get that u_λ is a local minimum of $J_{\varepsilon,\lambda}$ in $H_0^1(\Omega)$. ■

Since now $f \neq 0$, fix some small ball $B_{2r} \subset \Omega$ of radius $2r$, $r > 0$, so that $\int_{B_r} f(x)dx > 0$. Take a cut-off function χ so that $\chi = 1$ on B_r and $\chi = 0$ outside B_{2r}. Let $w_\varepsilon = (1-\varepsilon)\chi \in H_0^1(\Omega)$. We have that:

$$J_{\varepsilon,\lambda}(w_\varepsilon) \leq \frac{(1-\varepsilon)^2}{2} \int_\Omega |\nabla \chi|^2 dx - \frac{\lambda}{\varepsilon^2} \int_{B_r} f(x) \to -\infty$$

as $\varepsilon \to 0$, and uniformly for λ far away from zero. Since

$$J_{\varepsilon,\lambda}(u_\lambda) = \frac{1}{2} \int_\Omega |\nabla u_\lambda|^2 dx - \lambda \int_\Omega \frac{f(x)}{1-u_\lambda} dx \to \frac{1}{2} \int_\Omega |\nabla u^*|^2 dx - \lambda^* \int_\Omega \frac{f(x)}{1-u^*} dx$$

as $\lambda \to \lambda^*$, we can find that for $\varepsilon > 0$ small, the inequality

$$J_{\varepsilon,\lambda}(w_\varepsilon) < J_{\varepsilon,\lambda}(u_\lambda) \tag{2.8}$$

holds for any λ close to λ^*.

Fix now $\varepsilon > 0$ small enough in order that (2.8) holds for λ close to λ^*, and define

$$c_{\varepsilon,\lambda} = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} J_{\varepsilon,\lambda}(u),$$

where $\Gamma = \{\gamma : [0,1] \to H_0^1(\Omega); \gamma \text{ continuous and } \gamma(0) = u_\lambda,\ \gamma(1) = w_\varepsilon\}$. We can then apply the Mountain Pass Theorem [2] to get a solution $U_{\varepsilon,\lambda}$ of (2.2) for λ close to λ^*, provided the Palais-Smale condition holds at level c. We shall now prove this (PS)-condition in the following form:

Lemma 2.2. *Assume that $\{w_n\} \subset H_0^1(\Omega)$ satisfies*

$$J_{\varepsilon,\lambda_n}(w_n) \leq C\,, \qquad J'_{\varepsilon,\lambda_n}(w_n) \to 0 \quad \text{in } H^{-1} \tag{2.9}$$

for $\lambda_n \to \lambda > 0$. Then the sequence $(w_n)_n$ is uniformly bounded in $H_0^1(\Omega)$ and therefore admits a convergent subsequence in $H_0^1(\Omega)$.

Proof: By (2.9) we have that:

$$\int_\Omega |\nabla w_n|^2 dx - \lambda_n \int_\Omega f(x) g_\varepsilon(w_n) w_n dx = o(\| w_n \|_{H_0^1})$$

as $n \to +\infty$ and then,

$$\begin{aligned}
C &\geq \frac{1}{2} \int_\Omega |\nabla w_n|^2 dx - \lambda_n \int_\Omega f(x) G_\varepsilon(w_n) dx \\
&= \left(\frac{1}{2} - \frac{1}{\theta}\right) \int_\Omega |\nabla w_n|^2 dx + \lambda_n \int_\Omega f(x) \left(\frac{1}{\theta} w_n g_\varepsilon(w_n) - G_\varepsilon(w_n)\right) dx + o(\| w_n \|_{H_0^1}) \\
&\geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \int_\Omega |\nabla w_n|^2 dx + \lambda_n \int_{\{w_n \geq M_\varepsilon\}} f(x) \left(\frac{1}{\theta} w_n g_\varepsilon(w_n) - G_\varepsilon(w_n)\right) dx + o(\| w_n \|_{H_0^1}) - C_\varepsilon \\
&\geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \int_\Omega |\nabla w_n|^2 dx + o(\| w_n \|_{H_0^1}) - C_\varepsilon
\end{aligned}$$

in view of (2.5). Hence, $\sup_{n\in\mathbb{N}} \| w_n \|_{H_0^1} < +\infty$.

Since p is subcritical, the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ provides that, up to a subsequence, $w_n \to w$ weakly in $H_0^1(\Omega)$ and strongly in $L^{p+1}(\Omega)$, for some $w \in H_0^1(\Omega)$. By (2.9) we get that $\int_\Omega |\nabla w|^2 = \lambda \int_\Omega f(x) g_\varepsilon(w) w$, and then, by (2.4), we deduce that

$$\int_\Omega |\nabla(w_n - w)|^2 = \int_\Omega |\nabla w_n|^2 - \int_\Omega |\nabla w|^2 + o(1) = \lambda_n \int_\Omega f(x) g_\varepsilon(w_n) w_n - \lambda \int_\Omega f(x) g_\varepsilon(w) w + o(1) \to 0$$

as $n \to +\infty$. ∎

To conclude the proof of Theorem 1.2, we consider for any $\lambda \in (\lambda^* - \delta, \lambda^*)$ the mountain pass solution $U_{\varepsilon,\lambda}$ of (2.2) at energy level $c_{\varepsilon,\lambda}$, where $\delta > 0$ is small enough. Since $c_{\varepsilon,\lambda} \le c_{\varepsilon,\lambda^*-\delta}$ for any $\lambda \in (\lambda^* - \delta, \lambda^*)$, and applying again Lemma 2.2, we get that $\| U_{\varepsilon,\lambda} \|_{H_0^1} \le C$, for any λ close to λ^*. Then, by (2.4) and elliptic regularity theory, we get that $U_{\varepsilon,\lambda}$ is uniformly bounded in $C^{2,\alpha}(\bar{\Omega})$ for $\lambda \uparrow \lambda^*$, for $\alpha \in (0,1)$. Hence, we can extract a sequence $U_{\varepsilon,\lambda_n}$, $\lambda_n \uparrow \lambda^*$, converging in $C^2(\bar{\Omega})$ to some function U^*, where U^* is a solution for problem (2.2) at $\lambda = \lambda^*$. Also u^* is a solution for (2.2) at $\lambda = \lambda^*$ so that $\mu_1 \left(-\Delta - \lambda^* f(x) g_\varepsilon'(u^*)\right) = 0$. By convexity of $g_\varepsilon(u)$, it is classical to show that u^* is the unique solution of this equation and therefore $U^* = u^*$. Since along any convergent sequence of $U_{\varepsilon,\lambda}$ as $\lambda \uparrow \lambda^*$ the limit is always u^*, we get that $\lim_{\lambda\uparrow\lambda^*} U_{\varepsilon,\lambda} = u^*$ in $C^2(\bar{\Omega})$. Therefore, since $u^* \le 1 - 2\varepsilon$, there exists $\delta > 0$ so that for any $\lambda \in (\lambda^* - \delta, \lambda^*)$ $U_{\varepsilon,\lambda} \le u^* + \varepsilon \le 1 - \varepsilon$ and hence, $U_{\varepsilon,\lambda}$ is a solution of $(S)_\lambda$. Since the mountain pass energy level $c_{\varepsilon,\lambda}$ satisfies $c_{\varepsilon,\lambda} > J_{\varepsilon,\lambda}(u_\lambda)$, we have that $U_{\varepsilon,\lambda} \ne u_\lambda$ and then $U_{\varepsilon,\lambda} = U_\lambda$ for any $\lambda \in (\lambda^* - \delta, \lambda^*)$. Note that by [9], we know that u_λ, U_λ are the only solutions of $(S)_\lambda$ as $\lambda \uparrow \lambda^*$.

3 Minimal branch on the ball for power-like permittivity profiles

Let B be the unit ball. Let $(\lambda_n)_n$ be such that $\lambda_n \to \lambda \in [0, \lambda^*]$ and u_n be a solution of $(S)_{\lambda_n}$ on B so that $\mu_{1,n} := \mu_{1,\lambda_n}(u_n) \ge 0$. By Proposition 7.3 u_n coincides with the minimal solution u_{λ_n} and, by some symmetrization arguments, in [11] it is shown that the minimal solution u_n is radial and achieves its absolute maximum only at zero.

Given a permittivity profile $f(x)$ as in (1.3), in order to get Theorem 1.1 we want to show:

$$\sup_{n\in\mathbb{N}} \| u_n \|_\infty < 1, \tag{3.1}$$

provided $N \ge 8$ and $\alpha > \alpha_N = \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$. In particular, since u_λ is non decreasing in λ and

$$\sup_{\lambda\in[0,\lambda^*)} \| u_\lambda \|_\infty < 1,$$

the extremal solution $u^* = \lim_{\lambda\uparrow\lambda^*} u_\lambda$ would be a solution of $(S)_{\lambda^*}$ so that $\mu_{1,\lambda^*}(u^*) \le 0$. Property (1.2) must hold because otherwise, by Implicit Function Theorem, we could find solutions of $(S)_\lambda$ for $\lambda > \lambda^*$.

In order to prove (3.1), let us argue by contradiction. Up to a subsequence, assume that $u_n(0) = \max_B u_n \to 1$ as $n \to +\infty$. Since $\lambda = 0$ implies $u_n \to 0$ in $C^2(\bar{B})$, we can assume that $\lambda_n \to \lambda > 0$. Let $\varepsilon_n := 1 - u_n(0) \to 0$ as $n \to +\infty$ and introduce the following rescaled function:

$$U_n(y) = \frac{1 - u_n\left(\varepsilon_n^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}} y\right)}{\varepsilon_n}, \quad y \in B_n := B_{\varepsilon_n^{-\frac{3}{2+\alpha}} \lambda_n^{\frac{1}{2+\alpha}}}(0). \tag{3.2}$$

The function U_n satisfies:

$$\begin{cases} \Delta U_n = \dfrac{|y|^\alpha g\left(\varepsilon_n^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}} y\right)}{U_n^2} & \text{in } B_n, \\ U_n(y) \geq U_n(0) = 1, \end{cases} \tag{3.3}$$

and $B_n \to \mathbb{R}^N$ as $n \to +\infty$. We would get a contradiction to $\mu_{1,n} \geq 0$ by proving:

Proposition 3.1. *There exists a subsequence $\{U_n\}_n$ such that $U_n \to U$ in $C^1_{loc}(\mathbb{R}^N)$, where U is a solution of the equation:*

$$\begin{cases} \Delta U = g(0)\dfrac{|y|^\alpha}{U^2} & \text{in } R^N, \\ U(y) \geq U(0) = 1 & \text{in } \mathbb{R}^N. \end{cases} \tag{3.4}$$

Moreover, there exists $\phi_n \in C_0^\infty(B)$ such that:

$$\int_B \left(|\nabla \phi_n|^2 - \frac{2\lambda_n |x|^\alpha g(x)}{(1-u_n)^3} \phi_n^2 \right) < 0.$$

Proof: Let $R > 0$. For n large, decompose $U_n = U_n^1 + U_n^2$, where U_n^2 satisfies:

$$\begin{cases} \Delta U_n^2 = \Delta U_n & \text{in } B_R(0), \\ U_n^2 = 0 & \text{on } \partial B_R(0). \end{cases}$$

By (3.3) we get that on $B_R(0)$:

$$0 \leq \Delta U_n \leq R^\alpha \parallel g \parallel_\infty,$$

and then, standard elliptic regularity theory gives that U_n^2 is uniformly bounded in $C^{1,\beta}(B_R(0))$, $\beta \in (0,1)$. Up to a subsequence, we get that $U_n^2 \to U^2$ in $C^1(B_R(0))$. Since $U_n^1 = U_n \geq 1$ on $\partial B_R(0)$, by harmonicity $U_n^1 \geq 1$ in $B_R(0)$ and, by Harnack inequality:

$$\sup_{B_{R/2}(0)} U_n^1 \leq C_R \inf_{B_{R/2}(0)} U_n^1 \leq C_R U_n^1(0) = C_R \left(1 - U_n^2(0)\right) \leq C_R \left(1 + \sup_{n \in \mathbb{N}} |U_n^2(0)|\right) < \infty.$$

Hence, U_n^1 is uniformly bounded in $C^{1,\beta}(B_{R/4}(0))$, $\beta \in (0,1)$. Up to a further subsequence, we get that $U_n^1 \to U^1$ in $C^1(B_{R/4}(0))$ and then, $U_n \to U^1 + U^2$ in $C^1(B_{R/4}(0))$, for any $R > 0$. By a diagonal process and up to a subsequence, we find that $U_n \to U$ in $C^1_{\text{loc}}(\mathbb{R}^N)$, where U is a solution of the equation (3.4).

If $1 \leq N \leq 7$ or $N \geq 8$, $\alpha > \alpha_N$, since $g(0) > 0$ by Theorem 1.4 we get that $\mu_1(U) < 0$ and then, we find $\phi \in C_0^\infty(\mathbb{R}^N)$ so that:

$$\int \left(|\nabla \phi|^2 - 2g(0)\frac{|y|^\alpha}{U^3}\phi^2 \right) < 0.$$

Define now $\phi_n(x) = \left(\varepsilon_n^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}}\right)^{-\frac{N-2}{2}} \phi\left(\varepsilon_n^{-\frac{3}{2+\alpha}} \lambda_n^{\frac{1}{2+\alpha}} x\right)$. We have that:

$$\begin{aligned} \int_B \left(|\nabla \phi_n|^2 - \frac{2\lambda_n |x|^\alpha g(x)}{(1-u_n)^3}\phi_n^2 \right) &= \int \left(|\nabla \phi|^2 - \frac{2|y|^\alpha}{U_n^3} g(\varepsilon_n^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}} y)\phi^2 \right) \\ &\to \int \left(|\nabla \phi|^2 - 2g(0)\frac{|y|^\alpha}{U^3}\phi^2 \right) < 0 \end{aligned}$$

as $n \to +\infty$, since ϕ has compact support and $U_n \to U$ in $C^1_{\text{loc}}(\mathbb{R})$. The proof of Proposition 3.1 is now complete. ∎

4 Compactness along the second branch of solutions

In this Section we turn to the compactness result stated in Theorem 1.3. Assume that $f \in C(\bar{\Omega})$ is in the form (1.5), and let $(u_n)_n$ be a solution sequence for $(S)_{\lambda_n}$ where $\lambda_n \to \lambda \in [0, \lambda^*]$.

4.1 Blow-up analysis

Assume that the sequence $(u_n)_n$ is not compact, which means that up to passing to a subsequence, we may assume that $\max_{\Omega} u_n \to 1$ as $n \to \infty$. Let x_n be a maximum point of u_n in Ω (i.e., $u_n(x_n) = \max_{\Omega} u_n$) and set $\varepsilon_n = 1 - u_n(x_n)$. Let us assume that $x_n \to p$ as $n \to +\infty$. We have three different situations depending on the location of p and the rate of $|x_n - p|$:

1) blow up outside the zero set of $f(x)$ $\{p_1, \dots, p_k\}$, i.e. $p \notin \{p_1, \dots, p_k\}$;
2) "slow" blow up at some p_i in the zero set of $f(x)$, i.e. $x_n \to p_i$ and $\varepsilon_n^{-3} \lambda_n |x_n - p_i|^{\alpha+2} \to +\infty$ as $n \to +\infty$;
3) "fast" blow at some p_i in the zero set of $f(x)$, i.e. $x_n \to p_i$ and $\limsup_{n \to +\infty} \left(\varepsilon_n^{-3} \lambda_n |x_n - p_i|^{\alpha+2} \right) < +\infty$.

Accordingly, we discuss now each one of these situations.

1^{st} **Case** Assume that $p \notin \{p_1, \dots, p_k\}$. In general, we are not able to prove that a blow up point p is always far away from $\partial\Omega$, even though we suspect it to be true. However, some weaker estimate is available and –as explained later– will be sufficient for our purposes. We have that:

Lemma 4.1. *Let h_n be a function on a smooth bounded domain A_n in $\mathbb{R}^N$. Let W_n be a solution of:*

$$\begin{cases} \Delta W_n = \dfrac{h_n(x)}{W_n^2} & \text{in } A_n, \\ W_n(y) \geq C > 0 & \text{in } A_n, \\ W_n(0) = 1, \end{cases} \tag{4.1}$$

for some $C > 0$. Assume that $\sup_{n \in \mathbb{N}} \| h_n \|_\infty < +\infty$ and $A_n \to T_\mu$ as $n \to +\infty$ for some $\mu \in (0, +\infty)$, where T_μ is an hyperspace so that $0 \in T_\mu$ and dist $(0, \partial T_\mu) = \mu$. Then, either

$$\inf_{\partial A_n \cap B_{2\mu}(0)} W_n \leq C \tag{4.2}$$

or

$$\inf_{\partial A_n \cap B_{2\mu}(0)} \partial_\nu W_n \leq 0, \tag{4.3}$$

where ν is the unit outward normal of A_n.

Proof: Assume that $\partial_\nu W_n > 0$ on $\partial A_n \cap B_{2\mu}(0)$. Let

$$G(x) = \begin{cases} -\frac{1}{2\pi} \log \frac{|x|}{2\mu} & \text{if } N = 2 \\ c_N \left(\frac{1}{|x|^{N-2}} - \frac{1}{(2\mu)^{N-2}} \right) & \text{if } N \geq 3 \end{cases}$$

be the Green function at 0 of the operator $-\Delta$ in $B_{2\mu}(0)$ with homogeneous Dirichlet boundary condition, where $c_N = \frac{1}{(N-2)|\partial B_1(0)|}$ and $| \cdot |$ stands for the Lebesgue measure.

Here and in the sequel, when there is no ambiguity on the domain we are considering, ν and $d\sigma$ will denote the unit outward normal and the boundary integration element of the corresponding domain. By the representation formula we have that:

$$W_n(0) = -\int_{A_n\cap B_{2\mu}(0)} \Delta W_n(x) G(x)dx - \int_{\partial A_n\cap B_{2\mu}(0)} W_n(x)\partial_\nu G(x) d\sigma(x) + \int_{\partial A_n\cap B_{2\mu}(0)} \partial_\nu W_n(x) G(x) d\sigma(x) - \int_{\partial B_{2\mu}(0)\cap A_n} W_n(x)\partial_\nu G(x) d\sigma(x). \tag{4.4}$$

Since on ∂T_μ:

$$-\partial_\nu G(x) = \begin{cases} \frac{1}{2\pi}\frac{x}{|x|^2}\cdot\nu & \text{if } N=2 \\ (N-2)c_N\frac{x}{|x|^N}\cdot\nu & \text{if } N\geq 3 \end{cases} > 0 \tag{4.5}$$

and $\partial A_n \to \partial T_\mu$, we get that

$$\partial_\nu G(x) < 0 \quad \text{on } \partial A_n \cap B_{2\mu}(0). \tag{4.6}$$

Hence, by (4.4), (4.6) and the assumptions on W_n, we then get:

$$1 \geq -\int_{A_n\cap B_{2\mu}(0)} \frac{h_n(x)}{W_n^2(x)} G(x)dx - \left(\inf_{\partial A_n\cap B_{2\mu}(0)} W_n\right)\int_{\partial A_n\cap B_{2\mu}(0)} \partial_\nu G(x) d\sigma(x)$$

since $G(x)\geq 0$ in $B_{2\mu}(0)$ and $\partial_\nu G(x)\leq 0$ on $\partial B_{2\mu}(0)$. Now, we have that

$$\Big|\int_{A_n\cap B_{2\mu}(0)} \frac{h_n(x)}{W_n^2(x)} G(x)\Big| \leq C$$

and by (4.5)

$$-\int_{\partial A_n\cap B_{2\mu}(0)} \partial_\nu G(x) d\sigma(x) \to -\int_{\partial T_\mu\cap B_{2\mu}(0)} \partial_\nu G(x) d\sigma(x) > 0.$$

Then, $1\geq -C + C^{-1}\left(\inf_{\partial A_n\cap B_{2\mu}(0)} W_n\right)$ for some $C>0$ large enough. Hence, $\inf_{\partial A_n\cap B_{2\mu}(0)} W_n$ is uniformly bounded and the proof is complete. ∎

We are now ready to completely discuss this first case. Introduce the following rescaled function:

$$U_n(y) = \frac{1-u_n(\varepsilon_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}y + x_n)}{\varepsilon_n}, \qquad y\in\Omega_n = \frac{\Omega - x_n}{\varepsilon_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}}. \tag{4.7}$$

Then, U_n satisfies

$$\begin{cases} \Delta U_n = \dfrac{f(\varepsilon_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}y + x_n)}{U_n^2} & \text{in } \Omega_n, \\ U_n(0) = 1 & \text{in } \Omega_n. \end{cases} \tag{4.8}$$

In addition, we have that $U_n\geq U_n(0)=1$ as long as x_n is the maximum point of u_n in Ω.

We would like to prove the following:

Proposition 4.2. *Let $x_n\in\Omega$ and set $\varepsilon_n := 1-u_n(x_n)$. Assume that*

$$x_n \to p \notin \{p_1,\dots,p_k\}, \ \ \varepsilon_n^3\lambda_n^{-1}\to 0 \quad \text{as } n\to+\infty. \tag{4.9}$$

Let U_n, Ω_n be defined as in (4.7). Assume that

$$U_n \geq C > 0 \quad \text{in } \Omega_n\cap B_{R_n}(0) \tag{4.10}$$

for some $R_n \to +\infty$ as $n \to +\infty$. Then, there exists a subsequence of $(U_n)_n$ such that $U_n \to U$ in $C^1_{loc}(\mathbb{R}^N)$, where U is a solution of the equation:

$$\begin{cases} \Delta U = \dfrac{f(p)}{U^2} & \text{in } R^N \,, \\ U(y) \geq C > 0 & \text{in } \mathbb{R}^N \,. \end{cases} \tag{4.11}$$

Moreover, there exists a function $\phi_n \in C_0^\infty(\Omega)$ such that:

$$\int_\Omega \left(|\nabla \phi_n|^2 - \frac{2\lambda_n f(x)}{(1-u_n)^3} \phi_n^2 \right) < 0 \tag{4.12}$$

and Supp $\phi_n \subset B_{M\varepsilon_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}}(x_n)$ for some $M > 0$.

Proof: By (4.9) Lemma 4.1 provides us with a stronger estimate:

$$\varepsilon_n^3 \lambda_n^{-1} (\text{dist } (x_n, \partial\Omega))^{-2} \to 0 \quad \text{as } n \to +\infty. \tag{4.13}$$

Indeed, by contradiction and up to a subsequence, assume that $\varepsilon_n^3 \lambda_n^{-1} d_n^{-2} \to \delta > 0$ as $n \to +\infty$, where $d_n := \text{dist } (x_n, \partial\Omega)$. In view of (4.9) we get that $d_n \to 0$ as $n \to +\infty$. We introduce the following rescaling W_n:

$$W_n(y) = \frac{1 - u_n(d_n y + x_n)}{\varepsilon_n} \,, \quad y \in A_n = \frac{\Omega - x_n}{d_n} \,.$$

Since $d_n \to 0$, we get that $A_n \to T_1$ as $n \to +\infty$, where T_μ is an hyperspace containing 0 so that dist $(0, \partial T_\mu) = \mu$. The function W_n solves problem (4.1) with $h_n(y) = \frac{\lambda_n d_n^2}{\varepsilon_n^3} f(d_n y + x_n)$ and $C = W_n(0) = 1$. We have that:

$$\| h_n \|_\infty \leq \frac{\lambda_n d_n^2}{\varepsilon_n^3} \| f \|_\infty \leq \frac{2}{\delta} \| f \|_\infty$$

and $W_n = \frac{1}{\varepsilon_n} \to +\infty$ on ∂A_n. By Lemma 4.1 we get that (4.3) must hold. A contradiction to Hopf Lemma applied to u_n. Hence, the validity of (4.13).

We have proved that the blow up is "essentially" in the interior of Ω: (4.13) implies that $\Omega_n \to \mathbb{R}^N$ as $n \to +\infty$. Arguing as in the proof of Proposition 3.1, we get that $U_n \to U$ in $C^1_{\text{loc}}(\mathbb{R}^N)$, where U is a solution of (4.11) by means of (4.8)-(4.10).

If $1 \leq N \leq 7$, since $f(p) > 0$ by Theorem 1.4 we get that $\mu_1(U) < 0$ and then, we find $\phi \in C_0^\infty(\mathbb{R}^N)$ so that:

$$\int \left(|\nabla \phi|^2 - \frac{2f(p)}{U^3} \phi^2 \right) < 0.$$

Define now $\phi_n(x) = \left(\varepsilon_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}} \right)^{-\frac{N-2}{2}} \phi \left(\varepsilon_n^{-\frac{3}{2}} \lambda_n^{\frac{1}{2}} (x - x_n) \right)$. We have that:

$$\begin{aligned} \int_\Omega \left(|\nabla \phi_n|^2 - \frac{2\lambda_n f(x)}{(1-u_n)^3} \phi_n^2 \right) &= \int \left(|\nabla \phi|^2 - \frac{2f(\varepsilon_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}} y + x_n)}{U_n^3} \phi^2 \right) \\ &\to \int \left(|\nabla \phi|^2 - \frac{2f(p)}{U^3} \phi^2 \right) < 0 \end{aligned}$$

as $n \to +\infty$, since ϕ has compact support and $U_n \to U$ in $C^1_{\text{loc}}(\mathbb{R})$. The proof of Proposition 4.2 is now complete. ∎

2^{nd} **Case** Assume that $x_n \to p_i$ and $\varepsilon_n^{-3}\lambda_n|x_n - p_i|^{\alpha+2} \to +\infty$ as $n \to +\infty$. Define

$$f_i(x) = \left(\prod_{j=1,\, j\neq i}^{k} |x - p_j|^{\alpha_j} \right) g(x). \tag{4.14}$$

We rescale the function u_n in a different way:

$$U_n(y) = \frac{1 - u_n(\varepsilon_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}|x_n - p_i|^{-\frac{\alpha}{2}}y + x_n)}{\varepsilon_n}, \qquad y \in \Omega_n = \frac{\Omega - x_n}{\varepsilon_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}|x_n - p_i|^{-\frac{\alpha}{2}}}. \tag{4.15}$$

In this situation, U_n satisfies:

$$\begin{cases} \Delta U_n = \left|\varepsilon_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}|x_n - p_i|^{-\frac{\alpha+2}{2}}y + \dfrac{x_n - p_i}{|x_n - p_i|}\right|^{\alpha} \dfrac{f_i(\varepsilon_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}|x_n - p_i|^{-\frac{\alpha}{2}}y + x_n)}{U_n^2} & \text{in } \Omega_n, \\ U_n(0) = 1 & \text{in } \Omega_n. \end{cases} \tag{4.16}$$

The following result holds:

Proposition 4.3. *Let $x_n \in \Omega$ and set $\varepsilon_n := 1 - u_n(x_n)$. Assume that*

$$x_n \to p_i\,,\ \ \varepsilon_n^{-3}\lambda_n|x_n - p_i|^{\alpha+2} \to +\infty \quad \text{as } n \to +\infty. \tag{4.17}$$

Let U_n, Ω_n be defined as in (4.15). Assume that (4.10) holds. Then, up to a subsequence, $U_n \to U$ in $C^1_{loc}(\mathbb{R}^N)$, where U is a solution of the equation:

$$\begin{cases} \Delta U = \dfrac{f_i(p_i)}{U^2} & \text{in } R^N, \\ U(y) \geq C > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{4.18}$$

Moreover, there holds (4.12) for some $\phi_n \in C_0^\infty(\Omega)$ such that Supp $\phi_n \subset B_{M\varepsilon_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}|x_n-p_i|^{-\frac{\alpha}{2}}}(x_n)$, $M > 0$.

Proof: By (4.17) we get that $\Omega_n \to \mathbb{R}^N$ as $n \to +\infty$. As before, $U_n \to U$ in $C^1_{\text{loc}}(\mathbb{R}^N)$ and U is a solution of (4.18) in view of (4.10) and (4.16)-(4.17). Since $1 \leq N \leq 7$ and $f_i(p_i) > 0$, Theorem 1.4 implies $\mu_1(U) < 0$ and the existence of some $\phi \in C_0^\infty(\mathbb{R}^N)$ so that:

$$\int \left(|\nabla\phi|^2 - \frac{2f_i(p_i)}{U^3}\phi^2 \right) < 0.$$

Define now $\phi_n(x) = \left(\varepsilon_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}|x_n - p_i|^{-\frac{\alpha}{2}}\right)^{-\frac{N-2}{2}} \phi\left(\varepsilon_n^{-\frac{3}{2}}\lambda_n^{\frac{1}{2}}|x_n - p_i|^{\frac{\alpha}{2}}(x - x_n)\right)$. We have that:

$$\begin{aligned} &\int_\Omega \left(|\nabla\phi_n|^2 - \frac{2\lambda_n f(x)}{(1 - u_n)^3}\phi_n^2 \right) \\ &= \int \left(|\nabla\phi|^2 - \left|\varepsilon_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}|x_n - p_i|^{-\frac{\alpha+2}{2}}y + \frac{x_n - p_i}{|x_n - p_i|}\right|^{\alpha} \frac{2f_i(\varepsilon_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}|x_n - p_i|^{-\frac{\alpha}{2}}y + x_n)}{U_n^3}\phi^2 \right) \\ &\to \int \left(|\nabla\phi|^2 - \frac{2f_i(p_i)}{U^3}\phi^2 \right) < 0 \end{aligned}$$

as $n \to +\infty$. Proposition 4.3 is now completely proved. ∎

3^{rd} **Case** Assume that $x_n \to p_i$ as $n \to +\infty$ and $\varepsilon_n^{-3}\lambda_n|x_n - p_i|^{\alpha+2} \le C$. We rescale the function u_n in a still different way:

$$U_n(y) = \frac{1 - u_n(\varepsilon_n^{\frac{3}{2+\alpha}}\lambda_n^{-\frac{1}{2+\alpha}} y + x_n)}{\varepsilon_n}\,, \qquad y \in \Omega_n = \frac{\Omega - x_n}{\varepsilon_n^{\frac{3}{2+\alpha}}\lambda_n^{-\frac{1}{2+\alpha}}}\,. \tag{4.19}$$

The equation satisfied by U_n is:

$$\begin{cases} \Delta U_n = \left|y + \varepsilon_n^{-\frac{3}{2+\alpha}}\lambda_n^{\frac{1}{2+\alpha}}(x_n - p_i)\right|^\alpha \dfrac{f_i(\varepsilon_n^{\frac{3}{2+\alpha}}\lambda_n^{-\frac{1}{2+\alpha}} y + x_n)}{U_n^2} & \text{in } \Omega_n, \\ U_n(0) = 1 & \text{in } \Omega_n, \end{cases} \tag{4.20}$$

where f_i is defined in (4.14).

In this situation, the result we have is the following:

Proposition 4.4. *Let $x_n \in \Omega$ and set $\varepsilon_n := 1 - u_n(x_n)$. Assume that*

$$\varepsilon_n^3\lambda_n^{-1} \to 0\,,\ \ x_n \to p_i\,,\ \ \varepsilon_n^{-\frac{3}{\alpha+2}}\lambda_n^{\frac{1}{\alpha+2}}(x_n - p_i) \to y_0 \quad \textit{as } n \to +\infty. \tag{4.21}$$

Let U_n, Ω_n be defined as in (4.19). Assume that either (4.10) holds or

$$U_n \ge C\left(\varepsilon_n^{-\frac{3}{\alpha+2}}\lambda_n^{\frac{1}{\alpha+2}}|x_n - p_i|\right)^{-\frac{\alpha}{3}}\left|y + \varepsilon_n^{-\frac{3}{\alpha+2}}\lambda_n^{\frac{1}{\alpha+2}}(x_n - p_i)\right|^{\frac{\alpha}{3}} \quad \textit{in } \Omega_n \cap B_{R_n}(0) \tag{4.22}$$

for some $R_n \to +\infty$ as $n \to +\infty$ and $C > 0$. Then, up to a subsequence, $U_n \to U$ in $C^1_{loc}(\mathbb{R}^N)$ and U satisfies:

$$\begin{cases} \Delta U = |y + y_0|^\alpha \dfrac{f_i(p_i)}{U^2} & \textit{in } \mathbb{R}^N\,, \\ U(y) \ge C > 0 & \textit{in } \mathbb{R}^N\,. \end{cases} \tag{4.23}$$

Moreover, we have that (4.12) holds for some function $\phi_n \in C_0^\infty(\Omega)$ such that Supp $\phi_n \subset B_{M\varepsilon_n^{\frac{3}{2+\alpha}}\lambda_n^{-\frac{1}{2+\alpha}}}(x_n)$, $M > 0$.

Proof: By (4.21) we get that $\Omega_n \to \mathbb{R}^N$ as $n \to +\infty$. If (4.10) holds, as before $U_n \to U$ in $C^1_{\text{loc}}(\mathbb{R}^N)$ and, by (4.10) and (4.20)-(4.21), U solves (4.23).

We need to discuss the non trivial situation when we have the validity of (4.22). Arguing as in the proof of Proposition 3.1, fix $R > 2|y_0|$ and decompose $U_n = U_n^1 + U_n^2$, where U_n^2 satisfies:

$$\begin{cases} \Delta U_n^2 = \Delta U_n & \text{in } B_R(0)\,, \\ U_n^2 = 0 & \text{on } \partial B_R(0)\,. \end{cases}$$

By (4.20) and (4.22) we get that on $B_R(0)$:

$$\begin{aligned} 0 \le \Delta U_n &= \left|y + \varepsilon_n^{-\frac{3}{2+\alpha}}\lambda_n^{\frac{1}{2+\alpha}}(x_n - p_i)\right|^\alpha \frac{f_i(\varepsilon_n^{\frac{3}{2+\alpha}}\lambda_n^{-\frac{1}{2+\alpha}} y + x_n)}{U_n^2} \\ &\le C\left(\varepsilon_n^{-\frac{3}{\alpha+2}}\lambda_n^{\frac{1}{\alpha+2}}|x_n - p_i|\right)^{\frac{2\alpha}{3}}\left|y + \varepsilon_n^{-\frac{3}{2+\alpha}}\lambda_n^{\frac{1}{2+\alpha}}(x_n - p_i)\right|^{\frac{\alpha}{3}}. \end{aligned}$$

Since $\varepsilon_n^{-\frac{3}{\alpha+2}}\lambda_n^{\frac{1}{\alpha+2}}(x_n - p_i)$ is bounded, we get that $0 \le \Delta U_n \le C_R$ on $B_R(0)$ for n large, and then, standard elliptic regularity theory gives that U_n^2 is uniformly bounded in $C^{1,\beta}(B_R(0))$, $\beta \in (0,1)$. Up to a subsequence, we get that $U_n^2 \to U^2$ in $C^1(B_R(0))$. Since by (4.22) $U_n^1 = U_n \geq C(R-2|y_0|)^{\frac{\alpha}{3}} > 0$ on $\partial B_R(0)$, by harmonicity $U_n^1 \geq C_R$ in $B_R(0)$ and, by Harnack inequality:

$$\sup_{B_{R/2}(0)} U_n^1 \leq C_R \inf_{B_{R/2}(0)} U_n^1 \leq C_R U_n^1(0) = C_R\left(1 - U_n^2(0)\right) \leq C_R\left(1 + \sup_{n\in\mathbb{N}} |U_n^2(0)|\right) < \infty\,.$$

Hence, U_n^1 is uniformly bounded in $C^{1,\beta}(B_{R/4}(0))$, $\beta \in (0,1)$. Up to a further subsequence, we get that $U_n^1 \to U^1$ in $C^1(B_{R/4}(0))$ and then, $U_n \to U^1 + U^2$ in $C^1(B_{R/4}(0))$, for any $R > 0$. By a diagonal process and up to a subsequence, by (4.22) we find that $U_n \to U$ in $C^1_{\text{loc}}(\mathbb{R}^N)$, where $U \in C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{-y_0\})$ is a solution of the equation

$$\begin{cases} \Delta U = |y+y_0|^\alpha \dfrac{f_i(p_i)}{U^2} & \text{in } \mathbb{R}^N \setminus \{-y_0\}\,,\\ U(y) \geq C|y+y_0|^{\frac{\alpha}{3}} & \text{in } \mathbb{R}^N\,, \end{cases}$$

for some $C > 0$. In order to prove that U is a solution of (4.23), we need to prove that $U(-y_0) > 0$. Let B some ball so that $-y_0 \in \partial B$ and assume by contradiction that $U(-y_0) = 0$. Since

$$-\Delta U + c(y)U = 0 \text{ in } B\,,\ \ U \in C^2(B) \cap C(\bar{B})\,,\ \ U(y) > U(-y_0) \text{ in } B,$$

and $c(y) = f_i(p_i)\frac{|y+y_0|^\alpha}{U^3} \geq 0$ is a bounded function, by Hopf Lemma we get that $\partial_\nu U(-y_0) < 0$, where ν is the unit outward normal of B at $-y_0$. Hence, U becomes negative in a neighborhood of $-y_0$ in contradiction with the positivity of U. Hence, $U(-y_0) > 0$ and U satisfies (4.23).

Since $1 \leq N \leq 7$ and $f_i(p_i) > 0$, Theorem 1.4 implies $\mu_1(U) < 0$ and the existence of some $\phi \in C_0^\infty(\mathbb{R}^N)$ so that:

$$\int \left(|\nabla\phi|^2 - |y+y_0|^\alpha \frac{2f_i(p_i)}{U^3}\phi^2\right) < 0.$$

Let $\phi_n(x) = \left(\varepsilon_n^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}}\right)^{-\frac{N-2}{2}} \phi\left(\varepsilon_n^{-\frac{3}{2+\alpha}} \lambda_n^{\frac{1}{2+\alpha}}(x - x_n)\right)$. There holds:

$$\begin{aligned} \int_\Omega \left(|\nabla\phi_n|^2 - \frac{2\lambda_n f(x)}{(1-u_n)^3}\phi_n^2\right) &= \int \left(|\nabla\phi|^2 - \left|y + \varepsilon_n^{-\frac{3}{2+\alpha}} \lambda_n^{\frac{1}{2+\alpha}}(x_n - p_i)\right|^\alpha \frac{2f_i(\varepsilon_n^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}} y + x_n)}{U_n^3}\phi^2\right)\\ &\to \int \left(|\nabla\phi|^2 - |y+y_0|^\alpha \frac{2f_i(p_i)}{U^3}\phi^2\right) < 0 \end{aligned}$$

as $n \to +\infty$. Also Proposition 4.4 is established. ■

4.2 Spectral confinement

Let us assume now the validity of (1.6), namely $\mu_{2,n} := \mu_{2,\lambda_n}(u_n) \geq 0$ for any $n \in \mathbb{N}$. This information will play a crucial role in controlling the number k of "blow up points" (for $(1-u_n)^{-1}$) in terms of the spectral information on u_n. Indeed, roughly speaking, we can estimate k with the number of negative eigenvalues of L_{u_n,λ_n} (with multiplicities). In particular, assumption (1.6) implies that "blow up" can occur only along the sequence x_n of maximum points of u_n in Ω. The following pointwise estimate on u_n is available:

Proposition 4.5. *Assume $2 \leq N \leq 7$. Let $f \in C(\bar{\Omega})$ be as in (1.5). Let $\lambda_n \to \lambda \in [0, \lambda^*]$ and u_n be an associated solution. Assume that $u_n(x_n) = \max_\Omega u_n \to 1$ as $n \to +\infty$. Then, there exist constants $C > 0$ and $N_0 \in \mathbb{N}$ such that*

$$\left(1 - u_n(x)\right) \geq C\lambda_n^{\frac{1}{3}} d(x)^{\frac{\alpha}{3}} |x - x_n|^{\frac{2}{3}}\,, \quad \forall\, x \in \Omega\,,\ n \geq N_0\,, \tag{4.24}$$

where $d(x) = \min\{|x - p_i| : i = 1, \ldots, k\}$ is the distance function from the zero set of $f(x)$ $\{p_1, \ldots, p_k\}$.

Proof: Let $\varepsilon_n = 1 - u_n(x_n)$. Then, $\varepsilon_n \to 0$ as $n \to +\infty$ and, even more precisely:

$$\varepsilon_n^2 \lambda_n^{-1} \to 0 \quad \text{as } n \to +\infty. \tag{4.25}$$

Indeed otherwise, we would have along some subsequence:

$$0 \le \frac{\lambda_n f(x)}{(1-u_n)^2} \le \frac{\lambda_n}{\varepsilon_n^2} \parallel f \parallel_\infty \le C\ ,\ \ \lambda_n \to 0 \text{ as } n \to +\infty.$$

But if the right hand side of $(S)_{\lambda_n}$ is uniformly bounded, from elliptic regularity theory we get that u_n is uniformly bounded in $C^{1,\beta}(\bar\Omega)$, $\beta \in (0,1)$. Hence, up to a furhter subsequence, $u_n \to u$ in $C^1(\bar\Omega)$, where u is an harmonic function such that $u = 0$ on $\partial\Omega$, $\max\limits_{\Omega} u = 1$. A contradiction.

By (4.25) we get that $\varepsilon_n^3 \lambda_n^{-1} \to 0$ as $n \to +\infty$, as needed in (4.9), (4.21) respectively in Proposition 4.2, 4.4. Now, depending on the case corresponding to the blow up sequence x_n, we can apply one among Propositions 4.2-4.4 to get the existence of a function $\phi_n \in C_0^\infty(\Omega)$ such that (4.12) holds and with a specific control on Supp ϕ_n.

By contradiction, assume now that (4.24) is false: up to a subsequence, there exist a sequence $y_n \in \Omega$ such that

$$\lambda_n^{-\frac13} d(y_n)^{-\frac{\alpha}{3}} |y_n - x_n|^{-\frac23} \big(1 - u_n(y_n)\big) = \lambda_n^{-\frac13} \min_{x\in\Omega} \Big(d(x)^{-\frac{\alpha}{3}} |x - x_n|^{-\frac23} \left(1 - u_n(x)\right) \Big) \to 0 \quad as \quad n \to \infty\,. \tag{4.26}$$

Then, $\mu_n := 1 - u_n(y_n) \to 0$ as $n \to \infty$ and (4.26) rewrites as:

$$\frac{\mu_n^{\frac32} \lambda_n^{-\frac12}}{|x_n - y_n|\, d(y_n)^{\frac{\alpha}{2}}} \to 0 \quad \text{as } n \to +\infty. \tag{4.27}$$

We want now to explain the meaning of the crucial choice (4.26). Let β_n be a sequence of positive numbers so that

$$R_n := \beta_n^{-\frac12} \min\{d(y_n)^{\frac12}, |x_n - y_n|^{\frac12}\} \to +\infty \quad \text{as } n \to +\infty. \tag{4.28}$$

Let us introduce the following rescaled function:

$$\hat U_n(y) = \frac{1 - u_n(\beta_n y + y_n)}{\mu_n}\ ,\ \ y \in \hat\Omega_n = \frac{\Omega - y_n}{\beta_n}.$$

Formula (4.26) implies:

$$\begin{aligned} \mu_n &= d(y_n)^{\frac{\alpha}{3}} |y_n - x_n|^{\frac23} \min_{x\in\Omega} \Big(d(x)^{-\frac{\alpha}{3}} |x - x_n|^{-\frac23} \left(1 - u_n(x)\right) \Big) \\ &\le \mu_n d(y_n)^{\frac{\alpha}{3}} |y_n - x_n|^{\frac23} d(\beta_n y + y_n)^{-\frac{\alpha}{3}} |\beta_n y + y_n - x_n|^{-\frac23} \hat U_n(y). \end{aligned}$$

Since

$$\frac{d(\beta_n y + y_n)}{d(y_n)} = \min\{\Big|\frac{y_n - p_i}{d(y_n)} + \frac{\beta_n}{d(y_n)} y\Big| :\ i = 1, \dots, k\} \ge 1 - \frac{\beta_n}{d(y_n)} |y|$$

in view of $|y_n - p_i| \ge d(y_n)$, by (4.28) we get that:

$$\hat U_n(y) \ge \left(1 - \frac{\beta_n R_n}{d(y_n)}\right)^{\frac{\alpha}{3}} \left(1 - \frac{\beta_n R_n}{|x_n - y_n|}\right)^{\frac23} \ge \big(\frac12\big)^{\frac{2+\alpha}{3}}$$

for any $y \in \hat\Omega_n \cap B_{R_n}(0)$. Hence, whenever (4.28) holds, we get the validity of (4.10) for the rescaled function $\hat U_n$ at y_n with respect to β_n.

We need to discuss all the possible types of blow up at y_n.

1^{st} **Case** Assume that $y_n \to q \notin \{p_1, \ldots, p_k\}$. By (4.27) we get that $\mu_n^3 \lambda_n^{-1} \to 0$ as $n \to +\infty$. Since $d(y_n) \geq C > 0$, let $\beta_n = \mu_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}}$ and, by (4.27) we get that (4.28) holds. Associated to y_n, μ_n, define $\hat{U}_n$, $\hat{\Omega}_n$ as in (4.7). We have that (4.10) holds by the validity of (4.28) for our choice of β_n. Hence, Proposition 4.2 applies to $\hat{U}_n$ and give the existence of $\psi_n \in C_0^\infty(\Omega)$ such that (4.12) holds and Supp $\psi_n \subset B_{M\mu_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}}(y_n)$, $M > 0$. In the worst case $x_n \to q$, given U_n be as in (4.7) associated to x_n, ε_n, we get by scaling that for $x = \varepsilon_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}} y + x_n$:

$$\lambda_n^{-\frac{1}{3}} \big(d(x)^{-\frac{\alpha}{3}} |x - x_n|^{-\frac{2}{3}} (1 - u_n(x)) \big) \geq C \lambda_n^{-\frac{1}{3}} \big(|x - x_n|^{-\frac{2}{3}} (1 - u_n(x)) \big) = C|y|^{-\frac{2}{3}} U_n(y) \geq C_R > 0$$

uniformly in n and $y \in B_R(0)$, for any $R > 0$. Then,

$$\frac{\varepsilon_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}}}{|x_n - y_n|} \to 0 \quad \text{as } n \to +\infty.$$

Hence, in this situation ϕ_n and ψ_n have disjoint compact supports and obviously, it remains true when $x_n \to p \neq q$. Hence, $\mu_{2,n} < 0$ in contradiction with (1.6).

2^{nd} **Case** Assume that $y_n \to p_i$ in a "slow" way:

$$\mu_n^{-3} \lambda_n |y_n - p_i|^{\alpha+2} \to +\infty \quad \text{as } n \to +\infty.$$

Let now $\beta_n = \mu_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}} |y_n - p_i|^{-\frac{\alpha}{2}}$. Since $d(y_n) = |y_n - p_i|$ in this situation, we get that:

$$\frac{d(y_n)}{\beta_n} = \mu_n^{-\frac{3}{2}} \lambda_n^{\frac{1}{2}} |y_n - p_i|^{\frac{\alpha+2}{2}} \to +\infty,$$

and (4.27) gives exactly:

$$\frac{|x_n - y_n|}{\beta_n} = \frac{|x_n - y_n|}{\mu_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}} |y_n - p_i|^{-\frac{\alpha}{2}}} \to +\infty \tag{4.29}$$

as $n \to +\infty$. Hence, (4.28) holds. Associated to μ_n, y_n, define now $\hat{U}_n$, $\hat{\Omega}_n$ according to (4.15). Since (4.10) follows by (4.28), Proposition 4.3 for $\hat{U}_n$ gives some $\psi_n \in C_0^\infty(\Omega)$ such that (4.12) holds and Supp $\psi_n \subset B_{M\mu_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}|y_n - p_i|^{-\frac{\alpha}{2}}}(y_n)$, $M > 0$. If $x_n \to p \neq p_i$, then clearly ϕ_n, ψ_n have disjoint compact supports leading to $\mu_{2,n} < 0$ in contradiction with (1.6). If also $x_n \to p_i$, we can easily show by scaling that:

1) if $\varepsilon_n^{-3} \lambda_n |x_n - p_i|^{\alpha+2} \to +\infty$ as $n \to +\infty$, given U_n be as in (4.15) associated to x_n, ε_n, we get that for $x = \varepsilon_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}} |x_n - p_i|^{-\frac{\alpha}{2}} y + x_n$

$$\lambda_n^{-\frac{1}{3}} \big(d(x)^{-\frac{\alpha}{3}} |x - x_n|^{-\frac{2}{3}} (1 - u_n(x)) \big) = |y|^{-\frac{2}{3}} U_n(y) \Big| \varepsilon_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}} |x_n - p_i|^{-\frac{\alpha+2}{2}} y + \frac{x_n - p_i}{|x_n - p_i|} \Big|^{-\frac{\alpha}{3}} \geq C_R > 0$$

uniformly in n and $y \in B_R(0)$, for any $R > 0$. Then,

$$\frac{\varepsilon_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}} |x_n - p_i|^{-\frac{\alpha}{2}}}{|x_n - y_n|} \to 0 \quad \text{as } n \to +\infty,$$

and hence, by (4.29) ϕ_n, ψ_n have disjoint compact supports leading to $\mu_{2,n} < 0$ in contradiction with (1.6).

2) if $\varepsilon_n^{-3} \lambda_n |x_n - p_i|^{\alpha+2} \leq C$ as $n \to +\infty$, given U_n be as in (4.19) associated to x_n, ε_n, we get that for $x = \varepsilon_n^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}} y + x_n$

$$\begin{aligned} \lambda_n^{-\frac{1}{3}} \big(d(x)^{-\frac{\alpha}{3}} |x - x_n|^{-\frac{2}{3}} (1 - u_n(x)) \big) &= |y|^{-\frac{2}{3}} U_n(y) \Big| y + \varepsilon_n^{-\frac{3}{2+\alpha}} \lambda_n^{\frac{1}{2+\alpha}} (x_n - p_i) \Big|^{-\frac{\alpha}{3}} \\ &\geq D_R |y|^{-\frac{2}{3}} U_n(y) \geq C_R > 0 \end{aligned}$$

uniformly in n and $y \in B_R(0)$, for any $R > 0$. Then,

$$\frac{\varepsilon_n^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}}}{|x_n - y_n|} \to 0 \quad \text{as } n \to +\infty,$$

and hence, by (4.29) ϕ_n, ψ_n have disjoint compact supports leading to a contradiction.

$3^{\mathbf{rd}}$ **Case** Assume that $y_n \to p_i$ in a "fast" way:

$$\mu_n^{-3} \lambda_n |y_n - p_i|^{\alpha+2} \le C.$$

Since $d(y_n) = |y_n - p_i|$, by (4.27) we get that

$$\frac{|y_n - p_i|}{|x_n - y_n|} = \frac{\mu_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}}}{|x_n - y_n|\, |y_n - p_i|^{\frac{\alpha}{2}}} \left(\mu_n^{-3} \lambda_n |y_n - p_i|^{\alpha+2}\right)^{\frac{1}{2}} \to 0 \quad \text{as } n \to +\infty, \tag{4.30}$$

and then, for n large:

$$\frac{|x_n - p_i|}{|y_n - p_i|} \ge \frac{|x_n - y_n|}{|y_n - p_i|} - 1 \ge 1 \ , \quad \frac{|x_n - p_i|}{|x_n - y_n|} \ge 1 - \frac{|y_n - p_i|}{|x_n - y_n|} \ge \frac{1}{2}. \tag{4.31}$$

Since $\varepsilon_n \le \mu_n$, by (4.27) and (4.31) we get that

$$\begin{aligned} \varepsilon_n^{-3} \lambda_n |x_n - p_i|^{\alpha+2} &\ge \left(\frac{\mu_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}}}{|x_n - y_n|\, |y_n - p_i|^{\frac{\alpha}{2}}} \right)^{-2} \left(\frac{|x_n - p_i|}{|x_n - y_n|^{\frac{2}{\alpha+2}}\, |y_n - p_i|^{\frac{\alpha}{\alpha+2}}} \right)^{\alpha+2} \\ &\ge C \left(\frac{\mu_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}}}{|x_n - y_n|\, d(y_n)^{\frac{\alpha}{2}}} \right)^{-2} \to +\infty \quad \text{as } n \to +\infty. \end{aligned} \tag{4.32}$$

The meaning of (4.32) is the following: once y_n provides a fast blowing up sequence at p_i, then no other fast blow up at p_i can occurr as (4.32) states for x_n.

Let $\beta_n = \mu_n^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}}$. By (4.27) and (4.30) we get that

$$\frac{\beta_n}{|x_n - y_n|} = \mu_n^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}} |x_n - y_n|^{-1} = \left(\frac{\mu_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}}}{|x_n - y_n|\, d(y_n)^{\frac{\alpha}{2}}} \right)^{\frac{2}{2+\alpha}} \left(\frac{|y_n - p_i|}{|x_n - y_n|} \right)^{\frac{\alpha}{2+\alpha}} \to 0 \quad \text{as } n \to +\infty. \tag{4.33}$$

However, since u_n blows up fast at p_i along y_n, we have that $\beta_n^{-1} d(y_n) \le C$ and then, (4.28) does not hold. Letting as before

$$\hat{U}_n(y) = \frac{1 - u_n(\beta_n y + y_n)}{\mu_n} \ , \ \ y \in \hat{\Omega}_n = \frac{\Omega - y_n}{\beta_n},$$

we need to refine the analysis before in order to get some estimate for $\hat{U}_n$ even when only (4.33) does hold. Formula (4.26) gives that:

$$\begin{aligned} \hat{U}_n(y) &\ge |y_n - p_i|^{-\frac{\alpha}{3}} |y_n - x_n|^{-\frac{2}{3}} |\beta_n y + y_n - p_i|^{\frac{\alpha}{3}} |\beta_n y + y_n - x_n|^{\frac{2}{3}} \\ &= \left(\mu_n^{-\frac{3}{2+\alpha}} \lambda_n^{\frac{1}{2+\alpha}} |y_n - p_i| \right)^{-\frac{\alpha}{3}} |y + \mu_n^{-\frac{3}{2+\alpha}} \lambda_n^{\frac{1}{2+\alpha}} (y_n - p_i)|^{\frac{\alpha}{3}} \left|\frac{\beta_n}{|x_n - y_n|} y + \frac{y_n - x_n}{|x_n - y_n|}\right|^{\frac{2}{3}} \\ &\ge C \left(\mu_n^{-\frac{3}{2+\alpha}} \lambda_n^{\frac{1}{2+\alpha}} |y_n - p_i| \right)^{-\frac{\alpha}{3}} |y + \mu_n^{-\frac{3}{2+\alpha}} \lambda_n^{\frac{1}{2+\alpha}} (y_n - p_i)|^{\frac{\alpha}{3}} \end{aligned} \tag{4.34}$$

for $|y| \le R_n = (\frac{|x_n - y_n|}{\beta_n})^{\frac{1}{2}}$, and $R_n \to +\infty$ as $n \to +\infty$ by (4.33). Since (4.34) implies that (4.22) holds for μ_n, y_n, $\hat{U}_n$, Proposition 4.4 provides some $\psi_n \in C_0^\infty(\Omega)$ such that (4.12) holds and Supp $\psi_n \subset B_{M\mu_n^{\frac{3}{2+\alpha}} \lambda_n^{-\frac{1}{2+\alpha}}}(y_n)$, $M > 0$.

Since y_n cannot lie in any ball centered at x_n and radius of order of the scale parameter ($\varepsilon_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}}$ or $\varepsilon_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}} |x_n - p_i|^{-\frac{\alpha}{2}}$), by (4.33) we get that ϕ_n and ψ_n have disjoint compact supports leading to $\mu_{2,n} < 0$. A contradiction to (1.6). The proof of the Proposition is now complete. ■

4.3 Compactness issues

We are now in position to give the proof of Theorem 1.3. Assume $2 \le N \le 7$. Let $f \in C(\bar{\Omega})$ be as in (1.5). Let $(\lambda_n)_n$ be a sequence such that $\lambda_n \to \lambda \in [0, \lambda^*]$ and let u_n be an associated solution such that (1.6) holds, namely

$$\mu_{2,n} := \mu_{2,\lambda_n}(u_n) \ge 0.$$

The essential ingredient will be the estimate of Proposition 4.5 combined with the uniqueness result of Proposition 7.3.

Proof (of Theorem 1.3): Let x_n be the maximum point of u_n in Ω and, up to a subsequence, assume by contradiction that $u_n(x_n) = \max_\Omega u_n(x) \to 1$ as $n \to \infty$. Proposition 4.5 gives that:

$$u_n(x) \le 1 - C\lambda_n^{\frac{1}{3}} d(x)^{\frac{\alpha}{3}} |x - x_n|^{\frac{2}{3}}$$

for any $x \in \Omega$ and $n \ge N_0$, for some $C > 0$ and $N_0 \in \mathbb{N}$ large. Here, $d(x) = \min\{|x - p_i| : i = 1, \dots, k\}$ stands for the distance function from the zero set of $f(x)$. Thus, we have that:

$$0 \le \frac{\lambda_n f(x)}{(1 - u_n)^2} \le C \frac{f(x)}{d(x)^{\frac{2\alpha}{3}}} \frac{\lambda_n^{\frac{1}{3}}}{|x - x_n|^{\frac{4}{3}}} \tag{4.35}$$

for any $x \in \Omega$ and $n \ge N_0$. Since by (1.5)

$$|\frac{f(x)}{d(x)^{\frac{2\alpha}{3}}}| \le |x - p_i|^{\frac{\alpha}{3}} \|f_i\|_\infty \le C$$

for x close to p_i, f_i as in (4.14), we get that $\frac{f(x)}{d(x)^{\frac{2\alpha}{3}}}$ is a bounded function on Ω and then, by (4.35) $\lambda_n f(x)/(1 - u_n)^2$ is uniformly bounded in $L^s(\Omega)$, for any $1 < s < \frac{3N}{4}$. Standard elliptic regularity theory now implies that u_n is uniformly bounded in $W^{2,s}(\Omega)$. By Sobolev's imbedding theorem, u_n is uniformly bounded in $C^{0,\beta}(\bar{\Omega})$ for any $0 < \beta < 2/3$. Up to a subsequence, we get that $u_n \to u_0$ weakly in $H_0^1(\Omega)$ and strongly in $C^{0,\beta}(\bar{\Omega})$, $0 < \beta < 2/3$, where u_0 is an Hölderian function solving weakly in $H_0^1(\Omega)$ the equation:

$$\begin{cases} -\Delta u_0 = \dfrac{\lambda f(x)}{(1 - u_0)^2} & \text{in } \Omega\,, \\ 0 \le u_0 \le 1 & \text{in } \Omega\,, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.36}$$

Moreover, by uniform convergence

$$\max_\Omega u_0 = \lim_{n \to +\infty} \max_\Omega u_n = 1$$

and, in particular $u_0 > 0$ in Ω. Clearly, $\lambda > 0$ since any weak harmonic function in $H_0^1(\Omega)$ is identically zero. To reach a contradiction, we shall first show that $\mu_{1,\lambda}(u_0) \ge 0$ and then deduce from the uniqueness, stated in Proposition 7.3, of the semi-stable solution u_λ that $u_0 = u_\lambda$. But $\max_\Omega u_\lambda < 1$ for any $\lambda \in [0, \lambda^*]$, contradicting $\max_\Omega u_0 = 1$. Hence, the claimed compactness must hold.

In addition to (1.6), assume now that $\mu_{1,n} < 0$, then $\lambda > 0$. Indeed, if $\lambda_n \to 0$, then by compactness and standard regularity theory, we get that $u_n \to u_0$ in $C^2(\bar\Omega)$, where u_0 is an harmonic function so that $u_0 = 0$ on $\partial\Omega$. Then, $u_0 = 0$ and $u_n \to 0$ in $C^2(\bar\Omega)$. But the only branch of solutions for $(S)_\lambda$ bifurcating from 0 for λ small is the branch of minimal solutions u_λ and then, $u_n = u_{\lambda_n}$ for n large contradicting $\mu_{1,n} < 0$.

In order to complete the proof, we need only to show that

$$\mu_{1,\lambda}(u_0) = \inf\left\{\int_\Omega \left(|\nabla\phi|^2 - \frac{2\lambda f(x)}{(1-u_0)^3}\phi^2\right); \phi \in C_0^\infty(\Omega) \text{ and } \int_\Omega \phi^2 = 1\right\} \geq 0. \tag{4.37}$$

Indeed, first by Propositions 4.2-4.4 we get the existence of a function $\phi_n \in C_0^\infty(\Omega)$ so that

$$\int_\Omega \left(|\nabla\phi_n|^2 - \frac{2\lambda_n f(x)}{(1-u_n)^3}\phi_n^2\right) < 0. \tag{4.38}$$

Moreover, Supp $\phi_n \subset B_{r_n}(x_n)$ and $r_n \to 0$ as $n \to +\infty$. Up to a subsequence, assume that $x_n \to p \in \bar\Omega$ as $n \to +\infty$.

By contradiction, if (4.37) were false, then there exists $\phi_0 \in C_0^\infty(\Omega)$ such that

$$\int_\Omega \left(|\nabla\phi_0|^2 - \frac{2\lambda f(x)}{(1-u_0)^3}\phi_0^2\right) < 0. \tag{4.39}$$

We will replace ϕ_0 with a truncated function ϕ_δ with $\delta > 0$ small enough, and so that (4.39) is still true while $\phi_\delta = 0$ in $B_{\delta^2}(p) \cap \Omega$. In this way, ϕ_n and ϕ_δ would have disjoint compact supports in contradiction to $\mu_{2,n} \geq 0$.

Let $\delta > 0$ and set $\phi_\delta = \chi_\delta\phi_0$, where χ_δ is a cut-off function defined as:

$$\chi_\delta(x) = \begin{cases} 0 & |x-p| \leq \delta^2\,, \\ 2 - \dfrac{\log|x-p|}{\log\delta} & \delta^2 \leq |x-p| \leq \delta\,, \\ 1 & |x-p| \geq \delta\,. \end{cases}$$

By Lebesgue's theorem, we have:

$$\int_\Omega \frac{2\lambda f(x)}{(1-u_0)^3}\phi_\delta^2 \to \int_\Omega \frac{2\lambda f(x)}{(1-u_0)^3}\phi_0^2\,, \quad \text{as } \delta \to 0\,. \tag{4.40}$$

For the gradient term, we have the expansion:

$$\int_\Omega |\nabla\phi_\delta|^2 = \int_\Omega \phi_0^2|\nabla\chi_\delta|^2 + \int_\Omega \chi_\delta^2|\nabla\phi_0|^2 + 2\int_\Omega \chi_\delta\phi_0\nabla\chi_\delta\nabla\phi_0\,.$$

The following estimates hold:

$$0 \leq \int_\Omega \phi_0^2|\nabla\chi_\delta|^2 \leq \|\phi_0\|_\infty^2 \int_{\delta^2\leq|x-p|\leq\delta} \frac{1}{|x-p|^2\log^2\delta} \leq \frac{C}{\log\frac{1}{\delta}}$$

and

$$\left|2\int_\Omega \chi_\delta\phi_0\nabla\chi_\delta\nabla\phi_0\right| \le \frac{2\|\phi_0\|_\infty\|\nabla\phi_0\|_\infty}{\log\frac{1}{\delta}}\int_{B_1(0)}\frac{1}{|x|},$$

and provide:

$$\int_\Omega |\nabla\phi_\delta|^2 \to \int_\Omega |\nabla\phi_0|^2 \quad \text{as } \delta\to 0. \tag{4.41}$$

Combining (4.39)-(4.41), we get that:

$$\int_\Omega \left(|\nabla\phi_\delta|^2 - \frac{2\lambda f(x)}{(1-u_0)^3}\phi_\delta^2\right) < 0$$

for $\delta>0$ sufficiently small. This completes the proof of (4.37) and Theorem 1.3 is completely established.

5 The one dimensional problem

Let $I=(a,b)$ be a bounded interval in $\mathbb{R}$. Assume $f\in C^1(\bar I)$ so that $f\ge C>0$ in I. In Theorem 1.6 we study solutions u_n of the following problem:

$$\begin{cases} -\ddot u_n = \dfrac{\lambda_n f(x)}{(1-u_n)^2} & \text{in } I\,,\\ 0<u_n<1 & \text{in } I\,,\\ u_n(a)=u_n(b)=0. \end{cases} \tag{5.1}$$

Proof (of Theorem 1.6): Assume that u_n satisfy (1.9) and $\lambda_n\to\lambda\in(0,\lambda^*]$. Let $x_n\in I$ be a maximum point: $u_n(x_n)=\max\limits_I u_n$. If $(u_n)_n$ is not compact, then up to a subsequence, we may assume that $u_n(x_n)\to 1$ with $x_n\to x_0\in\bar I$ as $n\to+\infty$. Away from x_0, u_n is uniformly far away from 1. Otherwise, by the maximum principle we would have $u_n\to 1$ on an interval of positive measure, and then $\mu_{k,\lambda_n}(u_n)<0$, for any k and n large. A contradiction.

Assume, for example, that $a\le x_0<b$. By elliptic regularity theory, $\dot u_n$ is uniformly bounded far away from x_0. Let $\varepsilon>0$. We multiply (5.1) by $\dot u_n$ and integrate on $(x_n,x_0+\varepsilon)$:

$$\dot u_n^2(x_n)-\dot u_n^2(x_0+\varepsilon)=\int_{x_n}^{x_0+\varepsilon}\frac{2\lambda_n f(s)\dot u_n(s)}{(1-u_n(s))^2}ds=\frac{2\lambda_n f(x_0+\varepsilon)}{1-u_n(x_0+\varepsilon)}-\frac{2\lambda_n f(x_n)}{1-u_n(x_n)}-\int_{x_n}^{x_0+\varepsilon}\frac{2\lambda_n\dot f(s)}{1-u_n(s)}ds.$$

Then, for n large:

$$\begin{aligned}\dot u_n^2(x_n)+\frac{C\lambda}{1-u_n(x_n)} &\le \dot u_n^2(x_0+\varepsilon)+2\lambda_n\frac{f(x_0+\varepsilon)}{1-u_n(x_0+\varepsilon)}-2\lambda_n\int_{x_n}^{x_0+\varepsilon}\frac{\dot f(s)}{1-u_n(s)}ds\\ &\le C_\varepsilon+4\lambda\parallel \dot f\parallel_\infty\frac{x_0+\varepsilon-x_n}{1-u_n(x_n)}\end{aligned}$$

since $u_n(x_n)$ is the maximum value of u_n in I. Choosing $\varepsilon>0$ sufficiently small, we get that for any n large: $\frac{1}{1-u_n(x_n)}\le C_\varepsilon$, contradicting $u_n(x_n)\to 1$ as $n\to+\infty$. ∎

6 The second bifurcation point and the branch of unstable solutions

We now establish Theorem 1.5. First, let us recall the definition of λ_2^*:

$\lambda_2^* = \inf\{\beta > 0 : \exists$ a curve$V_\lambda \in C([\beta, \lambda^*]; C^2(\Omega))$ of solutions to $(S)_\lambda$ s.t. $\mu_{2,\lambda}(V_\lambda) \geq 0, V_\lambda \equiv U_\lambda \forall \lambda \in (\lambda^* - \delta, \lambda^*)\}$.

As for as Theorem 1.5 is concerned, for any $\lambda \in (\lambda_2^*, \lambda^*)$ by definition there exists a solution V_λ and it is such that:

$$\mu_{1,\lambda} := \mu_{1,\lambda}(V_\lambda) < 0 \quad \forall\, \lambda \in (\lambda_2^*, \lambda^*). \tag{6.1}$$

In particular, $V_\lambda \neq u_\lambda$ provides a second solution different from the minimal one.

Clearly (6.1) is true because first $\mu_{1,\lambda} < 0$ for λ close to λ^*. Moreover, if $\mu_{1,\lambda} = 0$ for some $\lambda \in (\lambda_2^*, \lambda^*)$, then by Proposition 7.3 $V_\lambda = u_\lambda$ contradicting the fact that $\mu_{1,\lambda}(u_\lambda) > 0$ for any $0 < \lambda < \lambda^*$.

Since by definition $\mu_{2,\lambda}(V_\lambda) \geq 0$ for any $\lambda \in (\lambda_2^*, \lambda^*)$, we can take a sequence $\lambda_n \downarrow \lambda_2^*$ and apply Theorem 1.3 to get that $\lambda_2^* = \lim\limits_{n\to+\infty} \lambda_n > 0$, $\sup\limits_{n\in\mathbb{N}} \| V_{\lambda_n} \|_\infty < 1$. By elliptic regularity theory, up to a subsequence $V_{\lambda_n} \to V^*$ in $C^2(\bar\Omega)$, where V^* is a solution for $(S)_{\lambda_2^*}$. As before, $\mu_{1,\lambda_2^*}(V^*) < 0$ and by continuity $\mu_{2,\lambda_2^*}(V^*) \geq 0$.

If $\mu_{2,\lambda_2^*}(V^*) > 0$, let us fix some $\varepsilon > 0$ small so that $0 \leq V^* \leq 1 - 2\varepsilon$ and consider the truncated nonlinearity $g_\varepsilon(u)$ as in (2.1). Clearly, V^* is a solution of (2.2) at $\lambda = \lambda_2^*$ so that $-\Delta - \lambda_2^* f(x) g_\varepsilon'(V^*)$ has no zero eigenvalues. Namely, V^* solves $N(\lambda_2^*, V^*) = 0$, where N is a map from $\mathbb{R} \times C^{2,\alpha}(\bar\Omega)$ into $C^{2,\alpha}(\bar\Omega)$, $\alpha \in (0,1)$, defined as:

$$N : (\lambda, V) \;\longrightarrow\; V + \Delta^{-1}\left(\lambda f(x) g_\varepsilon(V)\right).$$

Moreover,

$$\partial_V N(\lambda_2^*, V^*) = \text{Id } + \Delta^{-1}\left(\frac{2\lambda_2^* f(x)}{(1-V^*)^3}\right)$$

is an invertible map since $-\Delta - \lambda_2^* f(x) g_\varepsilon'(V^*)$ has no zero eigenvalues. The Implicit Function Theorem gives the existence of a curve W_λ, $\lambda \in (\lambda_2^* - \delta, \lambda_2^* + \delta)$, of solution for (2.2) so that $\lim_{\lambda\to\lambda_2^*} W_\lambda = V^*$ in $C^{2,\alpha}(\bar\Omega)$. Up to take δ smaller, this convergence implies that $\mu_{2,\lambda}(W_\lambda) > 0$ and $W_\lambda \leq 1 - \varepsilon$ for any $\lambda \in (\lambda_2^* - \delta, \lambda_2^* + \delta)$. Hence, W_λ is a solution of $(S)_\lambda$ so that $\mu_{2,\lambda}(W_\lambda) > 0$ contradicting the definition of λ_2^*. Hence, $\mu_{2,\lambda_2^*}(V^*) = 0$. A similar argument works for the radial problem $(S)_\lambda$ on the unit ball and $f(x)$ as in (1.3), provided either $2 \leq N \leq 7$ or $N \geq 8$, $\alpha > \alpha_N$. The proof of Theorem 1.5 is complete.

7 Appendix

We shall prove here the following Theorem already announced in the Introduction.

Theorem 7.1. *Assume either* $1 \leq N \leq 7$ *or* $N \geq 8$, $\alpha > \alpha_N$. *Let* U *be a solution of*

$$\begin{cases} \Delta U = \dfrac{|y|^\alpha}{U^2} & \text{in } \mathbb{R}^N, \\ U(y) \geq C > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{7.1}$$

Then,

$$\mu_1(U) = \inf\left\{\int_{\mathbb{R}^N} \left(|\nabla\phi|^2 - \frac{2|y|^\alpha}{U^3}\phi^2\right);\, \phi \in C_0^\infty(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} \phi^2 = 1\right\} < 0\,. \tag{7.2}$$

Moreover, if $N \geq 8$ *and* $0 \leq \alpha \leq \alpha_N$, *then there exists at least a solution* U *of (7.1) such that* $\mu_1(U) \geq 0$.

Proof: By contradiction, assume that

$$\mu_1(U)=\inf\left\{\int_{\mathbb{R}^N}\left(|\nabla\phi|^2-\frac{2|y|^\alpha}{U^3}\phi^2\right);\,\phi\in C_0^\infty(\mathbb{R}^N)\text{ and }\int_{\mathbb{R}^N}\phi^2\,dx=1\right\}\geq 0.$$

By the density of $C_0^\infty(\mathbb{R}^N)$ in $D^{1,2}(\mathbb{R}^N)$, we have that

$$\int|\nabla\phi|^2\geq 2\int\frac{|y|^\alpha}{U^3}\phi^2\,,\quad\forall\,\phi\in D^{1,2}(\mathbb{R}^N)\,. \tag{7.3}$$

In particular, the test function $\phi=\frac{1}{(1+|y|^2)^{\frac{N-2}{4}+\frac{\delta}{2}}}\in D^{1,2}(\mathbb{R}^N)$ applied in (7.3) gives that

$$\int\frac{|y|^\alpha}{(1+|y|^2)^{\frac{N-2}{2}+\delta}U^3}\leq C\int\frac{1}{(1+|y|^2)^{\frac{N}{2}+\delta}}<+\infty\,, \tag{7.4}$$

for any $\delta>0$.

Step 1. We want to show that (7.3) allows us to perform the following Moser-type iteration scheme: for any $0<q<4+2\sqrt{6}$ and β there holds

$$\int\frac{1}{(1+|y|^2)^{\beta-1-\frac{\alpha}{2}}U^{q+3}}\leq C_q\left(1+\int\frac{1}{(1+|y|^2)^\beta U^q}\right) \tag{7.5}$$

(provided the second integral is finite).

Indeed, let $R>0$ and consider a smooth radial cut-off function η so that: $0\leq\eta\leq 1$, $\eta=1$ in $B_R(0)$, $\eta=0$ in $\mathbb{R}^N\setminus B_{2R}(0)$. Multiplying (7.1) by $\dfrac{\eta^2}{(1+|y|^2)^{\beta-1}U^{q+1}}$, $q>0$, and integrating by parts we get:

$$\begin{aligned}\int\frac{|y|^\alpha\eta^2}{(1+|y|^2)^{\beta-1}U^{q+3}} &= \frac{4(q+1)}{q^2}\int\Big|\nabla\Big(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}U^{\frac{q}{2}}}\Big)\Big|^2-\frac{4(q+1)}{q^2}\int\frac{1}{U^q}\Big|\nabla\Big(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}}\Big)\Big|^2\\ &\quad-\frac{q+2}{q^2}\int\nabla(\frac{1}{U^q})\nabla\Big(\frac{\eta^2}{(1+|y|^2)^{\beta-1}}\Big)\\ &= \frac{4(q+1)}{q^2}\int\Big|\nabla\Big(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}U^{\frac{q}{2}}}\Big)\Big|^2-\frac{2}{q}\int\frac{1}{U^q}\Big|\nabla\Big(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}}\Big)\Big|^2\\ &\quad+\frac{2(q+2)}{q^2}\int\frac{1}{U^q}\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}}\Delta\Big(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}}\Big)\,,\end{aligned}$$

by means of the relation: $\Delta(\psi)^2=2|\nabla\psi|^2+2\psi\Delta\psi$.

Then, by (7.3) we deduce that

$$(8q+8-q^2)\int\frac{|y|^\alpha\eta^2}{(1+|y|^2)^{\beta-1}U^{q+3}}\leq C'_q\int\frac{1}{U^q}\Big(\big|\nabla(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}})\big|^2+\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}}\big|\Delta(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}})\big|\Big).$$

Assuming that $|\nabla\eta|\leq\frac{C}{R}$ and $|\Delta\eta|\leq\frac{C}{R^2}$, it is straightforward to see that:

$$\big|\nabla(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}})\big|^2+\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}}\big|\Delta(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}})\big|\leq C\Big(\frac{1}{(1+|y|^2)^\beta}+\frac{1}{R^2(1+|y|^2)^{\beta-1}}\chi_{B_{2R}(0)\setminus B_R(0)}\Big]$$

for some constant C independent on $R > 0$. Then,

$$(8q + 8 - q^2) \int \frac{|y|^\alpha \eta^2}{(1+|y|^2)^{\beta-1} U^{q+3}} \le C''_q \int \frac{1}{(1+|y|^2)^\beta U^q}.$$

Let $q_+ = 4 + 2\sqrt{6}$. For any $0 < q < q_+$, we have $8q + 8 - q^2 > 0$ and therefore:

$$\int \frac{|y|^\alpha \eta^2}{(1+|y|^2)^{\beta-1} U^{q+3}} \le C_q \int \frac{1}{(1+|y|^2)^\beta U^q}\,,$$

where C_q does not depend on $R > 0$. Taking the limit as $R \to +\infty$, we get that:

$$\int \frac{|y|^\alpha}{(1+|y|^2)^{\beta-1} U^{q+3}} \le C_q \int \frac{1}{(1+|y|^2)^\beta U^q}$$

and then, the validity of (7.5) easily follows.

Step 2. Let now $1 \le N \le 7$ or $N \ge 8$, $\alpha > \alpha_N$. We want to show that

$$\int \frac{1}{(1+|y|^2) U^q} < +\infty \tag{7.6}$$

for some $0 < q < q_+ = 4 + 2\sqrt{6}$.

Indeed, set $\beta_0 = \frac{N-2-\alpha}{2} + \delta$, $\delta > 0$, and $q_0 = 3$. By (7.4) we get that

$$\int \frac{1}{(1+|y|^2)^{\beta_0} U^{q_0}} < +\infty.$$

Let $\beta_i = \beta_0 - i(1 + \frac{\alpha}{2})$ and $q_i = q_0 + 3i$, $i \in \mathbb{N}$. Since $q_0 < q_1 < q_+ = 4 + 2\sqrt{6} < q_2$, we can iterate (7.5) exactly two times to get that:

$$\int \frac{1}{(1+|y|^2)^{\beta_2} U^{q_2}} < +\infty \tag{7.7}$$

where $\beta_2 = \frac{N-6-3\alpha}{2} + \delta$, $q_2 = 9$.

Let $0 < q < q_+ = 4 + 2\sqrt{6} < 9$. By (7.7) and Hölder inequality we get that:

$$\begin{aligned} \int \frac{1}{(1+|y|^2) U^q} &= \int \frac{(1+|y|^2)^{\frac{q}{9}(\frac{6-N}{2} - \delta + \frac{3}{2}\alpha)}}{U^q} \cdot \frac{1}{(1+|y|^2)^{\frac{q}{9}(\frac{6-N}{2} - \delta + \frac{3}{2}\alpha) + 1}} \\ &\le \Big(\int \frac{1}{(1+|y|^2)^{\beta_2} U^{q_2}} \Big)^{\frac{q}{9}} \Big(\int \frac{1}{(1+|y|^2)^{\frac{q}{9-q}(\frac{6-N}{2} - \delta + \frac{3}{2}\alpha) + \frac{9}{9-q}}} \Big)^{\frac{9-q}{9}} < +\infty \end{aligned}$$

provided $-\frac{2q}{9-q}\beta_2 + \frac{18}{9-q} > N$ or equivalently

$$q > \frac{9N - 18}{6 - 2\delta + 3\alpha}. \tag{7.8}$$

To have (7.8) for some $\delta > 0$ small and $q_< q_+$ at the same time, we need to require $\frac{3N-6}{2+\alpha} < q_+$ or equivalently

$$1 \le N \le 7 \quad \text{or} \quad N \ge 8\,,\ \alpha > \alpha_N = \frac{3N - 14 - 4\sqrt{6}}{4 + 2\sqrt{6}}.$$

Our assumptions then provide the existence of some $0 < q < q_+ = 4 + 2\sqrt{6}$ such that (7.6) holds.

Step 3. We are ready to obtain a contradiction. Let $0 < q < 4 + 2\sqrt{6}$ be such that (7.6) holds. Let η be the cut-off function of Step 1. Using equation (7.1) we compute:

$$\begin{aligned}
\int |\nabla(\frac{\eta}{U^{\frac{q}{2}}})|^2 - \int \frac{2|y|^\alpha}{U^3}\left(\frac{\eta}{U^{\frac{q}{2}}}\right) &= \frac{q^2}{4}\int \frac{\eta^2|\nabla U|^2}{U^{q+2}} + \int \frac{|\nabla \eta|^2}{U^q} + \frac{1}{2}\int \nabla(\eta^2)\nabla(\frac{1}{U^q}) - \int \frac{2|y|^\alpha \eta^2}{U^{q+3}} \\
&= -\frac{q^2}{4(q+1)}\int \nabla U\cdot\nabla(\frac{\eta^2}{U^{q+1}}) + \int \frac{|\nabla\eta|^2}{U^q} \\
&\quad + \frac{q+2}{4(q+1)}\int \nabla(\eta^2)\nabla(\frac{1}{U^q}) - \int \frac{2|y|^\alpha\eta^2}{U^{q+3}} \\
&= -\frac{8q+8-q^2}{4(q+1)}\int \frac{|y|^\alpha\eta^2}{U^{q+3}} + \int \frac{|\nabla\eta|^2}{U^q} - \frac{q+2}{4(q+1)}\int \frac{\Delta\eta^2}{U^q}.
\end{aligned}$$

Since $0 < q < 4 + 2\sqrt{6}$, $8q + 8 - q^2 > 0$ and

$$\begin{aligned}
\int |\nabla(\frac{\eta}{U^{\frac{q}{2}}})|^2 - \int \frac{2|y|^\alpha}{U^3}\left(\frac{\eta}{U^{\frac{q}{2}}}\right)^2 &\leq -\frac{8q+8-q^2}{4(q+1)}\int_{B_1(0)} \frac{|y|^\alpha\eta^2}{U^{q+3}} + O(\frac{1}{R^2}\int_{B_{2R}(0)\setminus B_R(0)} \frac{1}{U^q}) \\
&\leq -\frac{8q+8-q^2}{4(q+1)}\int_{B_1(0)} \frac{|y|^\alpha\eta^2}{U^{q+3}} + O(\int_{|y|\geq R} \frac{1}{(1+|y|^2)U^q}).
\end{aligned}$$

Since (7.6) implies: $\lim_{R\to+\infty}\int_{|y|\geq R} \frac{1}{(1+|y|^2)U^q} = 0$, we get that for R large

$$\int |\nabla(\frac{\eta}{U^{q/2}})|^2 - \int \frac{2|y|^\alpha}{U^3}\left(\frac{\eta}{U^{q/2}}\right)^2 \leq -\frac{8q+8-q^2}{4(q+1)}\int_{B_1(0)} \frac{|y|^\alpha}{U^{q+3}} + O(\int_{|y|\geq R}\frac{1}{(1+|y|^2)U^q}) < 0.$$

A contradiction to (7.3). Hence, (7.2) holds and the proof of the first part of Theorem 7.1 is complete.

■

To describe the counterexample, we want to compute explicitly u^* and λ^* on the unit ball with $f(x) = |x|^\alpha$ and $N \geq 8$, $0 \leq \alpha \leq \alpha_N$. This will then provide an example of an extremal function u^* which satisfies $\| u^* \|_\infty = 1$ and is therefore not a classical solution. The second part of Theorem 7.1 then follows by considering the limit profile around zero as $\lambda \to \lambda^*$ for the minimal solution u_λ for $(S)_\lambda$ on the unit ball with $f(x) = |x|^\alpha$.

We shall borrow ideas from [4, 6], where the authors deal with the case of regular nonlinearities. However, unlike these papers where solutions are considered in a very weak sense, we consider here a more focussed and much simpler situation. Our example is based on the following useful characterization of the extremal solution:

Theorem 7.2. *Let $f \in C(\bar{\Omega})$ be a nonnegative function. For $\lambda > 0$, consider $u \in H_0^1(\Omega)$ to be a weak solution of $(S)_\lambda$ (in the $H_0^1(\Omega)$-sense) such that $\| u \|_{L^\infty(\Omega)} = 1$. Then the following assertions are equivalent:*

1. *u satisfies*
$$\int_\Omega |\nabla\phi|^2 \geq \int_\Omega \frac{2\lambda f(x)}{(1-u)^3}\phi^2 \quad \forall \phi \in H_0^1(\Omega)\,, \tag{7.9}$$

2. *$\lambda = \lambda^*$ and $u = u^*$.*

Here and in the sequel, u will be called a $H_0^1(\Omega)$-weak solution of $(S)_\lambda$ if $0 \le u \le 1$ a.e. while u solves $(S)_\lambda$ in the weak sense of $H_0^1(\Omega)$. We need the following uniqueness result:

Proposition 7.3. *Let $f \in C(\bar\Omega)$ be a nonnegative function. Let u_1, u_2 be two $H_0^1(\Omega)$-weak solutions of $(S)_\lambda$ so that $\mu_{1,\lambda}(u_i) \ge 0$, $i = 1, 2$. Then, $u_1 = u_2$ a.e. in Ω.*

Proof: For any $\theta \in [0,1]$ and $\phi \in H_0^1(\Omega)$, $\phi \ge 0$, we have that:

$$\begin{aligned} I_{\theta,\phi} : &= \int_\Omega \nabla\big(\theta u_1 + (1-\theta)u_2\big)\nabla\phi - \int_\Omega \frac{\lambda f(x)}{\big(1-\theta u_1-(1-\theta)u_2\big)^2}\phi \\ &= \lambda \int_\Omega f(x)\Big(\frac{\theta}{(1-u_1)^2} + \frac{1-\theta}{(1-u_2)^2} - \frac{1}{\big(1-\theta u_1-(1-\theta)u_2\big)^2}\Big)\phi \ge 0 \end{aligned}$$

due to the convexity of $1/(1-u)$ with respect to u. Since $I_{0,\phi} = I_{1,\phi} = 0$, the derivative of $I_{\theta,\phi}$ at $\theta = 0, 1$ provides:

$$\int_\Omega \nabla\big(u_1-u_2\big)\nabla\phi - \int_\Omega \frac{2\lambda f(x)}{\big(1-u_2\big)^3}(u_1-u_2)\phi \ge 0$$
$$\int_\Omega \nabla\big(u_1-u_2\big)\nabla\phi - \int_\Omega \frac{2\lambda f(x)}{\big(1-u_1\big)^3}(u_1-u_2)\phi \le 0$$

for any $\phi \in H_0^1(\Omega)$, $\phi \ge 0$.

Testing the first inequality on $\phi = (u_1-u_2)^-$ and the second one on $(u_1-u_2)^+$ we get that:

$$\int_\Omega \Big[|\nabla(u_1-u_2)^-|^2 - \frac{2\lambda f(x)}{(1-u_2)^3}\big((u_1-u_2)^-\big)^2\Big] \le 0$$
$$\int_\Omega \Big[|\nabla(u_1-u_2)^+|^2 - \frac{2\lambda f(x)}{(1-u_1)^3}\big((u_1-u_2)^+\big)^2\Big] \le 0\,.$$

Since $\mu_{1,\lambda}(u_1) \ge 0$, we have that:
(1). if $\mu_{1,\lambda}(u_1) > 0$, then $u_1 \le u_2$ a.e.;
(2). if $\mu_{1,\lambda}(u_1) = 0$, then

$$\int_\Omega \nabla\big(u_1-u_2\big)\nabla\bar\phi - \int_\Omega \frac{2\lambda f(x)}{\big(1-u_1\big)^3}(u_1-u_2)\bar\phi = 0 \tag{7.10}$$

where $\bar\phi = (u_1-u_2)^+$. Since $I_{\theta,\bar\phi} \ge 0$ for any $\theta \in [0,1]$ and $I_{1,\bar\phi} = \partial_\theta I_{1,\bar\phi} = 0$, we get that:

$$\partial^2_{\theta\theta} I_{1,\bar\phi} = -\int_\Omega \frac{6\lambda f(x)}{(1-u_1)^4}\big((u_1-u_2)^+\big)^3 \ge 0\,.$$

Let $Z_0 = \{x \in \Omega :\ f(x) = 0\}$. Clearly, $(u_1-u_2)^+ = 0$ a.e. in $\Omega \setminus Z_0$ and, by (7.10) we get:

$$\int_\Omega |\nabla\big(u_1-u_2\big)^+|^2 = 0.$$

Hence, $u_1 \le u_2$ a.e. in Ω. The same argument applies to prove the reversed inequality: $u_2 \le u_1$ a.e. in Ω. Therefore, $u_1 = u_2$ a.e. in Ω and the proof is complete. ∎

Since $\| u_\lambda \| < 1$ for any $\lambda \in (0, \lambda^*)$, we need –in order to prove Theorem 7.2– only to show that $(S)_\lambda$ does not have any $H_0^1(\Omega)$-weak solution for $\lambda > \lambda^*$. By the definition of λ^*, this is already true for classical solutions. We shall now extend this property to the class of weak solutions by means of the following result:

Proposition 7.4. *If w is a $H_0^1(\Omega)$-weak solution of $(S)_\lambda$, then for any $\varepsilon \in (0,1)$ there exists a classic solution w_ε of $(S)_{\lambda(1-\varepsilon)}$.*

Proof: First of all, we prove that: for any $\psi \in C^2([0,1])$ concave function so that $\psi(0) = 0$, we have that

$$\int_\Omega \nabla\psi(w)\nabla\varphi \geq \int_\Omega \frac{\lambda f}{(1-w)^2}\dot{\psi}(w)\varphi \tag{7.11}$$

for any $\varphi \in H_0^1(\Omega)$, $\varphi \geq 0$. Indeed, by concavity of ψ we get:

$$\begin{aligned}\int_\Omega \nabla\psi(w)\nabla\varphi &= \int_\Omega \dot{\psi}(w)\nabla w\nabla\varphi = \int_\Omega \nabla w\nabla\left(\dot{\psi}(w)\varphi\right) - \int_\Omega \ddot{\psi}(w)\varphi|\nabla w|^2 \\ &\geq \int_\Omega \frac{\lambda f(x)}{(1-w)^2}\dot{\psi}(w)\varphi\end{aligned}$$

for any $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$. By density, we get (7.11).

Let $\varepsilon \in (0,1)$. Define

$$\psi_\varepsilon(w) := 1 - \left(\varepsilon + (1-\varepsilon)(1-w)^3\right)^{\frac{1}{3}}, \quad 0 \leq w \leq 1\,.$$

Since $\psi_\varepsilon \in C^2([0,1])$ is a concave function, $\psi_\varepsilon(0) = 0$ and

$$\dot{\psi}_\varepsilon(w) = (1-\varepsilon)\frac{g\left(\psi_\varepsilon(w)\right)}{g(w)}, \quad g(s) := (1-s)^{-2},$$

by (7.11) we obtain that for any $\varphi \in H_0^1(\Omega)$, $\varphi \geq 0$:

$$\int_\Omega \nabla\psi_\varepsilon(w)\nabla\varphi \geq \int_\Omega \frac{\lambda f(x)}{(1-w)^2}\dot{\psi}_\varepsilon(w)\varphi = \lambda(1-\varepsilon)\int_\Omega f(x)g\big(\psi_\varepsilon(w)\big)\varphi = \int_\Omega \frac{\lambda(1-\varepsilon)f(x)}{(1-\psi_\varepsilon(w))^2}\varphi\,.$$

Hence, $\psi_\varepsilon(w)$ is a $H_0^1(\Omega)$-weak supersolution of $(S)_{\lambda(1-\varepsilon)}$ so that $0 \leq \psi_\varepsilon(w) \leq 1 - \varepsilon^{\frac{1}{3}} < 1$. Since 0 is a subsolution for any $\lambda > 0$, we get the existence of a $H_0^1(\Omega)$-weak solution w_ε of $(S)_{\lambda(1-\varepsilon)}$ so that $0 \leq w_\varepsilon \leq 1 - \varepsilon^{\frac{1}{3}}$. By standard elliptic regularity theory, w_ε is a classical solution of $(S)_{\lambda(1-\varepsilon)}$. ■

We are now ready to provide the counterexample on $B = B_1(0)$. We want to show that $u^*(x) = 1 - |x|^{\frac{2+\alpha}{3}}$ and $\lambda_* = \frac{(2+\alpha)(3N+\alpha-4)}{9}$. It is easy to check that u^* is a $H_0^1(\Omega)$-weak solution of $(S)_{\lambda^*}$, provided $\alpha > 1$ if $N = 1$ and $\alpha \geq 0$ if $N \geq 2$. By the characterization of Theorem 7.2, we need only to prove (7.9). By Hardy's inequality, we have that for $N \geq 2$:

$$\int_B |\nabla\phi|^2 \geq \frac{(N-2)^2}{4}\int_B \frac{\phi^2}{|x|^2}$$

for any $\phi \in H_0^1(B)$, and then (7.9) holds if $2\lambda^* \leq \frac{(N-2)^2}{4}$, or equivalently, if

$$N \geq 8 \quad \text{and} \quad 0 \leq \alpha \leq \alpha_N.$$

References

[1] A. Ambrosetti, H. Brezis and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*. J. Funct. Anal. **122** (1994), no. 2, 519–543.

[2] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and application.* J. Funct. Anal. **14** (1973), 349–381.

[3] S. Alama and G. Tarantello, *Elliptic problems with nonlinearities indefinite in sign.* J. Funct. Anal. **141** (1996), no. 1, 159–215.

[4] H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa, *Blow up for $u_t - \Delta u = g(u)$ revisited.* Adv. Differential Equations **1** (1996), 73–90.

[5] H. Brezis and L. Nirenberg, *H^1 versus C^1 local minimizers.* C. R. Math. Acad. Sci. Paris **317** (1993), no. 5, 465–472.

[6] H. Brezis and J.L. Vazquez, *Blow-up solutions of some nonlinear elliptic problems.* Rev. Mat. Univ. Compl. Madrid **10** (1997), 443–469.

[7] X. Cabré, *Extremal solutions and instantaneous complete blow-up for elliptic and parabolic problems.* Preprint (2005).

[8] X. Cabré and A. Capella, *On the stability of radial solutions of semilinear elliptic equations in all of $\mathbb{R}^n$.* C. R. Math. Acad. Sci. Paris **338** (2004), no. 10, 769–774.

[9] M.G. Crandall and P.H. Rabinowitz, *Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems.* Arch. Ration. Mech. Anal. **58** (1975), 207–218.

[10] G. Flores, G.A. Mercado and J.A. Pelesko, *Dynamics and Touchdown in Electrostatic MEMS.* Proceedings of ICMENS 2003 (2003), 182–187.

[11] N. Ghoussoub and Y. Guo, *On the partial differential equations of electrostatic MEMS devices: stationary case.* Submitted (2005).

[12] N. Ghoussoub and Y. Guo, *On the partial differential equations of electrostatic MEMS devices II: dynamic case.* In preparation.

[13] N. Ghoussoub and Y. Guo, *On the partial differential equations of electrostatic MEMS devices III: refined touchdown behavior.* In preparation.

[14] Y. Guo, Z. Pan and M.J. Ward, *Touchdown and pull-in voltage behavior of a MEMS device with varying dielectric properties.* SIAM J. Appl. Math. **166** (2006), no. 1, 309–338 (electronic).

[15] D.D. Joseph and T.S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources.* Arch. Ration. Mech. Anal. **49** (1973), 241–268.

[16] J.P. Keener and H.B. Keller, *Positive solutions of convex nonlinear eigenvalue problems.* J. Differential Equations **16** (1974), 103–125.

[17] F. Mignot and J.P. Puel, *Sur une classe de problèmes non linéaires avec non linéarité positive, croissante, convexe.* Comm. Partial Differential Equations **5** (1980), no. 8, 791–836.

[18] J.A. Pelesko, *Mathematical Modeling of Electrostatic MEMS with Tailored Dielectric Properties.* SIAM J. Appl. Math. **62** (2002), no. 3, 888–908.

[19] J.A. Pelesko and D.H. Bernstein, *Modeling MEMS and NEMS.* Chapman Hall and CRC Press (2002).

[20] G. Zheng, *New results on the formation of singularities for parabolic problems.* PhD Thesis (2005).