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COMPACTNESS AND TIGHTNESS IN A SPACE OF MEASURES WITH THE TOPOLOGY OF WEAK CONVERGENCE

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1. Introduction.

Let X be a Hausdorff space and denote by $\mathscr{M}_+(X)$ the set of non-negative totally finite measures μ defined on the Borel σ-field $\mathscr{B}(X)$ and satisfying the condition

$$\mu A = \sup\{\mu K : K \subseteq A\}$$

for every Borel-set A, where the supremum is taken over all compact subsets K of A. Provide $\mathscr{M}_+(X)$ with the *topology of weak convergence*, that is, the weakest topology for which all maps

$$\mu \to \int f d\mu$$

with f a bounded real-valued upper semi-continuous function, is upper semi-continuous. In this topology a net (μ_α) on $\mathscr{M}_+(X)$ is convergent with limit $\mu \in \mathscr{M}_+(X)$ if and only if

$$\lim \mu_\alpha X = \mu X \tag{1}$$

and

$$\liminf \mu_\alpha G \geqq \mu G \tag{2}$$

for all open subsets G of X (cf. Theorem 8.1 of [20]).

Call a subset $\mathscr{P} \subseteq \mathscr{M}_+(X)$ *tight* if the measures in $\mathscr{P}$ live uniformly on compact sets except for all small error, i.e. if, for every positive ε, there exists a compact $K \subseteq X$ such that

$$\mu(\complement K) < \varepsilon$$

for all $\mu \in \mathscr{P}$.

It is the purpose of this paper to give a survey of results on the connections between tight and compact subset of $\mathscr{M}_+(X)$. The reader will also find some new results. Of particular interest are the *Prohorov spaces.* A regular Hausdorff space X is called a *Prohorov space* if the relatively

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compact subsets of $\mathscr{M}_+(X)$ are precisely the same as the tight subsets with

$$\sup\{\mu X : \mu \in \mathscr{P}\} < \infty .$$

Since, in any case, a tight subset with $\sup\{\mu X : \mu \in \mathscr{P}\} < \infty$ is relatively compact (see below), X is a Prohorov space if and only if every compact subset of $\mathscr{M}_+(X)$ is tight.

One may say that the first result on Prohorov spaces is the fact that the reals R, in its usual topology, is a Prohorov space (this is more or less equivalent to the classical selection theorem of Helly). It is also obvious from the known properties of $\mathscr{M}_+(X)$ in its socalled vague topology, that every locally compact space is a Prohorov space (cf. also Theorem V.4 of A. D. Aleksandroff [1]). The first result leaving the locally compact case states that every Polish space, i.e. every separable space metrizable with a complete metric, is a Prohorov space; this result is due to Prohorov, cf. Theorem 1.12 of [16]. The latest research in the area, due to David Preiss [15], indicates, that among the separable metrizable spaces it is very hard to find any Prohorov spaces other than the Polish ones.

Certain aspects of the theory have been left aside. Thus we have not discussed signed measures, a topic which is perhaps most effectively dealt with from the standpoint of functional analysis – we refer the reader to the two recent accounts [9] and [12].

2. Preliminaries.

We shall study compactness with the help of nets (rather than filters); for the basic notions on nets see Kelley [13].

A net $(x_\alpha) = (x_\alpha)_{\alpha \in D}$ on a topological space X is said to be *compact* (more accurately *compact in* X) provided every subnet of (x_α) contains a further subnet which converges in X (or, equivalently, if every universal subnet of (x_α) converges). The following easily established result is useful.

LEMMA 2.1. *The net (x_α) is a compact net in X if and only if, to every covering $X = \bigcup\{G_i : i \in I\}$ of X by open sets, there exists a finite subset I^* of the index set I such that*

$$x_\alpha \in \bigcup \{G_i :\ i \in I^*\}, \quad \textit{eventually} .$$

A subset A of X is called *net-compact* (more accurately *net-compact in* X) provided every net on A is compact or, equivalently, if every universal net on A is convergent, or, as a third characterization, if the diffuse net on A defined by considering the indentity map $\mathrm{id}: A \to A$ and the diffuse ordering on the domain of id ($a \leq b$ for all $a, b \in A$) is compact. From the last characterization we obtain from Lemma 2.1:

LEMMA 2.2. *A subset $A \subseteq X$ is net-compact if and only if the following holds:*

$$\forall X = \bigcup \{G_i : i \in I\} \; \exists I^* \textit{ finite}: \; A \subseteq \bigcup \{G_i : i \in I^*\}.$$

As a corollary to this we get

LEMMA 2.3. *In case X is regular, a subset is net-compact if and only if it is relatively compact.*

3. Necessary and sufficient conditions for compactness.

From the discussion in section 2, it is obvious that if we can decide which universal nets (μ_α) on $\mathcal{M}_+(X)$ are convergent, we will be able to characterize the net-compact subsets of $\mathcal{M}_+(X)$. Assume, as we may, that the universal net (μ_α) satisfies the condition

$$\limsup \mu_\alpha X < \infty .$$

If (μ_α) is convergent, it is natural to conjecture that the limit measure can be constructed from the set-function $A \to \lim \mu_\alpha A$. A careful inspection of the simplest cases in which (μ_α) is known to converge will soon lead to the conjecture that the desired limit measure is given by the formula

$$\mu A = \sup\nolimits_{K \subseteq A} \inf\nolimits_{G \supseteq K} \lim \mu_\alpha G; \quad A \in \mathcal{B}(X); \tag{3}$$

here, the letter K indicates a compact subset and the letter G indicates an open subset of X (in fact, to gain this insight, it is enough to consider the example in which (μ_α) is a universal subnet of the sequence (μ_n), where μ_n is an one-point probability measure on R concentrated at the point $1/n$).

Now it follows from Theorem 2 of [19] that the formula (3) always defines a measure $\mu \in \mathcal{M}_+(X)$ irrespectively of which universal net (μ_α) is being investigated so long as it obeys the condition $\limsup \mu_\alpha X < \infty$. (In fact, this is not surprising since a well-known feature of measure theory is that if we apply two limit operations to an additive set-function, we end up with a (countably additive) measure.) Since the constructed measure μ satisfies condition (2) for all open G, it follows that (μ_α) converges weakly to μ if μ satisfies (1) i.e. if

$$\sup_K \inf_{G \supseteq K} \lim \mu_\alpha G = \lim \mu_\alpha X \,. \tag{4}$$

We now obtain the result that if $\mathscr{P} \subseteq \mathscr{M}_+(X)$ is tight and if

$$\sup\{\mu X : \mu \in \mathscr{P}\} < \infty ,$$

then $\mathscr{P}$ is net-compact (in fact relatively compact since the closure of a tight set is tight); to see this, consider a universal net on $\mathscr{P}$ and verify (4).

Let us express the tightness condition in an equivalent but more complicated way! When we below write (G_K), it indicates that (G_K) is a family of open sets indexed by the compact subsets K of X such that each G_K contains K. It is easy to see that $\mathscr{P}$ is tight if and only if to every such family (G_K), and to every ε, there exists a set G_K in the family, such that $\mu(\complement G_K) < \varepsilon$ holds for all $\mu \in \mathscr{P}$. In short, we may express this condition as follows:

$$\forall (G_K)\, \forall \varepsilon > 0\, \exists G_K\, \forall \mu \in \mathscr{P}\colon\ \mu(\complement G_K) < \varepsilon \,. \tag{5}$$

A slightly more careful analysis than that we had in mind above will reveal the fact, that in case $\mathscr{P}$ satisfies the usual condition

$$\sup\{\mu X : \mu \in \mathscr{P}\} < \infty$$

and, furthermore, the following relaxed form of (5):

$$\forall (G_K)\, \forall \varepsilon > 0\, \exists G_{K_1}, G_{K_2}, \ldots, G_{K_n}\, \forall \mu \in \mathscr{P}\, \exists 1 \leqq i \leqq n\colon\ \mu(\complement G_{K_i}) < \varepsilon \,, \tag{6}$$

then $\mathscr{P}$ is net-compact.

The conditions we have now arrived at are necessary and sufficient, in other words we have (cf. Theorem 4 of [19])

THEOREM 3.1. *$\mathscr{P} \subseteq \mathscr{M}_+(X)$ is net-compact if and only if $\mathscr{P}$ is bounded, that is*

$$\sup\{\mu X : \mu \in \mathscr{P}\} < \infty ,$$

and condition (6) *holds.*

The proof of necessity is quite easy, just employ Lemma 2.2 together with the fact that for each family (G_K) the sets

$$\{\mu \in \mathscr{M}_+(X) : \mu(\complement G_K) < \varepsilon\}$$

constitute an open covering of $\mathscr{M}_+(X)$ (this proof is more neat than the one given in [19]).

If X is regular, $\mathscr{M}_+(X)$ is regular too (Theorem 11.2, (iv) of [20]) so that by Lemma 2.3, we may replace the word "net-compact" in the above result by "relatively compact".

Theorem 3.1 also generalizes to a characterization of compact nets on $\mathscr{M}_+(X)$, cf. Theorem 4 of [19].

4. First examples of non-Prohorov spaces.

The first such example was constructed around 1961 by Varadarajan [22, p. 225] and the second is from 1967 and due to Fernique (Example I.6.4 of [8]).

VARADARAJANS EXAMPLE. We consider the set of natural numbers with an added point ∞:

$$X = \mathsf{N} \cup \{\infty\} = \{1, 2, \ldots, \infty\}.$$

Denote by μ_n a uniform distribution on $\{1, 2, \ldots, n\}$ that is

$$\mu_n = n^{-1}(\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n),$$

ε_k denoting a unit mass at the point k, and let μ be a unit mass at the point ∞. Consider on X the finest topology for which the sequence (μ_n) converges weakly to μ. It is seen, that this makes sense, that each point in N is isolated and that a set G containing the point ∞ is open if and only if it is of "density" 1, that is, the number of points in G less than or equal to n divided by n tends to 1 as $n \to \infty$. The reader can easily see that X is a normal Hausdorff space for which only the finite subsets are compact. This last fact, at once tells us that X is not a Prohorov space since the compact set $\{\mu_1, \mu_2, \ldots, \mu\}$ is not tight. (The reader will have no difficulty in veryfying that condition (6) is satisfied for this set.)

A topological space of this type is useful for many other purposes; the idea to consider such spaces seems to be due to Appert cf. Sierpiński [18, p. 60].

FERNIQUE'S EXAMPLE. Let X be a separable Hilbert space and provide X with its weak topology. Denote by $(e_k)_{k \geq 1}$ an orthonormal basis for X (in its norm-topology). Consider the sequence $(\mu_n)_{n \geq 1}$ of measures on X where

$$\mu_n = n^{-3} \textstyle\sum_{k=1}^{n^3} \varepsilon_{ne_k},$$

ε_x denoting a unit mass at the point x. Clearly, the set $\{\mu_n : n \geq 1\}$ is not tight. We shall prove that it is relatively compact, in fact we shall prove that $\mu_n \to \varepsilon_0$. It is sufficient to prove that for all $x_0 \in X$

$$\mu_n G \to 1 \quad \text{where} \quad G = \{x : |\langle x, x_0\rangle| < 1\}.$$

But this follows immediately from the inequalities

$$\begin{aligned}\mu_n(\complement G) &\leqq \int_{\complement G} |\langle x,x_0\rangle|^2 d\mu_n(x) \leqq \int |\langle x,x_0\rangle|^2 d\mu_n(x) \\ &= n^{-3} \sum_1^{n^3} |\langle ne_k, x_0\rangle|^2 \leqq n^{-1}\|x_0\|^2 .\end{aligned}$$

This simplification in the presentation of Fernique's example is due to R. Dudley.

5. Invariance properties of Prohorov spaces.

All topological spaces occuring in this and in the remaining sections are assumed to be regular Hausdorff spaces. Which operations will allow us to construct new Prohorov spaces from old ones? The natural "building stones" are the obvious Prohorov spaces viz. the compact ones. The only really obvious operation preserving the Prohorov-property is that of taking closed subsets, thus, *a closed subset of a Prohorov space is a Prohorov space*. This is, however, too trivial to be of much help.

In the search for operations preserving the Prohorov-property, the following three lemmas seem particularly useful.

LEMMA 5.1. *If $\mathscr{P}$ is a relatively compact subset of $\mathscr{M}_+(X)$ and F a closed subset of X then $\mathscr{P}|F$, the family of measures obtained by restricting the measures in $\mathscr{P}$ to the subset F, is a relatively compact subset of $\mathscr{M}_+(F)$. In particular, if F is known to be a Prohorov-space, we can for each $\varepsilon > 0$ find a compact subset K of F such that*

$$\mu(F \setminus K) < \varepsilon, \quad \forall \mu \in \mathscr{P}.$$

It is interesting that there seems to exist no elementary and direct proof of this result, whereas, as soon as Theorem 3.1 is available, the proof reduces to a simple argument which we shall leave to the reader.

Before the next lemma, we introduce the following definition: $\mathscr{P} \subseteq \mathscr{M}_+(X)$ is said to be *τ-smooth* if

$$\inf_{F \in \mathscr{F}} \sup_{\mu \in \mathscr{P}} \mu F = 0$$

for every family $\mathscr{F}$ of closed subsets of X filtering downwards to the empty set. It is straight forward to deduce from Theorem 3.1. (a simple direct proof can also be deviced) the following:

LEMMA 5.2. *If $\mathscr{P} \subseteq \mathscr{M}_+(X)$ is relatively compact, then $\mathscr{P}$ is τ-smooth.*

LEMMA 5.3. *If $\mathscr{P} \subseteq \mathscr{M}(X)$ is compact, and F is a closed subset of X, and if the constant a is so that*

$$\mu F < a, \quad \forall \mu \in \mathscr{P}$$

then, to every $\varepsilon > 0$, there exists an open set G containing F such that

$$\mu G < \alpha + \varepsilon, \quad \forall \mu \in \mathscr{P} .$$

PROOF. To every $\mu \in \mathscr{P}$ we can find a compact set K_μ disjoint with F such that

$$\mu(\complement F \setminus K_\mu) < \varepsilon .$$

Now choose a closed set F_μ disjoint with K_μ and such that $\mathring{F}_\mu$, the interior of F_μ, contains F. Then

$$\mu(F_\mu) < a + \varepsilon .$$

The result now follow by consideration of the open covering of $\mathscr{P}$ constituted by the sets

$$\{\eta \in \mathscr{M}_+(X) : \eta(F_\mu) < a + \varepsilon\}$$

where $\mu \in \mathscr{P}$.

THEOREM 5.4. *If X can be covered with finitely many closed Prohorov subspaces, then X itself is a Prohorov space.*

This follows immediately from Lemma 5.1.

It may be remarked that the examples of Varadarajan and Fernique show that the result can not be generalized to a countable covering with closed Prohorov subspaces (even compact subspaces will not work).

Also note, that it is essential that the subspaces in Theorem 5.4 be closed. Indeed, Varadarajans space is the union of the two Prohorov spaces N and $\{\infty\}$. Perhaps it is true that if X can be covered with finitely many Prohorov spaces $A_1, \ldots, A_n$ and if the topology of X is the finest topology rendering the canonical imbeddings $A_i \to X$ continuous; $i = 1, \ldots, n$, then X is a Prohorov spaces. If true, this would be more general than Theorem 5.4. (Here is a result going in the direction indicated: If X is of countable type (see the definition further on) and if X can be written in the form $X = K \cup A$ with K compact and A a Prohorov space, then X is a Prohorov space. We do not know if the condition on K can be relaxed only assuming that this set is a closed Prohorov space.)

THEOREM 5.5. *The Prohorov property is a local property i.e. if, for every $x \in X$ there exists a subset A_x of X containing x in its interior and such that A_x (in its relative topology) is a Prohorov space, then X is a Prohorov space.*

This nice property of Prohorov spaces, due to Hoffmann-Jørgensen, is not difficult to prove utilizing Lemmas 5.1 and 5.2 (cf. Theorem 6 of [12]). As corollaries we obtain:

COROLLARY 5.6. *If X can be covered with open Prohorov subspaces, then X itself is a Prohorov space.*

COROLLARY 5.7. *Every open subset of a Prohorov space is a Prohorov space.*

PROOF. Let G be an open subset of the Prohorov space X. By regularity we can, for every $x \in G$ find a closed neighbourhood contained in G. Since this closed neighbourhood is a Prohorov space, G is a local Prohorov space, hence a Prohorov space.

The result can also be proved in a more elementary and direct way as an application of Lemma 5.3 (this remark we owe in part to R. O. Davies).

Corollary 5.7 can be generalized from open subsets to G_δ-subsets by a simple "$\varepsilon 2^{-n}$-argument" (cf. Theorem 5 of [12] and also Theorem 6.1 which offers a generalization). Thus we have:

THEOREM 5.8. *A G_δ-subset of a Prohorov space is a Prohorov space. In particular, it follows that every topological space X which is complete in the sense of Čech (i.e. X is homeomorphic to a G_δ-subset of a compact space) is a Prohorov space.*

Prohorovs result that Polish spaces are Prohorov spaces is a special case of this result. We remark, that Prohorovs original proof (cf. Theorem 1.12 of [16]) is very simple and direct; by Proposition 3 of [12], the same idea of proof can be carried over to cover the general case of spaces complete in the sense of Čech.

We shall now look at another corollary of Theorem 5.5. For this we need the notion of a space of *point-countable type*; these spaces were introduced by Henriksen and Isbell and by Arhangel'skiĭ and are spaces for which every point is contained in a compact set having a countable neighborhood basis; if we demand that every compact subset be contained in a compact subset having a countable neighbourhood basis, we obtain the spaces of *countable type*. Every metrizable and every locally compact space is of countable type (we refer the reader to Arhangel'skiĭ [2] for further information). Furthermore, we shall introduce the terminology, that a sequence (A_n) of subsets of X is *strongly increasing* if (A_n) is increasing and if every compact subset of X is contained in one of the A_n's. We can now state:

COROLLARY 5.9. *If X is of point-countable type and if there exists a strongly increasing sequence (A_n) of Prohorov subspaces, then X is a Prohorov space.*

PROOF. This follows from Theorem 5.5 since X is the union of the interiors of the A_n's (cf. Proposition 3 of [12].

The next result is related to the corollary just established, but can not be proved by a similar method. The question we ask is whether it is possible to relax the condition that X be of point countable type, only assuming that X is a k-space. We remind the reader that X is a k-*space* if every subset of X intersecting every compact set in a closed set is necessarily closed. The result that every space of point-countable type is a k-space is due to Arhangel'skiĭ (Theorem 3.7′ af [2]).

THEOREM 5.10. *If X is a* k-*space and if there exists a strongly increasing sequence (F_n) of closed Prohorov subspaces, then X is a Prohorov space.*

PROOF. Let $\mathscr{P} \subseteq \mathscr{M}_+(X)$ be compact. To prove that $\mathscr{P}$ is tight, we shall verify condition (5). Therefore, let (G_K) and $\varepsilon > 0$ be given. By Lemma 5.1 we can, for each $n \geq 1$ find a compact subset K_n of F_n such that

$$\mu(F_n \setminus K_n) < \varepsilon, \quad \forall \mu \in \mathscr{P}.$$

Then we also have

$$\mu(F_n \setminus G_{K_n}) < \varepsilon, \quad \forall \mu \in \mathscr{P}.$$

By Lemma 5.3 we can find open sets H_n such that H_n contains $F_n \setminus G_{K_n}$ and such that

$$\mu H_n < 2\varepsilon, \quad \forall \mu \in \mathscr{P}.$$

For every n, $G_{K_n} \cup H_n$ is an open set containing F_n. (At this place it is perhaps convenient to note the following simple fact: If X is a k-space, (A_n) a strongly increasing sequence, and if (G_n) is a sequence of open sets such that $G_n \supseteqq A_n$, $n \geqq 1$, then $\bigcap_1^\infty G_n$ is open.) It follows that the sets

$$V_n = \bigcap_{\nu=n}^{\infty} (G_{K_\nu} \cup H_\nu)$$

are open. Since $V_n \uparrow X$ there exists by Lemma 5.2 an n such that

$$\mu(\complement V_n) < \varepsilon, \quad \forall \mu \in \mathscr{P}.$$

For every $\mu \in \mathscr{P}$ we now have

$$\begin{aligned} \mu(\complement G_{K_n}) &= \mu(\complement G_{K_n} \cap \complement H_n) + \mu(\complement G_{K_n} \cap H_n) \\ &\leqq \mu(\complement V_n) + \mu H_n \\ &< 3\varepsilon \end{aligned}$$

and it follows, that (5) is satisfied.

In [12] the reader will find a different proof of this result; in fact Theorem 7 and Corollary 6 of [12] are more general than Theorem 5.10. It is not known whether the condition that the strongly increasing sequence in Theorem 5.10 consist of closed sets can be relaxed (compare with corollary 5.9).

Theorem 5.10, as opposed to the results derived so far, has the interesting feature that it allows the construction of Prohorov spaces not satisfying any completeness conditions, using compact spaces as "building stones". Indeed, as pointed out in Example 1 of [12], the dual of a metrizable, locally convex infinite dimensional linear vectorspace is a Prohorov space in the topology of uniform convergence on precompact sets; also, such a space is a k-space which is not of point-countable type. Furthermore, Example 2 of [12] shows that there exists a Prohorov space which is not even a k-space. The same kind of examples were also developed, independently in [9, pp. 126–129].

THEOREM 5.11. *Let K be a compact subset of X and (F_n) a sequence of closed Prohorov subspaces of X such that $F_n \cap K = \emptyset$ for $n \geqq 1$ and such that any closed subset of X disjoint with K is contained in at least one F_n (in other words, the complements of the F_n's constitute a neighborhoodbase for K.) Then X is a Prohorov space.*

PROOF. Let $\mathscr{P} \subseteqq \mathscr{M}_+(X)$ be relatively compact, and let $\varepsilon > 0$ be given. Choose a sequence (ε_n) of positive numbers with $\sum_1^\infty \varepsilon_n < \varepsilon$. We may assume that the sequence (F_n) is increasing. By Lemma 5.1 we can find compact sets $K_n \subseteqq F_n$ such that

$$\mu(F_n \setminus K_n) < \varepsilon_n, \quad \forall \mu \in \mathscr{P}.$$

Put

$$K_0 = K \cup \bigcup_1^\infty (K_n \setminus \mathring{F}_{n-1})$$

(where $F_0 = \emptyset$). By regularity of X, it is easy to see that the sets $\complement \mathring{F}_n$; $n \geqq 1$ constitute a neighbourhood basis for K. By Lemma 7.5 of [20] it follows that K_0 is compact. For every μ we have

$$\mu(\complement K_0) = \mu(\complement K_0 \cap \bigcup_1^\infty F_n) = \lim_{n\to\infty} \mu(F_n \setminus K_0) ,$$

and since

$$\begin{aligned} F_n \setminus K_0 &= \bigcup_{\nu=1}^n [(F_\nu \setminus F_{\nu-1}) \setminus K_0] \\ &\subseteqq \bigcup_{\nu=1}^n [(F_\nu \setminus F_{\nu-1}) \setminus (K_\nu \setminus \mathring{F}_{\nu-1})] \\ &\subseteqq \bigcup_{\nu=1}^n (F_\nu \setminus K_\nu) , \end{aligned}$$

we see that

$$\mu(\complement K_0) \leqq \textstyle\sum_1^\infty \varepsilon_n < \varepsilon$$

for all $\mu \in \mathscr{P}$.

It is not known what happens if we replace the condition

$$F \text{ closed, } F \cap K = \emptyset \Rightarrow \exists n\colon F \subseteqq F_n$$

by the condition that X be a k-space and that

$$K' \text{ compact, } K' \cap K = \emptyset \Rightarrow \exists n\colon K' \subseteqq F_n$$

holds. If the conclusion of the theorem — that X is a Prohorov space — still holds, we would have obtained a generalization of Theorem 5.10 (the latter would correspond to the case $K = \emptyset$).

6. The Prohorov property for spaces connected by mappings or by correspondences.

In this section we continue the type of investigations carried out in the last section. The first result is formulated in the language of correspondences (= multivalued mappings). We remind the reader that a *correspondence* $\varphi\colon X \to Y$ assigns to every point $x \in X$ a non-empty subset $\varphi(x)$ of Y. For $A \subseteqq X$ we define

$$\varphi(A) = \bigcup \{\varphi(x) : x \in A\}$$

and for $B \subseteqq Y$ we define the *weak inverse* $\varphi^w B$ by

$$\varphi^w B = \{x : \varphi(x) \cap B \neq \emptyset\}$$

and the *strong inverse* $\varphi^s B$ by

$$\varphi^s B = \{x : \varphi(x) \subseteq B\} .$$

Note that, in case φ is surjective ($\varphi X = Y$), φ^w is a correspondence $Y \to X$ and $(\varphi^w)^w = \varphi$. The correspondence $\varphi: X \to Y$ with X and Y topological spaces is *upper semicontinuous compact valued*, in short usco, provided

$$F \text{ closed in } Y \Rightarrow \varphi^w F \text{ closed in } X ,$$

and

$$x \in X \Rightarrow \varphi(x) \text{ compact in } Y ,$$

hold. If $\varphi: X \to Y$ is usco one has

$$K \text{ compact in } X \Rightarrow \varphi K \text{ compact in } Y .$$

Let $\varphi: X \to Y$ be usco. We define a new correspondence, which we shall denote be the same letter, $\varphi: \mathscr{M}_+(X) \to \mathscr{M}_+(Y)$ by

$$\varphi(\mu) = \{\eta \in \mathscr{M}_+(Y) : \mu X = \eta Y \text{ and } \mu(\varphi^s G) \leqq \eta G, \forall G \text{ open in } Y\}; \quad \mu \in \mathscr{M}_+(X) .$$

It can be shown that, $\varphi: \mathscr{M}_+(X) \to \mathscr{M}_+(Y)$ is again an usco correspondence when $\mathscr{M}_+(X)$ and $\mathscr{M}_+(Y)$ are provided with the topologies of weak convergence, in particular, $\varphi(\mu)$ is nonempty for every $\mu \in \mathscr{M}_+(X)$. For these results and for a more detailed discussion of measure preserving correspondences we refer the reader to the paper [21].

THEOREM 6.1. (Compare with Theorem 5 of [12].) *Let* $\varphi_n: X \to Y_n$; $n \geqq 1$ *be* usco *correspondences and assume that for each sequence* (L_n), *with* L_n *a compact subset of* Y_n, *the set*

$$\bigcap_{n=1}^{\infty} \varphi_n{}^w(L_n)$$

is a Prohorov subset of X.

Then, if the Y_n *are Prohorov spaces, so is* X.

PROOF. Let $\mathscr{P} \subseteq \mathscr{M}_+(X)$ be compact. Then $\varphi_n(\mathscr{P})$ is a compact subset of $\mathscr{M}_+(Y_n)$ for each $n \geqq 1$ (cf. Lemma 1.10 and Theorem 3.13 of [21]). To $\varepsilon > 0$ there then exist compact subsets L_n of Y_n such that

$$\eta(\complement L_n) < \varepsilon 2^{-n}, \quad \forall \eta \in \varphi_n(\mathscr{P}) .$$

Put

$$F = \bigcap_{n=1}^{\infty} \varphi_n{}^w(L_n) .$$

Then F is a closed Prohorov subspace of X, hence, by Lemma 5.1 there exists a compact subset K of F such that

$$\mu(F \setminus K) < \varepsilon, \quad \forall \mu \in \mathscr{P}.$$

We claim that

$$\mu(\complement F) < \varepsilon, \quad \forall \mu \in \mathscr{P}$$

holds. To see this, let $\mu \in \mathscr{P}$ be given and let (η_n) be a sequence of mesures with $\eta_n \in \varphi_n(\mu)$; $n \geqq 1$. Then we have

$$\begin{aligned}\mu(\complement F) &= \mu\left(\bigcup_{n=1}^{\infty} \varphi_n{}^s(\complement L_n)\right)\\ &\leqq \sum_1^{\infty} \mu(\varphi_n{}^s(\complement L_n))\\ &\leqq \sum_1^{\infty} \eta_n(\complement L_n)\\ &< \varepsilon\end{aligned}$$

as claimed above. It is now clear that

$$\mu(\complement K) < 2\varepsilon, \quad \forall \mu \in \mathscr{P},$$

thus $\mathscr{P}$ is tight.

As obvious corollaries we get that a countable product of Prohorov spaces is a Prohorov space, and that a countable intersection of Prohorov subspaces of a space X is again a Prohorov space. We wish to single out the following:

COROLLARY 6.2. *Let $\pi: X \to Y$ be a perfect map (that is, π is a continuous, surjective, and closed map for which $\pi^{-1}(y)$ is compact in X for each $y \in Y$). Then X is a Prohorov space if and only if Y is a Prohorov space.*

PROOF. Observe that π is perfect if and only if π as well as π^w considered as correspondences are usco. Then apply Theorem 6.1.

We mention the following open problem:

PROBLEM. *Is the continuous open image of a Prohorov space a Prohorov space?*

This Problem is related to the problem whether the continuous open image of a space complete in the sense of Čech is complete in the sense of Čech (recently, Z. Frolik has informed the author that he has settled this problem in the negative).

7. Tightness of special compact subsets.

Even though X may not be a Prohorov space, it may still be true that all compact subsets of $\mathcal{M}_+(X)$ of a special type are tight. Such results are usually established by imposing countability restrictions on the space X; we feel, it is a natural demand that the countability restrictions are not more severe than that they are satisfied by every separable metrizable space. The results of this section are interesting in the light of the fact, to be commented on in section 8, that among the metrizable spaces there are plenty of non-Prohorov spaces.

The first result of the desired type is due to LeCam (Theorem 4 of [14]) and states that if X is metrizable and if $\mathcal{P} \subseteq \mathcal{M}_+(X)$ consists of the elements in a convergent sequence, then $\mathcal{P}$ is tight. It is fairly obvious from the proof of LeCam that one may relax the metrizability condition only assuming that X is a space of countable type (cf. Theorem 9.3 of [20]). Thus LeCams theorem can be stated:

THEOREM 7.1. *If X is a space of countable type and if $(\mu_n)_{n \geqq 1}$ is a convergent sequence on $\mathcal{M}_+(X)$ then $\mathcal{P} = \{\mu_n : n \geqq 1\}$ is tight.*

Note that, in view of the examples of Varadarajan and Fernique, some restriction on the space X is necessary.

A refinement of LeCams result is due to Choquet (Proposition 31 of [5]) and, independently, to Hoffmann-Jørgensen (Theorem 9 of [12] and its corollaries), and consists in replacing the condition on $\mathcal{P}$, assuming only that $\mathcal{P}$ is compact and countable (see also proposition 6 of [9]).

THEOREM 7.2. *If X is a completely regular space of countable type, then every compact and countable subset of $\mathcal{M}_+(X)$ is tight.*

We remark that the results in Hoffmann-Jørgensen [12] are more general, firstly the basic space X is allowed to be slightly more general, and, secondly the condition that $\mathcal{P}$ be countable is replaced by the condition that $\mathcal{P}$ be scattered.

The measures in $\mathcal{P}$ in the above results are quite arbitrary. Another line of research is to investigate what happens if one imposes restrictions on the measures in the (relatively) compact family $\mathcal{P}$. At first one observes that in case $\mathcal{P}$ consists of one-point measures, it is obvious that $\mathcal{P}$ is tight. Secondly, one may note that in case, for some finite n, the support of each measures in $\mathcal{P}$ contains at most n points, then $\mathcal{P}$ is tight. (Observe that the sets $\{x : \exists \mu \in \mathcal{P}$ with $\mu(\{x\}) \geqq \varepsilon\}$ are net-compact. More generally, if $\mu_\alpha \to \mu$ and (F_α) is a net of closed sets with

$\mu_\alpha F_\alpha \geqq \varepsilon$, $\forall d$, then for every compact K with $\mu(\complement K) < \varepsilon$ we have $K \cap F \neq \emptyset$ where

$$F = \{x : \forall N(x) : N(x) \cap F_\alpha \neq \emptyset \text{ infinitely often}\}.)$$

A result more substantial than these remarks is the following due to Balkema [3]:

THEOREM 7.3. *If X is metrizable, and if the compact set $\mathscr{P}$ only contains measures with compact support, then $\mathscr{P}$ is tight.*

Note that this result is not true if we only assume that $\mathscr{P}$ is relatively compact, indeed, in any non-Prohorov space there exists a relatively compact set of probability measures $\mathscr{P}$ containing only measures with compact support, and such that there to any compact set K exists a $\mu \in \mathscr{P}$ with $\mu K = 0$. Also, as is seen from Varadarajan's and Fernique's examples, some restriction on the topological nature of X is needed.

8. The first examples of separable metrizable non-Prohorov spaces. The first such example is, to the best of our knowledge, due to C. Léger see [5, p. 6], who used a type of space first considered, for a completely different purpose by Sierpiński (cf. [18, p. 142]). Independently, Balkema also noted the existence of such an example (cf. [3]).

EXAMPLE. (Léger, Balkema). Let $S = [0,1] \times [0,1]$ denote the unit square. We shall "construct" X as the complement of a certain subset A of S. Denote by $\pi: S \to [0,1]$ the projection of S on the first coordinate space. For a subset $B \subseteq S$ denote by $B[x]$ the section

$$B[x] = \{y : (x,y) \in B\}.$$

We claim that there exists a set $A \subseteq S$ such that

1° $A[x]$ contains at most one point for all $x \in [0,1]$
2° K compact, $\pi K = [0,1] \Rightarrow K \cap A \neq \emptyset$.

To see this, consider the class $\mathscr{K}$ of compact subsets K of S with $\pi K = [0,1]$. A simple cardinality-argument tells us that there exists a bijection $x \to K_x$ from $[0,1]$ onto $\mathscr{K}$. Choose for each $x \in [0,1]$ a point $y(x)$ such that $(x, y(x)) \in K_x$. Clearly, the set

$$A = \{(x, y(x)) : x \in [0,1]\}$$

satisfies the requirements 1° and 2°.

In terms of the space $X = \complement A$, condition 2° can be expressed in the following way

2′° K compact, $K \subseteq X \Rightarrow \exists x\colon K[x] = \emptyset$.

Now consider the family

$$\mathscr{P} = \{\varepsilon_x \otimes \lambda : x \in [0,1]\},$$

λ denoting Lebesgue measure ($\varepsilon_x \otimes \lambda$ is thus Lebesgue measure concentrated on $\pi^{-1}(x)$). Since the mapping $x \to \varepsilon_x \otimes \lambda$ is continuous, $\mathscr{P}$ is compact. By 2′°, $\mathscr{P}$ is not tight.

It follows, that if $A = \complement X$ satisfies 1° and 2°, then X can not be Polish. The simplest purely topological proof of this is probably to remark that for the continuous open mapping π of X onto the compact set $[0,1]$ no compact subset K of X satisfies $\pi K = [0,1]$. That X can not be Polish then has been proved by D. H. Fremlin (private communication, September 1971).

It is natural to ask if one can avoid the axiom of choice and give a more effective construction of the set A. J. P. R. Christensen remarked, using some results from [6], that this can actually be done, and in this way a measurable A can be constructed. Thus there exists a Borel subset of S which is not a Prohorov space. A result of Mokobodzki (Théorème 23 of [5]) then assures the existence of a K_σ-subset (i.e. a countable union of compact sets) of S which is not a Prohorov space. These observations led R. O. Davies to give a very explicit and ingenious construction of a K_σ-set A satisfying 1° and 2° (cf. [7]). Thus we have

THEOREM 8.1. *There exists a K_σ-subset of $S = [0,1] \times [0,1]$ which is not a Prohorov space.*

As remarked by D. Monrad, it follows by the actual construction of Davies that the space X is a Baire space (by the results of the next section it is easy to see the existence of such examples since it is easy to construct separable metrizable Baire spaces which are not Polish).

There is a problem related to the Prohorov space problem which has been solved by the consideration of examples of the above type. Note, that if $\mathscr{P} \subseteq \mathscr{M}_+(X)$ is tight, then so is $\overline{\text{co}}(\mathscr{P})$, the closed convex hull of $\mathscr{P}$. Therefore, if X is a Prohorov space the implication

(7) $$\mathscr{P} \text{ compact} \Rightarrow \overline{\text{co}}(\mathscr{P}) \text{ compact}$$

holds. It is also known that if every τ-smooth measure on X is tight (i.e. contained in $\mathscr{M}_+(X)$) then (7) holds; this follows from a result essentially due V. S. Varadarajan (cf. Theorem 6 of [22]). We shall now give examples showing that (7) does not hold for every separable metrizable space.

The first example is due to C. Léger and uses the continuum hypothesis. With the aid of this hypotheses it is easy to prove the existance of a total ordering of $[0,1]$ such that

$$\{x:\ x<y\} \text{ is countable, } \forall y \in [0,1] .$$

Then it is easy to see that if

$$X = \{(x,y):\ x<y\}$$

and

$$\mathscr{P} = \{(\varepsilon_x \otimes \lambda:\ x \in [0,1]) ,$$

then $\mathscr{P}$ is compact but $\overline{\mathrm{co}}(\mathscr{P})$ is not compact.

The other example is due to D. H. Fremlin (private communication, april 1971) and does not make use of the continuum hypothesis. Fremlin remarks that if $A \subseteq S$ can be found satisfying 1° and

3° K compact, $K \subseteq S$, $\lambda K > 0 \ \Rightarrow\ K \cap A \neq \emptyset$,

λ denoting the 2-dimensional Lebesgue measure, then an example of the desired type can easily be constructed. To establish 1° and 3°, denote by ω_0 the first ordinal of cardinality $\aleph$ and let $\theta:\ \omega \to K_\omega$ be a bijection of ω_0 onto the compact subsets K of S with $\lambda K > 0$. We claim that there exists a family $(x_\omega, y_\omega)_{\omega<\omega_0}$ such that

$$(x_\omega, y_\omega) \in K_\omega, \quad \forall \omega < \omega_0$$

and

$$\omega_1 \neq \omega_2 \ \Rightarrow\ x_{\omega_1} \neq x_{\omega_2}$$

holds. This is proved by induction: If $\omega_1 < \omega_0$ and if for every $\omega < \omega_1$, (x_ω, y_ω) is constructed satisfying the required conditions, then, since $\lambda K_{\omega_1} > 0$ we find by Fubinis theorem that $\lambda(\pi K_{\omega_1}) > 0$ (now λ is the one-dimensional Lebesgue measure), hence the cardinality of πK_{ω_1} is $\aleph$. It is now obvious that $(x_{\omega_1}, y_{\omega_1}) \in K_{\omega_1}$ can be found such that $x_{\omega_1} \neq x_\omega$ for all $\omega < \omega_1$. This finishes the induction proof. It is now clear that

$$A = \{(x_\omega, y_\omega):\ \omega < \omega_0\}$$

satisfies 1° and 3°.

9. The results of David Preiss.

It is quite evident that the results of Preiss [15] are the deepest results on the Prohorov space-problem published so far – the results almost close the subject – as far as metrizable space are concerned – even though one can still pose unsolved problems. The beauty of the first result is partly due to its simplicity – in statement, not in proof! It reads:

THEOREM 9.1 *The rationals* Q *is not a Prohorov space.*

We shall now give the details of the proof. Even though the proof is very close to Preiss's original proof, some convenient changes in the details are due to D. H. Fremlin. Also, P. Billingsley have suggested some alterations in the presentation.

We need the following simple general lemma:

LEMMA 9.2 *Let* X^* *be compact and* X *a Borel subset of* X^*. *For each* $l=1,2,\ldots$, *let* $\mathscr{G}_l$ *be a family of open subsets of* X^* *such that*

$$\{G \setminus X : G \in \mathscr{G}_l\}$$

is upward filtering with union $\complement X$. *Then*

$$\mathscr{P} = \{P \in \mathscr{M}_+{}^1(X) : \forall l \, \forall G \in \mathscr{G}_l : P(G \cap X) \leqq 1/l\}$$

is compact in $\mathscr{M}_+{}^1(X)$.

PROOF OF THEOREM 9.1. Let $x \in [0,1]$. Let

$$x = 0,\varepsilon_1,\varepsilon_2,\ldots = \textstyle\sum_1^\infty \varepsilon_i 2^{-i}$$

be the expansion of x in binary digits (each ε_i is either 0 or 1 and infinitely many of the ε_i are 0). The exceptional number 1 has the expansion 1.00.... By $r=r(x)$, called the *rank* of x, we denote the number of 1's in the expansion of $x (r(0)=r(1)=0)$. By $n_1=n_1(x)$, $n_2=n_2(x)$, etc. we denote the *waiting times*; these are defined by the equations

$$\begin{aligned} n_1 &= \min\{i : \varepsilon_i=1\}\,, \\ n_1+n_2 &= \min\{i>n_1 : \varepsilon_i=1\}\,, \\ n_1+n_2+n_3 &= \min\{i>n_1+n_2 : \varepsilon_i = 1\}\,, \\ &\vdots \end{aligned}$$

In case $r(x)=\nu<\infty$, we have $n_{\nu+1}=n_{\nu+2}=\ldots=\infty$.

By $d_l(x)$ we denote the largest among the l first waiting times. In case $r(x)<l$, we have $d_l(x)=\infty$.

We shall employ the notation $x=\langle n_1 n_2 \ldots \rangle$, that is, we put

$$\langle n_1 n_2 \ldots \rangle = \sum_1^\infty 2^{-(n_1+n_2+\ldots+n_\nu)}.$$

We put

$$\langle n_1 n_2 \ldots n_\nu \rangle = \langle n_1 n_2 \ldots n_\nu \infty \infty \ldots \rangle$$

The following simple fact will be useful:

(8) Let $n_1, n_2, \ldots, n_\nu$ be natural numbers and $p \geqq 0$ an integer. Then the open interval

$$(\langle n_1 n_2 \ldots n_\nu \rangle, \langle n_1 n_2 \ldots n_\nu \rangle + 2^{-(n_1+n_2+\ldots+n_\nu+p)})$$

consists of those $y \in [0,1]$ which can be written in the form

$$y = \langle n_1 n_2 \ldots n_\nu n_{\nu+1} \ldots \rangle$$

with $n_{\nu+1} \geqq p+1$ and $n_{\nu+1} < \infty$ (hence $r(y) \geqq \nu+1$).

Let

$$X_k = \{x : r(x) \leqq k\}; \quad k \geqq 1$$

and

$$X = \bigcup_1^\infty X_k = \{x : r(x) < \infty\}.$$

Then X is the set of dyadic rationals in $[0,1]$. In its natural topology X is countable and without isolated points. Since X is also metrizable it follows by a well-known result of Sierpiński (cf. [17]) that X is homeomorphic to Q.

Define subsets of $[0,1]$ by

$$G_k^l = \{x : k+l < r(x) \leqq \infty, \ d_l(x) \leqq k\}; \quad k \geqq 1, \ l \geqq 1,$$

and put

$$U = \bigcup \{G_k^l : k \geqq 1, \ l \geqq 1\}.$$

Then:

(9) $$G_k^l \text{ is open},$$

(10) $$G_k^l \subseteqq [0,1] \setminus X_{k+l},$$

(11) $$G_k^l \setminus X = \{x : d_l(x) \leqq k\},$$

(12) $$G_k^l \setminus X \uparrow [0,1] \setminus X \text{ for each fixed } l \text{ as } k \uparrow \infty,$$

(13) $$U = \{x : r(x) \geqq n_1(x) + 2\}.$$

(9) follows from (8) (with $\nu = l$, $p = 0$) and (10)-(13) are obvious.

By Lemma 9.2, the set

$$\mathscr{P} = \{P \in \mathscr{M}_+^1(X) : P(G_k^l \cap X) \leqq 1/l, \ \forall k,l\}$$

is compact.

In the remaining part of the proof, K denotes a fixed compact subset of X. We shall prove that $\mathscr{P}$ is not tight by proving that there exists $P \in \mathscr{P}$ with $PK=0$. To achieve this, we shall construct a certain sequence $\mu_0, \mu_1, \mu_2, \ldots$ on $\mathscr{M}_+(X)$ such that

(a) μ_ν is supported by X_ν : $\mu_\nu(\complement X_\nu)=0$; $\nu \geqq 0$,

(b) $\mu_\nu K=0$,

(c) $\mu_\nu(G_k^l \cap X) \leqq 1/l, \ \forall k,l$.

If, for some ν, $\mu_\nu X \geqq 1$ then the measure P obtained by normalization of μ_ν meets the demands $P \in \mathscr{P}$ and $PK=0$. Thus in the construction of (μ_ν) we shall try to make the total masses $\mu_\nu X$ as large as possible.

We start the construction by putting $\mu_0=0$. If μ_ν has been defined, satisfying (a)-(c), $\mu_{\nu+1}$ is defined as follows. Denote by $V_{\nu+1}$ the set

$$V_{\nu+1} = X_{\nu+1} \cap \complement X_\nu \cap \complement K \cap U .$$

If $V_{\nu+1}=\emptyset$ we put $\mu_{\nu+1}=\mu_\nu$.
If $V_{\nu+1} \neq \emptyset$, we define a function $g_\nu : V_{\nu+1} \to [0,1]$ by

$$g_\nu(x) = \inf\{l^{-1}-\mu_\nu(G_k^l \cap X) : x \in G_k^l\} \tag{14}$$

(thus the infimum is taken over those (k,l) for which $x \in G_k^l$). Choose $x_{\nu+1} \in V_{\nu+1}$ such that

(d) $g_\nu(x_{\nu+1}) \geqq \frac{1}{2} \sup_{x \in V_{\nu+1}} g_\nu(x)$

and put

$$\mu_{\nu+1} = \mu_\nu + g_\nu(x_{\nu+1}) \cdot \varepsilon_{x_{\nu+1}} . \tag{15}$$

Clearly (a)-(c) continue to hold when ν is replaced by $\nu+1$. It will also be important, that the construction has been carried out so that (d) holds in case $V_{\nu+1} \neq \emptyset$.

Assume now, for the purpose of an indirect proof, that $\sup \mu_\nu X < \infty$. Then we can choose a sequence (n_ν) of natural numbers such that

$$\mu(X \setminus X_{n_\nu}) < 1/2\nu; \quad \nu = 1,2,\ldots,$$

where μ denotes the measure $\mu = \sup \mu_\nu$. Put

$$y_\nu = \langle n_1 n_2 \ldots n_\nu \rangle .$$

We claim that there exists ν_0 such that

$$y_{\nu+1} \in V_{\nu+1}, \quad \forall \nu \geqq \nu_0 . \tag{16}$$

To see this, put $y = \langle n_1 n_2 \ldots \rangle$. Then $y_\nu \to y$ as $\nu \to \infty$. As $y \in \complement K$, we have $y_\nu \in \complement K$, eventually. Clearly, $y_{\nu+1} \in X_{\nu+1} \setminus X_\nu$ for all ν. By (13) we have $y_{\nu+1} \in U$ for $\nu \geqq n_1+1$. This proves (16). It is interesting to note that the only fact about the structure of the compact subsets of X used in the entire proof, is the obvious fact that a subset of X is compact in X if and only if it is compact in $[0,1]$. Also, this fact is only used in the above proof of (16).

For $\nu > \nu_0$ we have

$$\mu_{\nu+1}X - \mu_\nu X = g_\nu(x_{\nu+1}) \geqq \tfrac{1}{2} g_\nu(y_{\nu+1}) .$$

Thus, if we can prove that

$$g_\nu(y_{\nu+1}) \geqq 1/2\nu, \quad \forall \nu \geqq \nu_0 , \tag{17}$$

we will have a contradiction.

To establish (17), let $\nu \geqq \nu_0$ and k,l with $y_{\nu+1} \in G_k^{\,l}$ be fixed. Clearly, $k+l \leqq \nu$, hence $l < \nu$. By (8), and the definition of $G_k^{\,l}$, it follows that

$$G_k^{\,l} \cap (y_{l-1}, y_{l-1} + 2^{-(n_1+\ldots+n_{l-1}+k)}) = \emptyset$$

and that

$$y_{\nu+1} \in G_k^{\,l} \cap (y_{l-1}, y_{l-1} + 2^{-(n_1+\ldots+n_{l-1}+n_l-1)}) .$$

We conclude that $k > n_l - 1$ from which $n_l \leqq k+l$ follows. Utilizing this and (10), we have

$$\begin{aligned} \mu_\nu(G_k^{\,l} \cap X) &\leqq \mu_\nu(\complement X_{k+l}) \leqq \mu_\nu(\complement X_{n_l}) \\ &\leqq \mu(\complement X_{n_l}) \quad \leqq (2l)^{-1} \end{aligned}$$

hence

$$l^{-1} - \mu_\nu(G_k^{\,l} \cap X) \geqq (2l)^{-1} \geqq (2\nu)^{-1} .$$

By definition of g_ν, this argument establishes (17) and we have arrived at the desired contradiction.

Having proved this result Preiss goes further to prove the very general:

THEOREM 9.2. *Let X be a coanalytic subset of* $[0,1]$. *Then, if X is a Prohorov space, X is necessarily Polish.*

The proof is based on Theorem 9.1 and the following result, interesting in its own right: *If X is coanalytic and not Polish then X contains a copy of the rationals* $\mathbb{Q}$ *as a G_δ-subset.*

But Preiss goes even further:

THEOREM 9.3. *If the continuum hypothesis is assumed then there exists a Prohorov subspace of* $[0,1]$ *which is not Polish.*

Of the 3 theorems mentioned, the last one is by far the simplest and we now give the details needed to establish that result. (The exposition, which is different from the one in Preiss's paper, was suggested by Roy O. Davies.) We take as basic fact the result that, assuming the continuum hypothesis, there exists an uncountable subset of the unit interval $[0,1]$ having a countable intersection with every nowhere dense subset of $[0,1]$ (cf. Besicovitch [4]). Let Y denote such a set and put $X=[0,1]\setminus Y$. We shall prove that X is a non-Polish Prohorov space. Let D be a countable dense subset of Y. Let $\mathscr{P}$ be a compact subset of $\mathscr{M}_+{}^1(X)$. Since $[0,1]\setminus D$ is a G_δ-subset of $[0,1]$, and as such a Prohorov space, we can, for a given $\varepsilon>0$, find a compact subset K_0 of $[0,1]\setminus D$ such that

$$\mu(K_0 \cap X) > 1-\varepsilon, \quad \forall \mu \in \mathscr{P} .$$

Since

$$(\overline{K_0 \cap Y})^\circ \subseteq \mathring{K}_0 \cap \overline{Y} = \mathring{K}_0 \cap \overline{D} = \emptyset ,$$

$K_0 \cap Y$ is nowhere dense and hence countable. Then $[0,1]\setminus K_0 \cap Y$ is a Prohorov space and there exists a compact subset K_1 of $[0,1]\setminus K_0 \cap Y$ such that

$$\mu(K_1 \cap X) > 1-\varepsilon, \quad \forall \mu \in \mathscr{P} .$$

Clearly, $K=K_0 \cap K_1$ is a compact subset of X for which $\mu K > 1-2\varepsilon$ holds for all $\mu \in \mathscr{P}$. This argument shows that X is a Prohorov space. If X were Polish, $Y\setminus D$ would be an uncountable analytic space and as such contain a copy of the Cantor space (cf. Theorem III.6.1. of [11]) which is a contradiction. Thus X is not Polish, in fact the argument shows that X can not be analytic.

The results of Preiss also imply the following result:

THEOREM 9.4. *Every separable metrizable Prohorov space is a Baire space.*

We remark that the result does not hold without the metrizability assumption – an explicit example of a non-Baire Prohorov space with the additional feature that every subspace of it is a Prohorov space was constructed by D. Monrad.

The author found it natural to ask whether a metrizable space is a Prohorov space if and only if every G_δ-subset of it is a Baire space. However, D. H. Fremlin has recently (private communication, december 1972) demolished this conjecture by exibiting an example, using the Sierpiński-Balkema-Leger type of construction, of a metrizable non-Prohorov space every G_δ-subset of which is a Baire space.

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REFERENCES

1. A. D. Alexandroff, *Additive set functions in abstract spaces*, Mat. Sb. 8 (1940), 307–348, 9 (1941), 563–628, 13 (1943), 169–238.
2. A. V. Arhangel'skiĭ, *Bicompact sets and the topology of spaces*, Trans. Moscow Math. Soc. 13 (1965), 1–26.
3. A. A. Balkema, *Tight and compact sets of probability measures on metric spaces*, to appear in the transactions of the sixth Prague conference, Prague 1971.
4. A. S. Besicovitch, *Concentrated and rarified sets of points*, Acta Math. 62 (1933), 289–300.
5. G. Choquet, *Sur les ensemble uniformément négligeables*, Séminaire Choquet (Initiation à l'analyse), 9ᵉ année no. 6 (1969/1970).
6. J. P. R. Christensen, *On some properties of Effros Borel Structure on spaces of closed subsets*, Math. Ann. 195 (1971), 1–23.
7. R. O. Davies, *A non-Prohorov space*, Bull. London Math. Soc. 3 (1971), 341–342.
8. X. Fernique, *Processus linéaires, processus généralisés*, Ann. Inst. Fourier (Grenoble) 17 (1967), 1–92.
9. D. H. Fremlin, J. H. Garling, and R. G. Haydon, *Bounded measures on topological spaces*, Proc. London Math. Soc. 25 (1972), 115–136.
10. B. R. Gelbaum, and J. M. H. Olmsted, *Counter examples in analysis*, Holden-Day, San Francisco, 1964.
11. J. Hoffmann-Jørgensen, *The theory of analytic spaces*, (various publication series no. 10) Matematisk Institut, Århus, 1970.
12. J. Hoffmann-Jørgensen, *Weak compactness and tightness of subsets of $M(X)$*, Math. Scand. 31 (1972), 127–150.
13. J. L. Kelley, *General Topology*, D. van Nostrand, New York, 1955.
14. L. LeCam, *Convergence in distribution of stochastic processes*, Univ. California publ. Statist. 2 (1957), 207–236.
15. D. Preiss, *Metric spaces in which Prohorovs theorem is not valid*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 23 (1972), 245–253.
16. YU. V. Prohorov, *Convergence of random processes and limit theorems in probability theory*, Theor. Probability Appl. 1 (1956), 159–214.
17. W. Sierpiñski, *Sur une propriété topologique des ensembles dénombrables denses ensoi*, Fund. Math. 1 (1920), 159–214.
18. W. Sierpiñski, *General Topology*, University of Toronto press, Toronto, 1956.
19. F. Topsøe, *Compactness in spaces of measures*, Studia Math. 36 (1970), 195–212.

20. F. Topsøe, *Topology and measure,* (Lecture notes in mathematics 133), Springer-Verlag, Berlin, Heidelberg, New York, 1970.
21. F. Topsøe, *Measure spaces connected by correspondences,* Math. Scand. 30 (1972), 5–45.
22. V. S. Varadarajan, *Measures on topological spaces,* Amer. Math. Soc. Transl. ser. II 48 (1965), 161–228 (Mat. Sb. 55 (1961)).

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