IMRN International Mathematics Research Notices
2004, No. 23

Compactness for Yamabe Metrics in Low Dimensions

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1 Introduction

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. A conformal metric to g is a metric $\tilde{g}$ which expresses as a smooth positive function multiplied by g. The conformal class $[g]$ of g is the set of such metrics. If $\tilde{g}$ is a conformal metric to g, we write that $\tilde{g} = u^{4/(n-2)}g$, where $u \in C^\infty(M)$, $u > 0$. The scalar curvatures S_g and $S_{\tilde{g}}$ of g and $\tilde{g}$ are then related by the equation

$$\Delta_g u + c_n S_g u = c_n S_{\tilde{g}} u^{2^\star - 1}, \tag{1.1}$$

where $\Delta_g = -\mathrm{div}_g(\nabla)$ is the Laplace-Beltrami operator, $2^\star = 2n/(n-2)$ is critical from the Sobolev viewpoint, and $c_n = (n-2)/4(n-1)$. The problem of finding a metric conformal to a given one with a constant scalar curvature is known as the Yamabe problem (see Yamabe [29]). The Yamabe invariant μ_g is defined by

$$\mu_g = \inf_{\tilde{g} \in [g]} V_{\tilde{g}}^{-(n-2)/n} \int_M S_{\tilde{g}} dv_{\tilde{g}}, \tag{1.2}$$

where $V_{\tilde{g}}$ denotes the volume of M with respect to $\tilde{g}$. Trudinger [28] solved the Yamabe problem for nonpositive Yamabe invariant μ_g. In this case, the solution is unique up to multiplication by a constant scale factor if the scalar curvature is not normalized. The positive case $\mu_g > 0$ is more intricate and the problem reduces to finding a smooth positive solution of the Yamabe equation

$$\Delta_g u + c_n S_g u = u^{2^\star - 1}. \tag{1.3}$$

Received 28 October 2003.
Communicated by Emmanuel Hebey.

The problem was solved in large dimensions when the manifold is not conformally flat by Aubin [2] and in the more difficult remaining cases by Schoen [20]. Moreover, there are examples of manifolds for which (1.3) possesses multiple solutions (see Hebey and Vaugon [13], Pollack [19], and Schoen [23]).

Schoen considered in [22, 23] the fascinating question of the compactness of Yamabe metrics. Let (M, g) be a smooth compact manifold of dimension $n \geq 3$ with $\mu_g > 0$. Let (u_i) be any sequence of smooth positive solutions of equations like

$$\Delta_g u_i + c_n S_g u_i = u_i^{q_i - 1}, \tag{1.4}$$

where $2 + \varepsilon_0 \leq q_i \leq 2^\star$, with $\varepsilon_0 > 0$ fixed. In [22], when the manifold is not the standard sphere (a necessary assumption), Schoen announced that the u_i's, if bounded in $H_1^2(M)$, are in fact bounded in $C^{2,\alpha}(M)$, $\alpha \in (0, 1)$, and thus precompact in $C^2(M)$. Here and below, $H_1^2(M)$ is the Sobolev space of functions in L^2 with one derivative in L^2. Schoen proved the result when the manifold is conformally flat in [22]. Then, still in the conformally flat case, Schoen proved in [23] that one can get rid of the bound on the H_1^2-norm. The proof in [23] uses the injectivity of the developing map and the Alexandrov method. In [21], Schoen also gave strong indications for the proof of the result for arbitrary manifolds. We refer also to Schoen and Zhang [27]. In [7], we proved compactness for sequences (u_i) of solutions of equations like

$$\Delta_g u_i + a_i u_i = u_i^{2^\star - 1} \tag{1.5}$$

when the u_i's are bounded in $H_1^2(M)$, and (a_i) is a converging sequence of functions on M. We refer to [7] for a precise statement and point out the fact that the H_1^2-bound is necessary for such general equations (see [9]). The proof in [7] is based on the very general C^0-theory for blowup developed by Druet, Hebey, and Robert in [10].

In this paper, we are interested in proving compactness results on general compact n-manifolds, $3 \leq n \leq 5$, when we do not assume any H_1^2-bound on the solutions. We follow Schoen's approach [21] and provide a detailed proof of his theorem. The 3-dimensional case was already written by Li and Zhu [18]. We let (M, g) be a smooth compact manifold of dimension $3 \leq n \leq 5$ and let (a_i) be a sequence of smooth positive functions on M such that

$$\lim_{i \to +\infty} a_i = a_\infty \quad \text{in } C^2(M), \tag{1.6}$$

where $a_\infty \in C^2(M)$ is such that the operator $\Delta_g + a_\infty$ is coercive, namely, such that its energy controls the H_1^2-norm. In the positive case of the Yamabe problem we discussed above, the conformal Laplacian $\Delta_g + c_n S_g$ in (1.3) is coercive. Also, we let (q_i) be a sequence of positive real numbers in $[2+\varepsilon_0; 2^\star]$, with $\varepsilon_0 > 0$ fixed, and consider equations like

$$\Delta_g u + a_i u = u^{q_i - 1}. \tag{1.7}$$

Equation (1.7) reduces to the geometric equation (1.3) when $a_i \equiv c_n S_g$ and $q_i = 2^\star$. A sequence (u_i) is said to be a sequence of solutions of (1.7) if for any i, u_i is a solution of (1.7). We prove here the following result.

Theorem 1.1. Let (M, g) be a smooth compact manifold of dimension $3 \le n \le 5$ with $\mu_g > 0$. We let (a_i) and (q_i) be as above. We assume that $a_i \le c_n S_g$ for all i and that (M, g) is not conformally diffeomorphic to the standard sphere if $a_\infty \equiv c_n S_g$ and $q_i \to 2^\star$ as $i \to +\infty$. Then compactness holds for (1.7) in the sense that any sequence (u_i) of solutions of (1.7) is bounded in $C^{2,\alpha}(M)$, $\alpha \in (0, 1)$, and thus precompact in $C^2(M)$. In particular, when $3 \le n \le 5$ and the manifold is not the standard sphere, the set of Riemannian metrics with constant scalar curvature 1 in a given conformal class is precompact in the C^2-topology. □

Note that compactness for (1.7) does not hold in general if the condition $a_i \le c_n S_g$ is false (see [6, 9, 15]). Independently, note that another proof of the theorem when $n = 4$, $a_i = c_n S_g$, and $q_i = 2^\star$ for all i follows from the combination of Druet [7] and Li and Zhang [17]. As a general remark, the blowup analysis we develop below, and in the related works of Druet [7] and Druet, Hebey, and Robert [10], is valid in any dimension. The proof we present here in dimensions $n = 3, 4, 5$, which, as already mentioned, mainly follows the approach developed by Schoen in [21, 22], should easily extend to higher dimensions with the difficulty that, in the final computation where the Pohozaev identity is involved, one more term (the Weyl tensor and then its derivatives) arises with each pair of dimensions $n = [2k, 2k+1]$, $k \ge 3$. The case of dimensions $n = 6, 7$ should follow from the material we develop here; the case of dimensions $n = 8, 9$ will be more involved; the case of dimensions $n = 10, 11$ is again more involved, and so on. The difficult problem would be to do the compactness for arbitrary dimensions without assumptions on the Weyl tensor. That pairs-of-dimensions occurrence was first noticed by Schoen [23, 24]. A very clear explanation of the phenomenon is given by Hebey and Vaugon [14].

The paper is organized as follows. In Section 2, we derive various asymptotic estimates for an arbitrary sequence (u_i) of solutions of equations like (1.7) around one of its possible concentration points. This section is divided into several claims. The first two ones are rather standard now: they provide fine asymptotic pointwise estimates on

(u_i) in a suitable neighborhood of a concentration point. In this neighborhood, u_i is controlled from above by a standard bubble. Claim 2.3 is purely technical and provides a rough estimate on the speed of convergence of q_i to $2^\star$ in the case of blowup. In Claims 2.4 and 2.5, we carry out a projection of u_i on the set of standard bubbles so as to write suitably u_i as the sum of a standard bubble and a rest. And we give sharp estimates on the H_1^2-norm of this rest as $i \to +\infty$. This technique was initiated in the Euclidean context by Adimurthi, Pacella, and Yadava [1]. Associated to strong pointwise estimates (like those of Claim 2.2), as in [10], it revealed to be powerful in a Riemannian setting (see, e.g., Druet and Hebey [8]). At last, Claims 2.6 and 2.7 make an intensive use of the Pohozaev identity derived in the appendix. The restriction on the dimension of the manifold appears in the computations involved in these claims (see also Remark 3.6). We get estimates relating the weight of the concentration point, the size of the neighborhood of this concentration point, where u_i is controlled by a standard bubble, and the underlying geometry of the manifold. Section 3 is devoted to the proof of the theorem. We prove the theorem by contradiction assuming that some sequence of solutions of equation (1.7) develops a concentration phenomenon. We first prove that concentration points are necessarily isolated. Such a fact follows mainly from Claim 2.7. Then the u_i's are bounded in $H_1^2(M)$ and we are in some sense back to Druet [7], with a slight difference from [7], where $q_i = 2^\star$ for all i. Compactness with the H_1^2-bound—which relies essentially on Claim 2.6 and thus on the Pohozaev identity—is proved at the end of Section 3.

2 Pointwise estimates around a concentration point

We let (M, g) be a smooth compact Riemannian manifold of dimension $3 \le n \le 5$ and we let (a_i) be a sequence of smooth functions on M such that (1.6) holds and such that

$$a_i \le c_n S_g. \tag{2.1}$$

We let also $q_i \in [2 + \varepsilon_0; 2^\star]$, with $\varepsilon_0 > 0$ fixed. We need to consider sequences of solutions of a slightly more general equation than (1.7). This will allow us to perform a suitable conformal change of the metric in Section 3. Thus we let $\varphi \in C^\infty(M)$, $\varphi > 0$, and we consider (u_i) a sequence of solutions of

$$\Delta_g u_i + a_i u_i = \varphi^{2^\star - q_i} u_i^{q_i - 1} \quad \text{in } M. \tag{2.2}$$

Throughout this section, we assume that there exist a sequence (x_i) of local maxima of u_i in M and a bounded sequence (ρ_i) of positive real numbers such that the following assertions hold:

(H1) $\rho_i u_i(x_i)^{(q_i-2)/2} \to +\infty$ as $i \to +\infty$;

(H2) there exists $C_0 > 0$ independent of i such that

$$d_g(x_i, x)^{2/(q_i-2)} u_i(x) \le C_0 \quad \text{in } B_{x_i}(3\rho_i). \tag{2.3}$$

We divide this section into many claims, being more and more precise in the estimates on u_i around x_i. We let $\mu_i > 0$ be defined by

$$u_i(x_i) = \mu_i^{-2/(q_i-2)} \tag{2.4}$$

so that

$$\mu_i \longrightarrow 0, \quad \frac{\rho_i}{\mu_i} \longrightarrow +\infty, \quad \text{as } i \longrightarrow +\infty, \tag{2.5}$$

thanks to assumption (H1). Claim 2.1 is really standard now.

Claim 2.1. We have that, after passing to a subsequence,

$$\mu_i^{2/(q_i-2)} u_i\left(\exp_{x_i}(\mu_i x)\right) \longrightarrow U(x) \tag{2.6}$$

in $C^2_{loc}(\mathbb{R}^n)$ as $i \to +\infty$, where

$$U(x) = \left(1 + \frac{|x|^2}{n(n-2)}\right)^{-(n-2)/2}. \tag{2.7}$$

Moreover, we have that $q_i \to 2^\star$ as $i \to +\infty$. □

Proof of Claim 2.1. We let (z_i) be a sequence of points in $\overline{B_{x_i}(\rho_i)}$ such that

$$u_i(z_i) = \sup_{B_{x_i}(\rho_i)} u_i, \tag{2.8}$$

and we set

$$u_i(z_i) = \tilde{\mu}_i^{-2/(q_i-2)}. \tag{2.9}$$

Thanks to (H2), we have that

$$d_g(x_i, z_i) = O(\tilde{\mu}_i). \tag{2.10}$$

Fix $0 < \delta < \text{inj}(M)$, with $\text{inj}(M)$ the injectivity radius of M. We set for $x \in B_0(\delta\tilde{\mu}_i^{-1})$ the Euclidean ball of center 0 and radius $\delta\tilde{\mu}_i^{-1}$,

$$\tilde{u}_i(x) = \tilde{\mu}_i^{2/(q_i-2)} u_i\big(\exp_{z_i}(\tilde{\mu}_i x)\big), \qquad \tilde{g}_i(x) = \exp_{z_i}^\star g\big(\tilde{\mu}_i x\big). \tag{2.11}$$

Since $\tilde{\mu}_i \to 0$ as $i \to +\infty$, we have that $\tilde{g}_i \to \xi$ in $C^2_{loc}(\mathbb{R}^n)$, with ξ the Euclidean metric. Independently, $\tilde{u}_i$ verifies

$$\begin{aligned} &\Delta_{\tilde{g}_i}\tilde{u}_i + \tilde{\mu}_i^2 a_i\big(\exp_{z_i}(\tilde{\mu}_i x)\big)\tilde{u}_i = \varphi\big(\exp_{z_i}(\tilde{\mu}_i x)\big)^{2^\star - q_i}\tilde{u}_i^{q_i-1} \quad \text{in } B_0\big(\delta\tilde{\mu}_i^{-1}\big),\\ &\tilde{u}_i(0) = \sup_{(1/\tilde{\mu}_i)\exp_{z_i}^{-1}(B_{x_i}(\rho_i))} \tilde{u}_i = 1. \end{aligned} \tag{2.12}$$

Thanks to (2.4), (2.5), (2.8), (2.9), and (2.10), we have that

$$\frac{1}{\tilde{\mu}_i}\exp_{z_i}^{-1}\big(B_{x_i}(\rho_i)\big) \longrightarrow \mathbb{R}^n \quad \text{as } i \longrightarrow +\infty. \tag{2.13}$$

It follows from the standard elliptic theory (see, e.g., [12]) that after passing to a subsequence,

$$\tilde{u}_i \longrightarrow U \text{ in } C^2_{loc}\big(\mathbb{R}^n\big) \quad \text{as } i \longrightarrow +\infty, \tag{2.14}$$

where

$$\Delta_\xi U = \varphi\big(z_0\big)^{2^\star - q_0} U^{q_0-1} \text{ in } \mathbb{R}^n, \quad U(0) = 1,\ 0 \le U \le 1 \text{ in } \mathbb{R}^n, \tag{2.15}$$

$q_0 = \lim_{i\to+\infty} q_i$, and $z_0 = \lim_{i\to+\infty} z_i$. Thanks to [11], it is possible if and only if $q_0 = 2^\star$, which proves the second assertion of Claim 2.1, and thanks to [5], we have that

$$U(x) = \left(1 + \frac{|x|^2}{n(n-2)}\right)^{1-n/2}. \tag{2.16}$$

Thus we have obtained that

$$\lim_{i\to+\infty} \tilde{\mu}_i^{2/(q_i-2)} u_i\big(\exp_{z_i}(\tilde{\mu}_i x)\big) = U(x) \quad \text{in } C^2_{loc}\big(\mathbb{R}^n\big). \tag{2.17}$$

Thanks to (2.10), we have that, up to the extraction of a new subsequence,

$$\frac{1}{\tilde{\mu}_i}\exp_{z_i}^{-1}\big(x_i\big) \longrightarrow x_0 \quad \text{as } i \longrightarrow +\infty, \tag{2.18}$$

for some $x_0 \in \mathbb{R}^n$. Moreover, since x_i is a local maximum of u_i for all i, we get that x_0 is a local maximum of U. This implies $x_0 = 0$. In turn, this clearly implies that $\tilde{\mu}_i/\mu_i \to 1$ as $i \to +\infty$. Claim 2.1 easily follows. ∎

For $0 \le r \le 3\rho_i$, we set

$$\psi_i(r) = r^{2/(q_i-2)} \frac{\int_{\partial B_{x_i}(r)} u_i \, d\sigma_g}{\int_{\partial B_{x_i}(r)} d\sigma_g}, \tag{2.19}$$

where $d\sigma_g$ denotes the $(n-1)$-dimensional Riemannian measure. If we let (X_i) be a sequence of positive real numbers converging to some $X > 0$ as $i \to +\infty$, it is easily checked, thanks to Claim 2.1, that

$$\begin{aligned} \psi_i(X_i\mu_i) &= \left(\frac{X}{1 + \frac{1}{n(n-2)} X^2} \right)^{(n-2)/2} + o(1), \\ \psi_i'(X_i\mu_i) &= \frac{n-2}{2} \left(\frac{X}{1 + \frac{1}{n(n-2)} X^2} \right)^{n/2} \left(\frac{1}{X^2} - \frac{1}{n(n-2)} \right) + o(1). \end{aligned} \tag{2.20}$$

We let $R_0 \ge 2\sqrt{n(n-2)}$ and we define r_i by

$$r_i = \max\left\{ r \in \left[R_0\mu_i; \rho_i\right] : \psi_i'(s) \le 0 \text{ for } s \in \left[R_0\mu_i; r\right] \right\}. \tag{2.21}$$

It follows from (2.20) that

$$\frac{r_i}{\mu_i} \longrightarrow +\infty \quad \text{as } i \longrightarrow +\infty. \tag{2.22}$$

Claim 2.2 provides strong pointwise estimates on u_i in $B_{x_i}(2r_i)$.

Claim 2.2. There exists $C_1 > 0$ such that for any i,

$$\begin{aligned} &u_i(x) \le C_1 \mu_i^{n-2-2/(q_i-2)} d_g\left(x_i, x\right)^{2-n} \quad \text{in } B_{x_i}\left(2r_i\right) \backslash \left\{x_i\right\}, \\ &\left|\nabla u_i(x)\right| \le C_1 \mu_i^{n-2-2/(q_i-2)} d_g\left(x_i, x\right)^{1-n} \quad \text{in } B_{x_i}\left(2r_i\right) \backslash \left\{x_i\right\}. \end{aligned} \tag{2.23}$$

□

Proof of Claim 2.2. First, we note that it follows from assumption (H2) and from Harnack's inequality (see, e.g., [12]) that there exists $C_2 > 1$ such that for all $r \in [0; (5/2)\rho_i]$ and all i,

$$\frac{1}{C_2} \max_{\partial B_{x_i}(r)} u_i \le r^{-2/(q_i-2)} \psi_i(r) \le C_2 \min_{\partial B_{x_i}(r)} u_i. \tag{2.24}$$

As a consequence, we can write, thanks to (2.20) and (2.21), that for all $R > R_0$, all $r \in [R\mu_i; r_i]$, all i, and all $x \in \partial B_{x_i}(r)$,

$$
\begin{aligned}
d_g(x_i, x)^{2/(q_i-2)} u_i(x) &\leq C_2 \psi_i(r) \\
&\leq C_2 \psi_i(R\mu_i) \\
&= C_2 \left(\frac{R}{1 + \frac{1}{n(n-2)} R^2} \right)^{(n-2)/2} + o(1).
\end{aligned}
\tag{2.25}
$$

Thus we have that

$$
\sup_{B_{x_i}(r_i) \setminus B_{x_i}(R\mu_i)} \left(d_g(x_i, x)^2 u_i(x)^{q_i-2} \right) = \varepsilon(R) + o(1),
\tag{2.26}
$$

where $\varepsilon(R) \to 0$ as $R \to +\infty$. We introduce the operator

$$
L_i : u \longmapsto \Delta_g u + a_i u - \varphi^{2^\star - q_i} u_i^{q_i-2} u
\tag{2.27}
$$

which verifies the maximum principle since $L_i u_i = 0$ and $u_i > 0$ (see [3]). We let G_i be the Green function of $\Delta_g + a_i$. Standard properties of the Green function (see, e.g., [10, Appendix A]) give that there exist $\tilde{\rho} > 0$, $C_3 > 1$, and $C_4 > 1$ such that for all $x, y \in M$, $x \neq y$,

$$
d_g(x, y) \leq \tilde{\rho} \Longrightarrow
\begin{cases}
\dfrac{1}{C_3} \leq d_g(x, y)^{n-2} G_i(x, y) \leq C_3, \\[2mm]
\dfrac{1}{C_4} \leq d_g(x, y) \dfrac{\left| \nabla G_i(x, y) \right|_g}{G_i(x, y)} \leq C_4.
\end{cases}
\tag{2.28}
$$

For $0 < \sigma < 1$, we write that

$$
\begin{aligned}
L_i G_i(x_i, \cdot)^\sigma &= G_i(x_i, \cdot)^\sigma \left[(1-\sigma) a_i + \sigma(1-\sigma) \frac{\left| \nabla G_i(x_i, \cdot) \right|_g^2}{G_i(x_i, \cdot)^2} - \varphi^{2^\star - q_i} u_i^{q_i-2} \right] \\
&\geq G_i(x_i, \cdot)^\sigma \left[(1-\sigma) \min_M a_i + \frac{\sigma(1-\sigma)}{C_4^2 d_g(x_i, \cdot)^2} - \varphi^{2^\star - q_i} u_i^{q_i-2} \right]
\end{aligned}
\tag{2.29}
$$

in $B_{x_i}(\tilde{\rho}) \setminus \{x_i\}$ thanks to (2.28). We then obtain, thanks to (2.26) and to the fact that $q_i \to 2^\star$ as $i \to +\infty$ (Claim 2.1), that

$$
L_i G_i(x_i, \cdot)^\sigma \geq \frac{G_i(x_i, \cdot)^\sigma}{d_g(x_i, \cdot)^2} \left[(1-\sigma) d_g(x_i, \cdot)^2 \min_M a_i + \frac{\sigma(1-\sigma)}{C_4^2} - \varepsilon(R) + o(1) \right]
\tag{2.30}
$$

in $(B_{x_i}(\tilde\rho) \cap B_{x_i}(r_i)) \backslash B_{x_i}(R\mu_i)$. Fix $0 < \nu < 1/2$. We choose $0 < \bar\rho < \tilde\rho$ such that

$$\bar\rho^2 \min_M a_i \geq -\frac{\nu}{2C_4^2} \tag{2.31}$$

for all i. Note that this is possible thanks to (1.6). Applying (2.30) with $\sigma = \nu$ and $\sigma = 1-\nu$, it is easily checked that we can choose $R_\nu > R_0$ large enough such that

$$L_i G_i(x_i,\cdot)^\nu \geq 0, \quad L_i G_i(x_i,\cdot)^{1-\nu} \geq 0 \quad \text{in } B_{x_i}(\bar r_i)\backslash B_{x_i}(R_\nu \mu_i) \tag{2.32}$$

for i large, where $\bar r_i$ is given by

$$\bar r_i = \min\{r_i; \bar\rho\}. \tag{2.33}$$

Thanks to Claim 2.1 and (2.28), we have that

$$u_i \leq \left(C_3 R_\nu^{n-2}\right)^{1-\nu} \mu_i^{(n-2)(1-\nu)-2/(q_i-2)} G_i(x_i,\cdot)^{1-\nu} \quad \text{on } \partial B_{x_i}(R_\nu \mu_i), \tag{2.34}$$

while

$$u_i \leq C_3^\nu \bar r_i^{(n-2)\nu}\left(\sup_{\partial B_{x_i}(\bar r_i)} u_i\right) G_i(x_i,\cdot)^\nu \quad \text{on } \partial B_{x_i}(\bar r_i). \tag{2.35}$$

Applying the maximum principle, we thus get, thanks to (2.32), to the fact that $L_i u_i = 0$ in M, and to these last two relations, that

$$\begin{aligned} u_i \leq\ & \left(C_3 R_\nu^{n-2}\right)^{1-\nu} \mu_i^{(n-2)(1-\nu)-2/(q_i-2)} G_i(x_i,\cdot)^{1-\nu} \\ & + C_3^\nu \bar r_i^{(n-2)\nu}\left(\sup_{\partial B_{x_i}(\bar r_i)} u_i\right) G_i(x_i,\cdot)^\nu \end{aligned} \tag{2.36}$$

in $B_{x_i}(\bar r_i)\backslash B_{x_i}(R_\nu\mu_i)$, which gives with (2.28) that

$$\begin{aligned} u_i \leq\ & \left(C_3^2 R_\nu^{n-2}\right)^{1-\nu} \mu_i^{(n-2)(1-\nu)-2/(q_i-2)} d_g(x_i,\cdot)^{-(n-2)(1-\nu)} \\ & + C_3^{2\nu} \bar r_i^{(n-2)\nu}\left(\sup_{\partial B_{x_i}(\bar r_i)} u_i\right) d_g(x_i,\cdot)^{-(n-2)\nu} \end{aligned} \tag{2.37}$$

in $B_{x_i}(\bar r_i)\backslash B_{x_i}(R_\nu\mu_i)$. Let $0 < \beta < 1$. Thanks to definitions (2.21) and (2.33) of r_i and $\bar r_i$, respectively, and to (2.24), we can write that

$$\begin{aligned} \max_{\partial B_{x_i}(\bar r_i)} u_i &\leq C_2 \bar r_i^{-2/(q_i-2)} \psi_i(\bar r_i) \\ &\leq C_2 \bar r_i^{-2/(q_i-2)} \psi_i(\beta \bar r_i) \\ &\leq C_2 \beta^{2/(q_i-2)} \max_{\partial B_{x_i}(\beta \bar r_i)} u_i, \end{aligned} \tag{2.38}$$

which leads with (2.37) to

$$\max_{\partial B_{x_i}(\bar{r}_i)} u_i \le C_2\left(C_3^2 R_\nu^{n-2}\right)^{1-\nu} \beta^{2/(q_i-2)-(n-2)(1-\nu)} \bar{r}_i^{-(n-2)(1-\nu)} \mu_i^{(n-2)(1-\nu)-2/(q_i-2)} + C_2 C_3^{2\nu} \beta^{2/(q_i-2)-(n-2)\nu} \max_{\partial B_{x_i}(\bar{r}_i)} u_i. \tag{2.39}$$

Since $q_i \to 2^\star$ and $\nu < 1/2$, we can choose $\beta > 0$ small enough such that

$$C_2 C_3^{2\nu} \beta^{2/(q_i-2)-(n-2)\nu} \le \frac{1}{2} \tag{2.40}$$

for i large in order to obtain that

$$\max_{\partial B_{x_i}(\bar{r}_i)} u_i \le 2C_2\left(C_3^2 R_\nu^{n-2}\right)^{1-\nu} \beta^{2/(q_i-2)-(n-2)(1-\nu)} \bar{r}_i^{-(n-2)(1-\nu)} \mu_i^{(n-2)(1-\nu)-2/(q_i-2)}. \tag{2.41}$$

Coming back to (2.37) with this last relation and using the fact that $d_g(x_i, x) \le \bar{r}_i$ in $B_{x_i}(\bar{r}_i)$, we get the existence of some $C_\nu > 0$ such that

$$u_i \le C_\nu \mu_i^{(n-2)(1-\nu)-2/(q_i-2)} d_g\left(x_i, \cdot\right)^{-(n-2)(1-\nu)} \tag{2.42}$$

in $B_{x_i}(\bar{r}_i) \backslash B_{x_i}(R_\nu \mu_i)$. Since this relation obviously holds in $B_{x_i}(R_\nu \mu_i) \backslash \{x_i\}$ thanks to Claim 2.1 and in $B_{x_i}(r_i) \backslash B_{x_i}(\bar{r}_i)$ thanks to (2.21), (2.24), and (2.33), we have obtained the following result: for any $0 < \nu < 1/2$, there exists $C_\nu > 0$ such that

$$u_i \le C_\nu \mu_i^{(n-2)(1-\nu)-2/(q_i-2)} d_g\left(x_i, \cdot\right)^{-(n-2)(1-\nu)} \quad \text{in } B_{x_i}\left(r_i\right) \backslash \left\{x_i\right\}, \tag{2.43}$$

for all i. We claim now that the following assertion implies Claim 2.2:

(A) for any sequence (s_i), $0 \le s_i \le r_i$, $s_i \to 0$ as $i \to +\infty$,

$$\psi_i(s_i) \left(\frac{s_i}{\mu_i}\right)^{n-2-2/(q_i-2)} = O(1). \tag{2.44}$$

Indeed, let (y_i) be a sequence of points in $B_{x_i}(r_i) \backslash B_{x_i}(2R_0 \mu_i)$. Thanks to (2.24), we have that

$$u_i(y_i) \le C_2 d_g\left(x_i, y_i\right)^{-2/(q_i-2)} \psi_i\left(d_g\left(x_i, y_i\right)\right). \tag{2.45}$$

We let $0 \le s_i \le d_g(x_i, y_i)$, $s_i \to 0$ as $i \to +\infty$, to be chosen later. Then definition (2.21) of r_i implies that

$$u_i(y_i) \le C_2 d_g(x_i, y_i)^{-2/(q_i-2)} \psi_i(s_i). \tag{2.46}$$

Applying (A), we get that

$$\mu_i^{-(n-2)+2/(q_i-2)} d_g(x_i, y_i)^{n-2} u_i(y_i) = O\left(\left(\frac{d_g(x_i, y_i)}{s_i}\right)^{n-2-2/(q_i-2)}\right). \tag{2.47}$$

Assume by contradiction that the left-hand side of this equation goes to $+\infty$ as $i \to +\infty$. Then it will always be possible to choose a sequence (s_i), $0 \le s_i \le d_g(x_i, y_i)$, $s_i \to 0$ as $i \to +\infty$, which violates the above equation. Just take, for instance, s_i such that

$$s_i^{n-2-2/(q_i-2)} = \frac{d_g(x_i, y_i)^{n-2-2/(q_i-2)}}{\sqrt{\mu_i^{-(n-2)+2/(q_i-2)} d_g(x_i, y_i)^{n-2} u_i(y_i)}}. \tag{2.48}$$

Thus we have proved that

$$\mu_i^{-(n-2)+2/(q_i-2)} d_g(x_i, y_i)^{n-2} u_i(y_i) = O(1) \tag{2.49}$$

for all sequences (y_i) of points in $B_{x_i}(r_i) \backslash B_{x_i}(2R_0\mu_i)$. Since the first estimate of Claim 2.2 obviously holds in $B_{x_i}(2R_0\mu_i) \backslash \{0\}$ thanks to Claim 2.1, we have proved that assertion (A) implies the first estimate of Claim 2.2 in $B_{x_i}(r_i) \backslash \{0\}$. Then Harnack's inequality gives, thanks to (H2), that the first estimate of Claim 2.2 holds in $B_{x_i}((5/2)r_i) \backslash \{0\}$, while standard elliptic theory leads then to the second estimate of Claim 2.2. The rest of the proof is devoted to the proof of (A). Let (s_i) be a sequence of real numbers, $0 \le r_i \le s_i$, $s_i \to 0$ as $i \to +\infty$. We assume that

$$\frac{s_i}{\mu_i} \longrightarrow +\infty \quad \text{as } i \longrightarrow +\infty. \tag{2.50}$$

Otherwise, (A) obviously holds for (s_i) thanks to (2.20). We set for $x \in B_0(1)$ the Euclidean ball of center 0 and radius 1,

$$\bar{u}_i(x) = s_i^{2/(q_i-2)} u_i\left(\exp_{x_i}(s_i x)\right), \qquad \bar{g}_i(x) = \exp_{x_i}^{\star} g(s_i x). \tag{2.51}$$

Then

$$\Delta_{\bar{g}_i} \bar{u}_i + s_i^2 a_i\left(\exp_{x_i}(s_i x)\right)\bar{u}_i = \varphi\left(\exp_{x_i}(s_i x)\right)^{2^\star - q_i} \bar{u}_i^{q_i - 1} \quad \text{in } B_0(1). \tag{2.52}$$

Thanks to (2.43) and (2.50), we have that

$$\bar{u}_i \longrightarrow 0 \quad \text{in } C^0_{loc}\left(B_0(1)\backslash\{0\}\right). \tag{2.53}$$

Then standard elliptic theory gives the existence of some $\lambda_i \to +\infty$ such that

$$\lambda_i \bar{u}_i \longrightarrow H \quad \text{in } C^2_{loc}\left(B_0(1)\backslash\{0\}\right), \tag{2.54}$$

where $H \not\equiv 0$ verifies $\Delta_\xi H = 0$ in $B_0(1)\backslash\{0\}$. Note here that $s_i \to 0$ as $i \to +\infty$ and that, as a consequence, $\bar{g}_i \to \xi$ in $C^2(B_0(1))$. By definition (2.21) of r_i, we also have that

$$r^{n/2-1} \frac{\displaystyle\int_{\partial B_0(r)} H \, d\sigma_\xi}{\displaystyle\int_{\partial B_0(r)} d\sigma_\xi} \tag{2.55}$$

is nonincreasing in $(0; 1]$ so that H must be singular at the origin. Thus we can write H as

$$H = \frac{\lambda}{|x|^{n-2}} + h, \tag{2.56}$$

where $h \in C^2(B_0(1))$ is harmonic and $\lambda > 0$ is some constant. We let $\eta \in C^\infty(B_0(1))$ be the first positive eigenfunction of the Euclidean Laplacian in the unit ball with Dirichlet boundary condition, that is, $\Delta_\xi \eta = \lambda_1 \eta$, $\eta > 0$ in $B_0(1)$, with λ_1 the first Dirichlet eigenvalue of Δ_ξ. We multiply equation (2.52) by η and integrate on $B_0(\delta)$, $0 < \delta < 1$. This leads after integration by parts to the following:

$$\begin{aligned} &\left[\int_{\partial B_0(\delta)} \bar{u}_i \partial_\nu \eta \, d\sigma_{\bar{g}_i} - \int_{\partial B_0(\delta)} \eta \partial_\nu \bar{u}_i \, d\sigma_{\bar{g}_i}\right] \\ &\quad = \int_{B_0(\delta)} \varphi\left(\exp_{x_i}(s_i x)\right)^{2^\star - q_i} \bar{u}_i^{q_i - 1} \eta \, dv_{\bar{g}_i} \\ &\qquad - \int_{B_0(\delta)} \left(\Delta_{\bar{g}_i} \eta + s_i^2 a_i\left(\exp_{x_i}(s_i x)\right)\eta\right)\bar{u}_i \, dv_{\bar{g}_i}. \end{aligned} \tag{2.57}$$

Since $\bar{g}_i \to \xi$ in $C^2(B_0(1))$ and $s_i \to 0$ as $i \to +\infty$, we obtain that

$$\Delta_{\bar{g}_i} \eta + s_i^2 a_i\left(\exp_{x_i}(s_i x)\right)\eta > 0 \tag{2.58}$$

in $B_0(\delta)$ for i large. Thus the above equation leads, thanks to (2.54) and to the fact that $q_i \to 2^\star$ as $i \to +\infty$ (Claim 2.1), to

$$\frac{1}{\lambda_i}\left[\int_{\partial B_0(\delta)} H\partial_\nu \eta \, d\sigma_\xi - \int_{\partial B_0(\delta)} \eta \partial_\nu H \, d\sigma_\xi + o(1)\right] \le (1+o(1)) \int_{B_0(\delta)} \bar{u}_i^{q_i-1} \eta \, dv_{\bar{g}_i}. \tag{2.59}$$

It is easily checked, thanks to Claim 2.1 and (2.43) (applied with $\nu > 0$ small enough), that

$$\int_{B_0(\delta)} \bar{u}_i^{q_i-1} \eta \, dv_{\bar{g}_i} = \left(\frac{\mu_i}{s_i}\right)^{n-2-2/(q_i-2)} \left(\eta(0)\frac{(n(n-2))^{n/2}}{n}\omega_{n-1} + o(1)\right), \tag{2.60}$$

while

$$\lim_{\delta \to 0}\left[\int_{\partial B_0(\delta)} H\partial_\nu \eta \, d\sigma_\xi - \int_{\partial B_0(\delta)} \eta \partial_\nu H \, d\sigma_\xi\right] = \lambda(n-2)\omega_{n-1}\eta(0), \tag{2.61}$$

where ω_{n-1} is the volume of the standard unit sphere in $\mathbb{R}^n$. We thus obtain that

$$\frac{1}{\lambda_i} = O\left(\left(\frac{\mu_i}{s_i}\right)^{n-2-2/(q_i-2)}\right). \tag{2.62}$$

This leads with (2.54) to the estimate (A) for the sequence $(s_i/2)$. Then it holds for (s_i) thanks to (2.21). This ends the proof of assertion (A). As already said, this also ends the proof of Claim 2.2. ∎

Lack of compactness can occur only if the equation is almost critical as proved in Claim 2.1 ($q_i \to 2^\star$ as $i \to +\infty$). Here we prove that q_i must go to $2^\star$ quite fast. More precise information on this speed of convergence may be deduced from Claim 2.6 but the following claim is easier to prove and sufficient for the moment.

Claim 2.3. We have that

$$2^\star - q_i = \begin{cases} O\left(\dfrac{\mu_i}{r_i}\right), & \text{if } n = 3, \\ O\left(\dfrac{\mu_i^2}{r_i^2}\right) + O\left(\mu_i^2 \ln\left(\dfrac{r_i}{\mu_i}\right)\right), & \text{if } n = 4, \\ O(\mu_i^2) + O\left(\left(\dfrac{\mu_i}{r_i}\right)^{n-2}\right), & \text{if } n \ge 5. \end{cases} \tag{2.63}$$

□

Proof of Claim 2.3. We write the Pohozaev identity (see the appendix) applied to u_i in $B_{x_i}(r_i)$ with test function $f_i = (1/2)d_g(x_i, x)^2$:

$$\begin{aligned}
&\left(\frac{n-2}{2} - \frac{n}{q_i}\right)\int_{B_{x_i}(r_i)} \varphi^{2^\star - q_i} u_i^{q_i}\, dv_g \\
&\quad = -\int_{B_{x_i}(r_i)} \left(a_i + \frac{1}{2}(\nabla f_i, \nabla a_i)_g + \frac{1}{4}(\Delta_g^2) f_i\right) u_i^2\, dv_g \\
&\qquad + \left(\frac{1}{2} - \frac{1}{q_i}\right)\int_{B_{x_i}(r_i)} (\Delta_g f + n)\varphi^{2^\star - q_i} u_i^{q_i}\, dv_g \\
&\qquad + \frac{1}{q_i}\int_{B_{x_i}(r_i)} \left(\nabla f, \nabla\left(\varphi^{2^\star - q_i}\right)\right)_g u_i^{q_i}\, dv_g \\
&\qquad + \int_{B_{x_i}(r_i)} (\nabla^2 f - g)(\nabla u_i, \nabla u_i)\, dv_g + A_i,
\end{aligned} \tag{2.64}$$

where A_i is the boundary term:

$$\begin{aligned}
A_i &= \frac{1}{2}\int_{\partial B_{x_i}(r_i)} (\nabla f_i, \nu)_g |\nabla u_i|_g^2\, d\sigma_g \\
&\quad - \int_{\partial B_{x_i}(r_i)} (\nabla f_i, \nu)_g \left(\frac{1}{q_i}\varphi^{2^\star - q_i} u_i^{q_i} - \frac{1}{2} a_i u_i^2\right) d\sigma_g \\
&\quad - \frac{n-2}{2}\int_{\partial B_{x_i}(r_i)} (\nabla u_i, \nu)_g u_i\, d\sigma_g \\
&\quad - \int_{\partial B_{x_i}(r_i)} (\nabla u_i, \nabla f_i)_g (\nabla u_i, \nu)_g\, d\sigma_g \\
&\quad + \frac{1}{2}\int_{\partial B_{x_i}(r_i)} (\Delta_g f_i + n)(\nabla u_i, \nu)_g u_i\, d\sigma_g \\
&\quad - \frac{1}{4}\int_{\partial B_{x_i}(r_i)} (\nabla(\Delta_g f_i), \nu)_g u_i^2\, d\sigma_g.
\end{aligned} \tag{2.65}$$

Thanks to Claim 2.2, we have that

$$A_i = O\left(\mu_i^{2(n-2)-4/(q_i-2)} r_i^{2-n}\right). \tag{2.66}$$

It is also easily checked that

$$\begin{aligned}
&\Delta_g f_i + n = O\left(d_g(x_i, x)^2\right), \\
&(\Delta_g^2) f_i = O(1), \\
&(\nabla^2 f_i - g)(\nabla u_i, \nabla u_i) = O\left(d_g(x_i, x)^2 |\nabla u_i|_g^2\right),
\end{aligned} \tag{2.67}$$

so that, since $q_i \to 2^\star$ as $i \to +\infty$, (2.64) becomes

$$\begin{aligned}
&\big(1+o(1)\big)\big(2^\star - q_i\big)\int_{B_{x_i}(r_i)} u_i^{q_i}\,dv_g \\
&\quad = O\Big(\mu_i^{2(n-2)-4/(q_i-2)} r_i^{2-n}\Big) + O\bigg(\int_{B_{x_i}(r_i)} d_g\big(x_i,x\big)^2 |\nabla u_i|_g^2\,dv_g\bigg) \\
&\qquad + O\bigg(\int_{B_{x_i}(r_i)} u_i^2\,dv_g\bigg) + O\bigg(\int_{B_{x_i}(r_i)} d_g\big(x_i,x\big)^2 u_i^{q_i}\,dv_g\bigg) \\
&\qquad + O\bigg((2^\star - q_i)\int_{B_{x_i}(r_i)} d_g\big(x_i,x\big) u_i^{q_i}\,dv_g\bigg).
\end{aligned} \tag{2.68}$$

Thanks to Claims 2.1 and 2.2, we have that

$$\begin{aligned}
&\int_{B_{x_i}(r_i)} u_i^{q_i}\,dv_g \ge \big(K_n^{-n/2} + o(1)\big)\mu_i^{n-2-4/(q_i-2)}, \\
&\int_{B_{x_i}(r_i)} d_g\big(x_i,x\big) u_i^{q_i}\,dv_g = o\Big(\mu_i^{n-2-4/(q_i-2)}\Big), \\
&\int_{B_{x_i}(r_i)} u_i^2\,dv_g = \begin{cases} O\Big(r_i\mu_i^{2-4/(q_i-2)}\Big), & \text{if } n=3, \\ O\bigg(\mu_i^{4-4/(q_i-2)}\ln\Big(\dfrac{r_i}{\mu_i}\Big)\bigg), & \text{if } n=4, \\ O\Big(\mu_i^{n-4/(q_i-2)}\Big), & \text{if } n\ge 5, \end{cases} \\
&\int_{B_{x_i}(r_i)} d_g\big(x_i,x\big)^2 u_i^{q_i}\,dv_g = O\Big(\mu_i^{n-4/(q_i-2)}\Big), \\
&\int_{B_{x_i}(r_i)} d_g\big(x_i,x\big)^2 |\nabla u_i|_g^2\,dv_g = \begin{cases} O\Big(r_i\mu_i^{2-4/(q_i-2)}\Big), & \text{if } n=3, \\ O\bigg(\mu_i^{4-4/(q_i-2)}\ln\Big(\dfrac{r_i}{\mu_i}\Big)\bigg), & \text{if } n=4, \\ O\Big(\mu_i^{n-4/(q_i-2)}\Big), & \text{if } n\ge 5. \end{cases}
\end{aligned} \tag{2.69}$$

Coming back to (2.68) with all these estimates, we obtain Claim 2.3. ∎

We project u_i on a set of bubbles (defined below). Let $\eta \in C^\infty(\mathbb{R})$ be such that $\eta \equiv 1$ on $[0;1/4]$ and $\eta \equiv 0$ on $[1/2;+\infty)$. We consider the function

$$J_i : M \times \mathbb{R}_+^\star \times \mathbb{R} \longmapsto \mathbb{R} \tag{2.70}$$

defined by

$$J_i(y,\nu,\theta) = \int_M \bigg|\nabla\bigg(\eta\Big(\frac{d_g(y,\cdot)}{r_i}\Big)\big(u_i - (1+\theta)B_{y,\nu}^i\big)\bigg)\bigg|_g^2\,dv_g, \tag{2.71}$$

where

$$B^i_{y,\nu}(x) = \nu^{(n-2)/2-2/(q_i-2)} \left(\frac{\nu}{\nu^2 + \frac{1}{n(n-2)} d_g(y,x)^2} \right)^{(n-2)/2} \tag{2.72}$$

for $y \in M$ and $\nu > 0$. We define the set Λ_i by

$$\Lambda_i = \left\{ (y,\nu,\theta) \in M \times \mathbb{R}^\star_+ \times \mathbb{R} : \frac{1}{2} \leq \frac{\nu}{\mu_i} \leq 2,\ -1 \leq \theta \leq 1,\ d_g\big(x_i, y\big) \leq \mu_i \right\}. \tag{2.73}$$

Since Λ_i is compact and J_i is continuous, there exists $(y_i, \nu_i, \theta_i) \in \Lambda_i$ such that

$$\min_{(y,\nu,\theta)\in\Lambda_i} J_i(y,\nu,\theta) = J_i\big(y_i, \nu_i, \theta_i\big). \tag{2.74}$$

Claim 2.4. We have that

$$\theta_i \longrightarrow 0, \quad \frac{\mu_i}{\nu_i} \longrightarrow 1, \quad \frac{d_g\big(x_i, y_i\big)}{\mu_i} \longrightarrow 0, \quad \text{as } i \longrightarrow +\infty. \tag{2.75}$$

□

Proof of Claim 2.4. First, we note that $(x_i, \mu_i, 0) \in \Lambda_i$. Moreover, we can write with (2.22) that

$$\begin{aligned} J_i\big(x_i, \mu_i, 0\big) &= \int_M \left| \nabla \left(\eta \left(\frac{d_g(y,\cdot)}{r_i} \right) \big(u_i - B^i_{x_i,\mu_i}\big) \right) \right|^2_g dv_g \\ &= \int_{B_{x_i}(R\mu_i)} \big|\nabla\big(u_i - B^i_{x_i,\mu_i}\big)\big|^2_g dv_g \\ &\quad + \int_{M\setminus B_{x_i}(R\mu_i)} \left| \nabla \left(\eta \left(\frac{d_g(y,\cdot)}{r_i} \right) \big(u_i - B^i_{x_i,\mu_i}\big) \right) \right|^2_g dv_g \end{aligned} \tag{2.76}$$

for all $R > 0$. Thanks to Claim 2.1, we have that, for all $R > 0$,

$$\int_{B_{x_i}(R\mu_i)} \big|\nabla\big(u_i - B^i_{x_i,\mu_i}\big)\big|^2_g dv_g = o\Big(\mu_i^{n-2-4/(q_i-2)}\Big), \tag{2.77}$$

while, thanks to Claim 2.2, to (2.22), and to some computations, we have that

$$\int_{M\setminus B_{x_i}(R\mu_i)} \left| \nabla \left(\eta \left(\frac{d_g(y,\cdot)}{r_i} \right) \big(u_i - B^i_{x_i,\mu_i}\big) \right) \right|^2_g dv_g \leq \big(\varepsilon_R + o(1)\big) \mu_i^{n-2-4/(q_i-2)}, \tag{2.78}$$

where $\varepsilon_R \to 0$ as $R \to +\infty$. Thus we obtain that

$$J_i(x_i, \mu_i, 0) = o\left(\mu_i^{n-2-4/(q_i-2)}\right). \tag{2.79}$$

By definition (2.74) of (y_i, ν_i, θ_i), we thus have that

$$J_i(y_i, \nu_i, \theta_i) = o\left(\mu_i^{n-2-4/(q_i-2)}\right). \tag{2.80}$$

We set

$$\eta_i = \eta\left(\frac{d_g(y_i, \cdot)}{r_i}\right) \tag{2.81}$$

and we write that

$$\begin{aligned} J_i(y_i, \nu_i, \theta_i) = &\int_M |\nabla(\eta_i u_i)|_g^2 dv_g + (1+\theta_i)^2 \int_M |\nabla(\eta_i B^i_{y_i,\nu_i})|_g^2 dv_g \\ &- 2(1+\theta_i)\int_M (\nabla(\eta_i u_i), \nabla(\eta_i B^i_{y_i,\nu_i}))_g dv_g. \end{aligned} \tag{2.82}$$

This leads first to

$$J_i(y_i, \nu_i, \theta_i) \geq \left[\left(\int_M |\nabla(\eta_i u_i)|_g^2 dv_g\right)^{1/2} - (1+\theta_i)\left(\int_M |\nabla(\eta_i B^i_{y_i,\nu_i})|_g^2 dv_g\right)^{1/2}\right]^2. \tag{2.83}$$

It is easily checked, thanks to Claims 2.1 and 2.2 and to (2.22), that

$$\mu_i^{2/(q_i-2)-(n-2)/2}\left(\int_M |\nabla(\eta_i u_i)|_g^2 dv_g\right)^{1/2} \longrightarrow K_n^{-n/4} \quad \text{as } i \longrightarrow +\infty, \tag{2.84}$$

while direct computations give also that

$$\mu_i^{2/(q_i-2)-(n-2)/2}\left(\int_M |\nabla(\eta_i B^i_{y_i,\nu_i})|_g^2 dv_g\right)^{1/2} \longrightarrow K_n^{-n/4} \quad \text{as } i \longrightarrow +\infty. \tag{2.85}$$

Thanks to (2.80) and (2.83), we can conclude that $\theta_i \to 0$ as $i \to +\infty$. Coming back to (2.82) with (2.80) and these last results, we also obtain that

$$\mu_i^{4/(q_i-2)-(n-2)} \int_M (\nabla(\eta_i u_i), \nabla(\eta_i B^i_{y_i,\nu_i}))_g dv_g \longrightarrow K_n^{-n/2} \quad \text{as } i \longrightarrow +\infty, \tag{2.86}$$

which leads, thanks to (2.79), to

$$\mu_i^{4/(q_i-2)-(n-2)} \int_M \left(\nabla\left(\eta_i B^i_{x_i,\mu_i}\right), \nabla\left(\eta_i B^i_{y_i,\nu_i}\right)\right)_g dv_g \longrightarrow K_n^{-n/2} \quad \text{as } i \longrightarrow +\infty. \tag{2.87}$$

It is easily checked to be possible if and only if the two remaining assertions of Claim 2.4 hold. This ends the proof of Claim 2.4. ∎

We set for $x \in B_0(2)$ the Euclidean ball of center 0 and radius 2,

$$\begin{aligned} v_i(x) &= r_i^{2/(q_i-2)} u_i\left(\exp_{y_i}(r_i x)\right),\\ h_i(x) &= \exp_{y_i}^{\star} g\left(r_i x\right),\\ \tilde{a}_i(x) &= a_i\left(\exp_{y_i}(r_i x)\right),\\ \tilde{\varphi}_i(x) &= \varphi\left(\exp_{y_i}(r_i x)\right). \end{aligned} \tag{2.88}$$

Then

$$\Delta_{h_i} v_i + r_i^2 \tilde{a}_i v_i = \tilde{\varphi}_i^{2^\star - q_i} v_i^{q_i - 1} \quad \text{in } B_0(2). \tag{2.89}$$

We let

$$\gamma_i = \frac{\nu_i}{r_i}. \tag{2.90}$$

As a consequence of Claims 2.1 and 2.4, we have

$$\gamma_i^{2/(q_i-2)} v_i\left(\gamma_i x\right) \longrightarrow U(x) \text{ in } C^2_{loc}\left(\mathbb{R}^n\right) \quad \text{as } i \longrightarrow +\infty, \tag{2.91}$$

while Claim 2.2 together with (2.22), and Claims 2.3 and 2.4 give

$$v_i(x) \le C\gamma_i^{(n-2)/2}|x|^{2-n} \quad \text{in } B_0(2)\backslash\{0\}, \tag{2.92}$$

$$\left|\nabla v_i(x)\right| \le C\gamma_i^{(n-2)/2}|x|^{1-n} \quad \text{in } B_0(2)\backslash\{0\}, \tag{2.93}$$

for some $C > 0$ independent of i. Thanks to standard elliptic theory (see, e.g., [12]), equations (2.89) and (2.92) give that

(B) $(\gamma_i^{-(n-2)/2} v_i)$ is bounded in $C^2_{loc}(B_0(2)\backslash\{0\})$.

We set

$$R_i = \eta\left(v_i - \left(1 + \theta_i\right) B_i\right), \tag{2.94}$$

where

$$B_i(x) = \gamma_i^{(n-2)/2-2/(q_i-2)} \left(\frac{\gamma_i}{\gamma_i^2 + \dfrac{|x|^2}{n(n-2)}} \right)^{(n-2)/2}. \tag{2.95}$$

The first variation formula associated to (2.74) gives after some computations that

$$\int_{B_0(1)} \left(\nabla(\eta B_i), \nabla R_i\right)_{h_i} dv_{h_i} = 0, \tag{2.96}$$

$$\int_{B_0(1)} \left(\nabla\left(\eta |x|^2 \left(1 + \frac{1}{n(n-2)} \frac{|x|^2}{\gamma_i^2} \right)^{-n/2} \right), \nabla R_i \right)_{h_i} dv_{h_i} = 0, \tag{2.97}$$

and, thanks to Claim 2.3,

$$\int_{B_0(1)} \left(\nabla\left(\eta \frac{\partial B_i}{\partial x_\alpha} \right), \nabla R_i \right)_{h_i} dv_{h_i} = O\left(\gamma_i^{n-2}\right) \tag{2.98}$$

for all $\alpha = 1, \dots, n$.

The next claim provides fine integral estimates on R_i. We state this claim only for dimensions $n = 3, 4, 5$. Similar estimates hold, and follow from the proof given here, in higher dimensions. These kinds of estimates were first obtained by Druet and Hebey [8].

Claim 2.5. The following estimates hold:

$$\int_{B_0(1)} |\nabla R_i|_{h_i}^2 dv_{h_i} = O\left(\gamma_i^{n-2}\right), \tag{2.99}$$

$$\theta_i = \begin{cases} O\left(\gamma_i \ln \dfrac{1}{\gamma_i}\right), & \text{if } n = 3, \\ O\left(\gamma_i^2 \ln \dfrac{1}{\gamma_i}\right) + O\left(r_i^2 \gamma_i^2 \left(\ln \dfrac{1}{\gamma_i}\right)^2\right), & \text{if } n = 4, \\ O\left(\gamma_i^3 \ln \dfrac{1}{\gamma_i}\right) + O\left(r_i^2 \gamma_i^2 \ln \dfrac{1}{\gamma_i}\right), & \text{if } n = 5. \end{cases} \tag{2.100}$$

□

Proof of Claim 2.5. We write with (2.94) and (2.96) that

$$\int_{B_0(1)} |\nabla R_i|_{h_i}^2 dv_{h_i} = \int_{B_0(1)} \left(\nabla R_i, \nabla(\eta v_i)\right)_{h_i} dv_{h_i} = \int_{B_0(1)} R_i \Delta_{h_i}(\eta v_i) dv_{h_i}, \tag{2.101}$$

which leads with (2.89) and (B) to

$$\begin{aligned} &\int_{B_0(1)} |\nabla R_i|_{h_i}^2 dv_{h_i} \\ &\quad = \int_{B_0(1)} \tilde{\varphi}_i^{2^\star - q_i} (\eta v_i)^{q_i - 1} R_i dv_{h_i} + O\left(\gamma_i^{n-2}\right) - r_i^2 \int_{B_0(1)} S_i(\eta v_i) R_i dv_{h_i}. \end{aligned} \tag{2.102}$$

Thanks to (2.91), (2.92), (2.94), and (2.95), it is easily checked that

$$r_i^2 \int_{B_0(1)} S_i(\eta v_i) R_i \, dv_{h_i} \longrightarrow 0 \quad \text{as } i \longrightarrow +\infty. \tag{2.103}$$

Independently, we write that

$$\begin{aligned} \left| \int_{B_0(1)} \tilde{\varphi}_i^{2^\star - q_i} (\eta v_i)^{q_i - 1} R_i \, dv_{h_i} \right| \le 2 \int_{B_0(R\gamma_i)} v_i^{q_i-1} \left| v_i - (1+\theta_i) B_i \right| dv_{h_i} \\ + 2 \int_{B_0(1) \setminus B_0(R\gamma_i)} (\eta v_i)^{q_i-1} \eta (B_i + v_i) \, dv_{h_i} \end{aligned} \tag{2.104}$$

for all $R > 0$ and i large. It is easily checked, thanks to Claims 2.3 and 2.4 and to (2.91), that

$$\int_{B_0(R\gamma_i)} v_i^{q_i-1} \left| v_i - (1+\theta_i) B_i \right| dv_{h_i} \longrightarrow 0 \quad \text{as } i \longrightarrow +\infty, \tag{2.105}$$

for all $R > 0$. Thanks to (2.92) and to Claim 2.3, we also have that

$$\lim_{R \to +\infty} \limsup_{i \to +\infty} \int_{B_0(1) \setminus B_0(R\gamma_i)} (\eta v_i)^{q_i-1} \eta (B_i + v_i) \, dv_{h_i} = 0. \tag{2.106}$$

Coming back to (2.102) with all these relations, we obtain that

$$\int_{B_0(1)} |\nabla R_i|_{h_i}^2 \, dv_{h_i} \longrightarrow 0 \quad \text{as } i \longrightarrow +\infty. \tag{2.107}$$

Let us be more precise now. We write, thanks to not only (2.94) and (2.107) but also Claims 2.3 and 2.4, that

$$\begin{aligned} \int_{B_0(1)} \tilde{\varphi}_i^{2^\star - q_i} (\eta v_i)^{q_i-1} R_i \, dv_{h_i} &= (1+\theta_i)^{q_i-1} \int_{B_0(1)} (\eta B_i)^{q_i-1} R_i \, dv_{h_i} \\ &\quad + \frac{n+2}{n-2} \int_{B_0(1)} (\eta B_i)^{2^\star - 2} R_i^2 \, dv_{h_i} \\ &\quad + O\left(\gamma_i^{(n-2)/2} \left(\int_{B_0(1)} |\nabla R_i|_{h_i}^2 \, dv_{h_i} \right)^{1/2} \right) \\ &\quad + o\left(\int_{B_0(1)} |\nabla R_i|_{h_i}^2 \, dv_{h_i} \right). \end{aligned} \tag{2.108}$$

Relations (B), (2.94), and (2.96) together with Claim 2.3 lead to

$$0 = \int_{B_0(1)} \left(\nabla(\eta B_i), \nabla R_i \right)_{h_i} dv_{h_i} = \int_{B_0(1)} \eta \Delta_{h_i} B_i R_i \, dv_{h_i} + O\left(\gamma_i^{n-2}\right). \tag{2.109}$$

Since

$$\begin{aligned}\Delta_{h_i} B_i &= \gamma_i^{(2/(q_i-2)-(n-2)/2)(2^\star-2)} B_i^{2^\star-1} + O\big(r_i^2|x|\,|\nabla B_i|\big)\\ &= B_i^{q_i-1} + O\Big(\big(2^\star - q_i\big)\Big(\ln\frac{1}{\gamma_i}\Big)B_i^{q_i-1}\Big) + O\big(r_i^2|x|\,|\nabla B_i|\big)\end{aligned} \tag{2.110}$$

thanks to Claim 2.3, we obtain with Hölder's and Sobolev's inequalities that

$$\begin{aligned}&\int_{B_0(1)} \big(\eta B_i\big)^{q_i-1} R_i \, dv_{h_i}\\ &\quad = O\Bigg(r_i^2\bigg(\int_{B_0(1)} |x|^{2^\star/(2^\star-1)}\big|\nabla B_i\big|^{2^\star/(2^\star-1)} dv_{h_i}\bigg)^{(2^\star-1)/2^\star}\\ &\qquad\quad \times\bigg(\int_{B_0(1)} \big|\nabla R_i\big|_{h_i}^2 dv_{h_i}\bigg)^{1/2}\Bigg)\\ &\qquad + O\bigg(\big(2^\star-q_i\big)\Big(\ln\frac{1}{\gamma_i}\Big)\int_{B_0(1)} B_i^{q_i-1}\big|R_i\big| dv_{h_i}\bigg) + O\big(\gamma_i^{n-2}\big),\end{aligned} \tag{2.111}$$

which leads after computations, thanks once again to Claim 2.3, and to Hölder's and Sobolev's inequalities, to

$$\int_{B_0(1)} \big(\eta B_i\big)^{q_i-1} R_i \, dv_{h_i} = O\Bigg(\gamma_i^{(n-2)/2}\bigg(\int_{B_0(1)} \big|\nabla R_i\big|_{h_i}^2 dv_{h_i}\bigg)^{1/2}\Bigg) + O\big(\gamma_i^{n-2}\big). \tag{2.112}$$

Using Hölder's and Sobolev's inequalities, one also gets after some computations

$$r_i^2 \int_{B_0(1)} S_i\big(\eta v_i\big) R_i \, dv_{h_i} = O\Bigg(r_i^2\gamma_i^{(n-2)/2}\bigg(\int_{B_0(1)} \big|\nabla R_i\big|_{h_i}^2 dv_{h_i}\bigg)^{1/2}\Bigg). \tag{2.113}$$

Coming back to (2.102) with (2.108), (2.112), and this last relation, we obtain the following:

$$\begin{aligned}\big(1+o(1)\big)\int_{B_0(1)} \big|\nabla R_i\big|_{h_i}^2 dv_{h_i} &= \frac{n+2}{n-2}\int_{B_0(1)} \big(\eta B_i\big)^{2^\star-2} R_i^2 \, dv_{h_i} + O\big(\gamma_i^{n-2}\big)\\ &\quad + O\Bigg(\gamma_i^{(n-2)/2}\bigg(\int_{B_0(1)} \big|\nabla R_i\big|_{h_i}^2 dv_{h_i}\bigg)^{1/2}\Bigg).\end{aligned} \tag{2.114}$$

We now consider the following eigenvalue problem:

$$\begin{aligned}&\Delta_{h_i}\zeta_{i,\alpha} = \tau_{i,\alpha}\big(\eta B_i\big)^{2^\star-2}\zeta_{i,\alpha} \quad \text{in } B_0(1),\\ &\zeta_{i,\alpha} = 0 \quad \text{on } \partial B_0(1),\\ &\int_{B_0(1)} \big(\eta B_i\big)^{2^\star-2}\zeta_{i,\alpha}\zeta_{i,\beta} \, dv_{h_i} = K_n^{-n/2}\delta_{\alpha\beta},\end{aligned} \tag{2.115}$$

with $\tau_{i,1} \le \cdots \le \tau_{i,\alpha} \le \cdots$. By the result of [7, Appendix 1], we know that

$$\lim_{i\to+\infty} \tau_{i,\alpha} = \tau_\alpha \quad \forall \alpha \in \mathbb{N}^\star, \tag{2.116}$$

and that

$$\lim_{i\to+\infty} \int_{B_0(1)} \left|\nabla\left(\zeta_{i,\alpha} - \tilde{\zeta}_{i,\alpha}\right)\right|_{h_i}^2 dv_{h_i} = 0 \quad \forall \alpha \in \mathbb{N}^\star, \tag{2.117}$$

where

$$\tilde{\zeta}_{i,\alpha} = \gamma_i^{1-n/2} \zeta_\alpha\left(\frac{x}{\gamma_i}\right) \tag{2.118}$$

with $(\zeta_\alpha, \tau_\alpha)$ the solutions of the following eigenvalue problem:

$$\begin{aligned} &\Delta_\xi \zeta_\alpha = \tau_\alpha U^{2^\star-2}\zeta_\alpha \quad \text{in } \mathbb{R}^n, \\ &\int_{\mathbb{R}^n} U^{2^\star-2}\zeta_\alpha\zeta_\beta \, dv_\xi = K_n^{-n/2}\delta_{\alpha\beta}, \end{aligned} \tag{2.119}$$

where $U(x) = (1 + |x|^2/n(n-2))^{1-n/2}$.

Thanks to the work of Bianchi and Egnell [4], we know that

$$\begin{aligned} &\zeta_1 = U, \qquad \tau_1 = 1, \\ &\zeta_\alpha = \lambda_\alpha \frac{\partial U}{\partial x_{\alpha-1}}, \quad \tau_\alpha = \frac{n+2}{n-2}, \quad \text{for } \alpha = 2, \ldots, n+1, \\ &\zeta_{n+2} = \lambda_{n+2}\left(U - \frac{2}{n(n-2)}|x|^2 U^{n/(n-2)}\right), \qquad \tau_{n+2} = \frac{n+2}{n-2}, \end{aligned} \tag{2.120}$$

where $\lambda_2, \ldots, \lambda_{n+2}$ are some positive real numbers, and that

$$\tau_{n+3} > \frac{n+2}{n-2}. \tag{2.121}$$

We now write that

$$R_i = \sum_{\alpha=1}^{n+2} D_{i,\alpha}\zeta_{i,\alpha} + \tilde{R}_i \tag{2.122}$$

with

$$D_{i,\alpha} = \frac{\int_{B_0(1)} \left(\nabla R_i, \nabla \zeta_{i,\alpha}\right)_{h_i} dv_{h_i}}{\int_{B_0(1)} \left|\nabla \zeta_{i,\alpha}\right|_{h_i}^2 dv_{h_i}}. \tag{2.123}$$

so that

$$\int_{B_0(1)} \left(\nabla \tilde{R}_i, \nabla \zeta_{i,\alpha}\right)_{h_i} dv_{h_i} = 0 \tag{2.124}$$

for $\alpha = 1, \ldots, n+2$. In particular, we obtain, thanks to (2.116), that

$$\int_{B_0(1)} \left|\nabla \tilde{R}_i\right|_{h_i}^2 dv_{h_i} \geq \left(\tau_{n+3} + o(1)\right) \int_{B_0(1)} \left(\eta B_i\right)^{2^\star - 2} \tilde{R}_i^2 dv_{h_i}. \tag{2.125}$$

We also have

$$\int_{B_0(1)} \left|\nabla R_i\right|_{h_i}^2 dv_{h_i} = K_n^{-n/2} \sum_{\alpha=1}^{n+2} \tau_{i,\alpha} D_{i,\alpha}^2 + \int_{B_0(1)} \left|\nabla \tilde{R}_i\right|_{h_i}^2 dv_{h_i} \tag{2.126}$$

thanks to (2.115). At last, we can write that

$$\int_{B_0(1)} \left(\eta B_i\right)^{2^\star - 2} R_i^2 dv_{h_i} = K_n^{-n/2} \sum_{\alpha=1}^{n+2} D_{i,\alpha}^2 + \int_{B_0(1)} \left(\eta B_i\right)^{2^\star - 2} \tilde{R}_i^2 dv_{h_i}. \tag{2.127}$$

We now estimate the $D_{i,\alpha}$'s. We write, thanks to (2.115), (2.117), and (2.123), that

$$\begin{aligned} K_n^{-n/2} \tau_{i,\alpha} D_{i,\alpha} &= \int_{B_0(1)} \left(\nabla R_i, \nabla\left(\zeta_{i,\alpha} - \tilde{\zeta}_{i,\alpha}\right)\right)_{h_i} dv_{h_i} + \int_{B_0(1)} \left(\nabla R_i, \nabla \tilde{\zeta}_{i,\alpha}\right)_{h_i} dv_{h_i} \\ &= \int_{B_0(1)} \left(\nabla R_i, \nabla \tilde{\zeta}_{i,\alpha}\right)_{h_i} dv_{h_i} + o\left(\left(\int_{B_0(1)} \left|\nabla R_i\right|_{h_i}^2 dv_{h_i}\right)^{1/2}\right). \end{aligned} \tag{2.128}$$

It is then easily checked that

$$\int_{B_0(1)} \left(\nabla R_i, \nabla \tilde{\zeta}_{i,\alpha}\right)_{h_i} dv_{h_i} = O\left(\gamma_i^{n-2}\right) \tag{2.129}$$

for $\alpha = 1, \ldots, n+2$, thanks to (2.96), (2.97), (2.98), (2.118), (2.120), and Claim 2.3. Thus we obtain that

$$D_{i,\alpha}^2 = o\left(\int_{B_0(1)} \left|\nabla R_i\right|_{h_i}^2 dv_{h_i}\right) + o\left(\gamma_i^{n-2}\right). \tag{2.130}$$

Then (2.126) becomes

$$\left(1 + o(1)\right) \int_{B_0(1)} \left|\nabla R_i\right|_{h_i}^2 dv_{h_i} = \int_{B_0(1)} \left|\nabla \tilde{R}_i\right|_{h_i}^2 dv_{h_i} + o\left(\gamma_i^{n-2}\right) \tag{2.131}$$

and (2.127) becomes

$$\int_{B_0(1)} (\eta B_i)^{2^\star-2} R_i^2 \, dv_{h_i} = \int_{B_0(1)} (\eta B_i)^{2^\star-2} \tilde{R}_i^2 \, dv_{h_i} + o\left(\int_{B_0(1)} |\nabla R_i|_{h_i}^2 \, dv_{h_i} \right) + o(\gamma_i^{n-2}). \tag{2.132}$$

Using (2.114), (2.121), and (2.125), we deduce (2.99). It remains to prove (2.100). For that purpose, we first write that

$$\int_{B_0(1)} |\nabla(\eta v_i)|_{h_i}^2 \, dv_{h_i} = (1+\theta_i)^2 \int_{B_0(1)} |\nabla(\eta B_i)|_{h_i}^2 \, dv_{h_i} + \int_{B_0(1)} |\nabla R_i|_{h_i}^2 \, dv_{h_i} \tag{2.133}$$

thanks to (2.94) and (2.96). Direct computations lead then with the Cartan expansion of the metric h_i around 0 and with Claim 2.3 to

$$\int_{B_0(1)} |\nabla(\eta B_i)|_{h_i}^2 \, dv_{h_i} = K_n^{-n/2} + O(\gamma_i^{n-2}) + \begin{cases} O\left(\gamma_i \ln \frac{1}{\gamma_i}\right), & \text{if } n = 3, \\ O\left(r_i^2 \gamma_i^2 \left(\ln \frac{1}{\gamma_i}\right)^2\right) + O\left(\gamma_i^2 \ln \frac{1}{\gamma_i}\right), & \text{if } n = 4, \\ O\left(r_i^2 \gamma_i^2 \ln \frac{1}{\gamma_i}\right) + O\left(\gamma_i^3 \ln \frac{1}{\gamma_i}\right), & \text{if } n = 5. \end{cases} \tag{2.134}$$

We thus obtain, thanks to (2.99), that

$$\int_{B_0(1)} |\nabla(\eta v_i)|_{h_i}^2 \, dv_{h_i} = K_n^{-n/2} (1+\theta_i)^2 + O(\gamma_i^{n-2}) + \begin{cases} O\left(\gamma_i \ln \frac{1}{\gamma_i}\right), & \text{if } n = 3, \\ O\left(r_i^2 \gamma_i^2 \left(\ln \frac{1}{\gamma_i}\right)^2\right) + O\left(\gamma_i^2 \ln \frac{1}{\gamma_i}\right), & \text{if } n = 4, \\ O\left(r_i^2 \gamma_i^2 \ln \frac{1}{\gamma_i}\right) + O\left(\gamma_i^3 \ln \frac{1}{\gamma_i}\right), & \text{if } n = 5. \end{cases} \tag{2.135}$$

Independently, using equation (2.89) satisfied by v_i and the estimate (B), we have, thanks to Claim 2.3, that

$$\begin{aligned}&\int_{B_0(1)} |\nabla(\eta v_i)|^2_{h_i} dv_{h_i}\\&\quad = \int_{B_0(1)} \tilde{\varphi}_i^{2^\star - q_i}(\eta v_i)^{q_i} dv_{h_i} + O(\gamma_i^{n-2}) - r_i^2 \int_{B_0(1)} S_i(\eta v_i)^2 dv_{h_i}.\end{aligned} \tag{2.136}$$

Using (2.94), (2.95), and (2.99), some computations give that

$$r_i^2 \int_{B_0(1)} S_i(\eta v_i)^2 dv_{h_i} = \begin{cases} O(r_i^2\gamma_i), & \text{if } n = 3,\\ O(r_i^2\gamma_i^2|\ln\gamma_i|), & \text{if } n = 4,\\ O(r_i^2\gamma_i^2), & \text{if } n = 5,\end{cases} \tag{2.137}$$

so that

$$\begin{aligned}\int_{B_0(1)} |\nabla(\eta v_i)|^2_{h_i} dv_{h_i} &= \int_{B_0(1)} \tilde{\varphi}_i^{2^\star - q_i}(\eta v_i)^{q_i} dv_{h_i} + O(\gamma_i^{n-2})\\ &\quad + \begin{cases} O(r_i^2\gamma_i), & \text{if } n = 3,\\ O(r_i^2\gamma_i^2|\ln\gamma_i|), & \text{if } n = 4,\\ O(r_i^2\gamma_i^2), & \text{if } n = 5.\end{cases}\end{aligned} \tag{2.138}$$

We now write with (2.94) that

$$\begin{aligned}\int_{B_0(1)} (\eta v_i)^{q_i} dv_{h_i} &= (1+\theta_i)^{q_i} \int_{B_0(1)} (\eta B_i)^{q_i} dv_{h_i}\\ &\quad + q_i(1+\theta_i)^{q_i-1} \int_{B_0(1)} (\eta B_i)^{q_i-1} R_i dv_{h_i}\\ &\quad + O\left(\int_{B_0(1)} |\nabla R_i|^2_{h_i} dv_{h_i}\right).\end{aligned} \tag{2.139}$$

This leads, thanks to (2.99), (2.112), Claim 2.3, and direct computations, to

$$\begin{aligned}&\int_{B_0(1)} \tilde{\varphi}_i^{2^\star - q_i}(\eta v_i)^{q_i} dv_{h_i}\\&\quad = (1+\theta_i)^{q_i} K_n^{-n/2} + O((2^\star - q_i)|\ln\gamma_i|) + O(\gamma_i^{n-2}) + O(r_i^2\gamma_i^2).\end{aligned} \tag{2.140}$$

Combining (2.135), (2.138), and (2.140), we obtain (2.100) thanks to Claim 2.3. This ends the proof of Claim 2.5. ∎

We let $0 < \delta < 1/2$. We apply the Pohozaev identity to v_i in $B_0(\delta)$ with test function $f = (1/2)|x|^2$ (see the appendix):

$$\begin{aligned} M_i = {} & \left(\frac{n-2}{2} - \frac{n}{q_i}\right) \int_{B_0(\delta)} \tilde{\varphi}_i^{2^\star - q_i} v_i^{q_i} \, dv_{h_i} \\ & + \int_{B_0(\delta)} \left(r_i^2 \tilde{a}_i + \frac{1}{2} r_i^2 \left(\nabla f, \nabla \tilde{a}_i\right)_{h_i} + \frac{1}{4} \left(\Delta_{h_i}^2\right) f \right) v_i^2 \, dv_{h_i} \\ & - \left(\frac{1}{2} - \frac{1}{q_i}\right) \int_{B_0(\delta)} \left(\Delta_{h_i} f + n\right) \tilde{\varphi}_i^{2^\star - q_i} v_i^{q_i} \, dv_{h_i} \\ & - \frac{1}{q_i} \int_{B_0(\delta)} \left(\nabla f, \nabla \tilde{\varphi}_i^{2^\star - q_i}\right)_{h_i} v_i^{q_i} \, dv_{h_i} \\ & - \int_{B_0(\delta)} \left(\nabla^2 f - h_i\right) \left(\nabla v_i, \nabla v_i\right) dv_{h_i}, \end{aligned} \tag{2.141}$$

where M_i is the boundary term

$$\begin{aligned} M_i = {} & \frac{1}{2} \int_{\partial B_0(\delta)} (\nabla f, \nu)_{h_i} \left|\nabla v_i\right|_{h_i}^2 d\sigma_{h_i} \\ & - \int_{\partial B_0(\delta)} (\nabla f, \nu)_{h_i} \left(\frac{\tilde{\varphi}_i^{2^\star - q_i}}{q_i} v_i^{q_i} - \frac{1}{2} r_i^2 \tilde{a}_i v_i^2 \right) d\sigma_{h_i} \\ & - \frac{n-2}{2} \int_{\partial B_0(\delta)} \left(\nabla v_i, \nu\right)_{h_i} v_i \, d\sigma_{h_i} \\ & - \int_{\partial B_0(\delta)} \left(\nabla v_i, \nabla f\right)_{h_i} \left(\nabla v_i, \nu\right)_{h_i} d\sigma_{h_i} \\ & + \frac{1}{2} \int_{\partial B_0(\delta)} \left(\Delta_{h_i} f + n\right) \left(\nabla v_i, \nu\right)_{h_i} v_i \, d\sigma_{h_i} \\ & - \frac{1}{4} \int_{\partial B_0(\delta)} \left(\nabla \left(\Delta_{h_i} f\right), \nu\right)_{h_i} v_i^2 \, d\sigma_{h_i}. \end{aligned} \tag{2.142}$$

In the next claim, we estimate M_i thanks to (2.141). In Claim 2.7 and Section 3, we will estimate M_i thanks to (2.142) in order to get contradictions (in different settings).

Claim 2.6. We have that

$$\begin{aligned} M_i = {} & -\left(\frac{(n-2)^2}{4n} K_n^{-n/2} + o(1) \right) \left(2^\star - q_i\right) + O\left(\delta r_i^2 \gamma_i^{n-2}\right) + o\left(\gamma_i^{n-2}\right) \\ & + \left(a_i(x_i) - c_n S_g(x_i)\right) \begin{cases} 64 \omega_3 r_i^2 \gamma_i^2 \left|\ln \gamma_i\right|, & \text{if } n = 4, \\ 16 K_5^{-5/2} r_i^2 \gamma_i^2, & \text{if } n = 5, \end{cases} \end{aligned} \tag{2.143}$$

where

$$\lim_{i \to +\infty} \frac{o(X_i)}{X_i} = 0, \qquad \left|O\left(X_{i,\delta}\right)\right| \le C \left|X_{i,\delta}\right| \tag{2.144}$$

for some $C > 0$ independent of i and δ. □

Proof of Claim 2.6. Thanks to (2.140), we have that

$$\left(\frac{n-2}{2}-\frac{n}{q_i}\right)\int_{B_0(\delta)}\tilde{\varphi}_i^{2^\star-q_i}v_i^{q_i}\,dv_{h_i}=\left(\frac{(n-2)^2}{4n}K_n^{-n/2}+o(1)\right)(q_i-2^\star), \tag{2.145}$$

while (2.91) and (2.92) lead to

$$\int_{B_0(\delta)}\left(\nabla f,\nabla\tilde{\varphi}_i^{2^\star-q_i}\right)_{h_i}v_i^{q_i}\,dv_{h_i}=o(2^\star-q_i). \tag{2.146}$$

Since B_i is radially symmetrical and $\eta\equiv 1$ in $B_0(\delta)$, we have that

$$\begin{aligned}\int_{B_0(\delta)}\left(\nabla^2 f-h_i\right)(\nabla v_i,\nabla v_i)\,dv_{h_i}&=\int_{B_0(\delta)}\left(\nabla^2 f-h_i\right)(\nabla R_i,\nabla R_i)\,dv_{h_i}\\&=O\left(r_i^2\int_{B_0(\delta)}|x|^2\left|\nabla R_i\right|_{h_i}^2dv_{h_i}\right)\end{aligned} \tag{2.147}$$

thanks to the Cartan expansion of h_i around 0. We get then, thanks to Claim 2.5, that

$$\int_{B_0(\delta)}\left(\nabla^2 f-h_i\right)(\nabla v_i,\nabla v_i)\,dv_{h_i}=O\left(\delta^2 r_i^2\gamma_i^{n-2}\right). \tag{2.148}$$

Since $\Delta_{h_i}f+n=O(r_i^2|x|^2)$, using (2.94), we write that

$$\begin{aligned}\int_{B_0(\delta)}\left(\Delta_{h_i}f+n\right)v_i^{q_i}\,dv_{h_i}&=\left(1+\theta_i\right)^{q_i}\int_{B_0(\delta)}\left(\Delta_{h_i}f+n\right)B_i^{q_i}\,dv_{h_i}\\&\quad+O\left(r_i^2\int_{B_0(\delta)}|x|^2B_i^{q_i-1}\left|R_i\right|dv_{h_i}\right)\\&\quad+O\left(r_i^2\int_{B_0(\delta)}|x|^2\left|R_i\right|^{q_i}dv_{h_i}\right).\end{aligned} \tag{2.149}$$

By Hölder's and Sobolev's inequalities, thanks to Claims 2.3 and 2.5, we get after some computations that

$$\int_{B_0(\delta)}\left(\Delta_{h_i}f+n\right)v_i^{q_i}\,dv_{h_i}=\left(1+\theta_i\right)^{q_i}\int_{B_0(\delta)}\left(\Delta_{h_i}f+n\right)B_i^{q_i}\,dv_{h_i}+o\left(\gamma_i^{n-2}\right). \tag{2.150}$$

We write now, with the Cartan expansion of h_i around 0, and since B_i is radially symmetrical, that

$$\begin{aligned}\int_{B_0(\delta)}\left(\Delta_{h_i}f+n\right)B_i^{q_i}\,dv_{h_i}&=\frac{1}{3}\mathrm{Ric}_{h_i}(0)_{\alpha\beta}\int_{B_0(\delta)}x^\alpha x^\beta B_i^{q_i}\,dv_\xi\\&\quad+A_{\alpha\beta\gamma}\int_{B_0(\delta)}x^\alpha x^\beta x^\gamma B_i^{q_i}\,dv_\xi\\&\quad+O\left(r_i^4\int_{B_0(\delta)}|x|^4B_i^{q_i}\,dv_\xi\right)\end{aligned} \tag{2.151}$$

which gives after some computations, and thanks to Claim 2.3, that

$$\int_{B_0(\delta)} (\Delta_{h_i} f + n) B_i^{q_i} dv_{h_i} = \frac{n}{3} K_n^{-n/2} S_{h_i}(0) \gamma_i^2 + o(\gamma_i^{n-2}) = \frac{n}{3} K_n^{-n/2} S_g(y_i) r_i^2 \gamma_i^2 + o(\gamma_i^{n-2}). \tag{2.152}$$

Coming back to (2.150) with this last relation and Claims 2.3 and 2.5, we get that

$$\left(\frac{1}{2} - \frac{1}{q_i}\right) \int_{B_0(\delta)} (\Delta_{h_i} f + n) \tilde{\varphi}_i^{2^\star - q_i} v_i^{q_i} dv_{h_i} = \frac{1}{3} K_n^{-n/2} S_g(y_i) r_i^2 \gamma_i^2 + o(\gamma_i^{n-2}) + o(2^\star - q_i). \tag{2.153}$$

We write now, thanks to the expansion of the metric h_i around 0, that

$$\left(r_i^2 \tilde{a}_i + \frac{1}{2} r_i^2 (\nabla f, \nabla \tilde{a}_i)_{h_i} + \frac{1}{4} (\Delta_{h_i}^2) f \right) dv_{h_i} = \left(r_i^2 \left(a_i(y_i) - \frac{1}{6} S_g(y_i) \right) + B_\alpha x^\alpha + O(r_i^4 |x|^2) \right) dv_\xi, \tag{2.154}$$

where $B_\alpha = ((3/2) r_i^2 \partial_\alpha \tilde{a}_i(0) + (1/4) \partial_\alpha (\Delta_{h_i}^2 f)(0))$. Using the fact that B_i is radially symmetrical, we get then with (2.94) that

$$\begin{aligned} &\int_{B_0(\delta)} \left(r_i^2 \tilde{a}_i + \frac{1}{2} r_i^2 (\nabla f, \nabla \tilde{a}_i)_{h_i} + \frac{1}{4} (\Delta_{h_i}^2) f \right) v_i^2 dv_{h_i} \\ &\quad = r_i^2 \left(a_i(y_i) - \frac{1}{6} S_g(y_i) \right) (1 + \theta_i^2) \int_{B_0(\delta)} B_i^2 dv_\xi + O\left(r_i^2 \int_{B_0(\delta)} R_i^2 dv_\xi \right) \\ &\quad + O\left(r_i^4 \int_{B_0(\delta)} |x|^2 B_i^2 dv_\xi \right) + O\left(r_i^2 \int_{B_0(\delta)} |R_i| B_i dv_\xi \right). \end{aligned} \tag{2.155}$$

This leads after some computations, thanks to Hölder's and Sobolev's inequalities and to Claim 2.5, to

$$\begin{aligned} &\int_{B_0(\delta)} \left(r_i^2 \tilde{a}_i + \frac{1}{2} r_i^2 (\nabla f, \nabla \tilde{a}_i)_{h_i} + \frac{1}{4} (\Delta_{h_i}^2) f \right) v_i^2 dv_{h_i} \\ &\quad = \left(a_i(y_i) - \frac{1}{6} S_g(y_i) \right) \begin{cases} 64 \omega_3 r_i^2 \gamma_i^2 |\ln \gamma_i|, & \text{if } n = 4, \\ 16 K_5^{-5/2} r_i^2 \gamma_i^2, & \text{if } n = 5, \end{cases} \\ &\quad + o(\gamma_i^{n-2}) + O(\delta r_i^2 \gamma_i^{n-2}). \end{aligned} \tag{2.156}$$

Combining (2.141) with (2.145), (2.146), (2.148), (2.153), and this last estimate, we obtain, thanks to Claim 2.4, the estimate of Claim 2.6. ∎

The next step is crucial in order to prove during Section 3 that concentration points are isolated and thus the energy of solutions of (2.2) is a priori bounded.

Claim 2.7. If $r_i \to 0$ as $i \to +\infty$, then we necessarily have that $r_i = \rho_i$ for i large. Moreover, we have that

$$\gamma_i^{-(n-2)/2} v_i \longrightarrow H \text{ in } C^2_{loc}\left(B_0(2)\backslash\{0\}\right) \quad \text{as } i \longrightarrow +\infty, \tag{2.157}$$

where

$$H(x) = \frac{\lambda}{|x|^{n-2}} + h(x) \tag{2.158}$$

with

$$\lambda = \left(n(n-2)\right)^{-(n+2)/2} \tag{2.159}$$

and h some smooth harmonic function in $B_0(2)$ such that $h(0) \leq 0$. □

Proof of Claim 2.7. Assume that $r_i \to 0$ as $i \to +\infty$. Thanks to (2.88), (2.89), and (B), after passing to a subsequence, we have (2.157), where H satisfies

$$\Delta_\xi H = 0 \quad \text{in } B_0(2)\backslash\{0\}. \tag{2.160}$$

The classification of singularities of harmonic functions then gives the existence of some $\lambda \in \mathbb{R}$ and of some smooth harmonic function h such that

$$H(x) = \frac{\lambda}{|x|^{n-2}} + h(x) \quad \text{in } B_0(2)\backslash\{0\}. \tag{2.161}$$

In order to compute λ, we integrate equation (2.89) on $B_0(1)$ to obtain

$$\begin{aligned} &\gamma_i^{-(n-2)/2} \int_{B_0(1)} \tilde{\varphi}_i^{2^\star - q_i} v_i^{q_i - 1} dv_{h_i} \\ &\quad = -\int_{\partial B_0(1)} \partial_\nu H \, d\sigma_\xi + r_i^2 \gamma_i^{-(n-2)/2} \int_{B_0(1)} \tilde{a}_i v_i dv_{h_i} + o(1). \end{aligned} \tag{2.162}$$

Thanks to (2.91), (2.92), and Claim 2.3, we get that

$$\begin{aligned} &\gamma_i^{-(n-2)/2} \int_{B_0(1)} \tilde{\varphi}_i^{2^\star - q_i} v_i^{q_i - 1} dv_{h_i} = \frac{(n-2)\omega_{n-1}}{\left(n(n-2)\right)^{(n+2)/2}} + o(1), \\ &r_i^2 \gamma_i^{-(n-2)/2} \int_{B_0(1)} \tilde{a}_i v_i dv_{h_i} = o(1). \end{aligned} \tag{2.163}$$

Thus we obtain that

$$-\int_{\partial B_0(1)} \partial_\nu H \, d\sigma_\xi = \frac{(n-2)\omega_{n-1}}{\left(n(n-2)\right)^{(n+2)/2}}, \tag{2.164}$$

which leads to

$$\lambda = \frac{1}{\left(n(n-2)\right)^{(n+2)/2}}. \tag{2.165}$$

Thanks to (2.157), we can estimate M_i, given by (2.142): since $r_i \to 0$ as $i \to +\infty$,we obtain that

$$\begin{aligned} \lim_{i\to+\infty} \gamma_i^{2-n} M_i &= \int_{\partial B_0(\delta)} \left(\frac{\delta}{2} |\nabla H|_\xi^2 - \delta \left(\partial_\nu H\right)^2 - \frac{n-2}{2} H \partial_\nu H \right) d\sigma_\xi \\ &= \frac{(n-2)^2}{2} \lambda \omega_{n-1} h(0). \end{aligned} \tag{2.166}$$

Claim 2.6 independently gives that

$$M_i \le O\left(\delta r_i^2 \gamma_i^{n-2}\right) + o\left(\gamma_i^{n-2}\right) = o\left(\gamma_i^{n-2}\right) \tag{2.167}$$

since $a_i \le c_n S_g$, $q_i \le 2^\star$, and $r_i \to 0$ as $i \to +\infty$. Thus we obtain that

$$h(0) \le 0. \tag{2.168}$$

It remains to prove that $r_i = \rho_i$ for i large. Assume that, on the contrary, there is a subsequence such that $r_i < \rho_i$ for i large. Then, by definition (2.21) of r_i, we have that

$$\psi_i'(r_i) = 0, \tag{2.169}$$

where ψ_i is defined by (2.19). Thanks to Claim 2.4, to (2.22), and to (2.157), this leads to

$$\left(\frac{\int_{\partial B_0(r)} H \, d\sigma_\xi}{\omega_{n-1} r^{n/2}} \right)' (1) = 0. \tag{2.170}$$

Thanks to (2.161), we have that

$$\frac{\int_{\partial B_0(r)} H \, d\sigma_\xi}{\omega_{n-1} r^{n/2}} = \frac{\lambda}{r^{(n-2)/2}} + h(0) r^{(n-2)/2} \tag{2.171}$$

so that we obtain $h(0) = \lambda$ which is in contradiction with (2.168). This ends the proof of Claim 2.7. ∎

3 Proof of Theorem 1.1

We prove the theorem in this section. The notations of this section are independent of those of the previous one. We use the results of Section 2 with different sequences (x_i) and (ρ_i) satisfying assumptions (H1) and (H2) at the beginning of Section 2. We let (M, g) be a smooth compact Riemannian manifold of dimension $3 \leq n \leq 5$ and we let (a_i), (q_i), and (u_i) be as in the theorem. If (u_i) is bounded in $L^\infty(M)$, then (u_i) is bounded in $C^2(M)$ thanks to standard elliptic theory (see, e.g., [12]), and the conclusion of the theorem holds. We assume by contradiction that

$$\max_M u_i \longrightarrow +\infty \quad \text{as } i \longrightarrow +\infty. \tag{3.1}$$

We claim first the following.

Claim 3.1. We have that $q_i \to 2^\star$ as $i \to +\infty$. □

Proof of Claim 3.1. We let $x_i \in M$ be a point where u_i achieves its maximum. By (3.1), we have that

$$u_i(x_i) = \max_M u_i \longrightarrow +\infty \quad \text{as } i \longrightarrow +\infty. \tag{3.2}$$

Fix $0 < \delta < \text{inj}(M)$. We set for $x \in B_0(\delta u_i(x_i)^{(q_i-2)/2})$ the Euclidean ball of center 0 and radius $\delta u_i(x_i)^{(q_i-2)/2}$,

$$\begin{aligned} \tilde{u}_i(x) &= u_i(x_i)^{-1} u_i\Big(\exp_{x_i}\Big(u_i(x_i)^{-(q_i-2)/2} x\Big)\Big), \\ g_i(x) &= \exp_{x_i}^\star g\Big(u_i(x_i)^{-(q_i-2)/2} x\Big) \end{aligned} \tag{3.3}$$

so that

$$\begin{aligned} \Delta_{g_i} \tilde{u}_i + u_i(x_i)^{2-q_i} a_i\Big(\exp_{x_i}\Big(u_i(x_i)^{-(q_i-2)/2} x\Big)\Big)\tilde{u}_i = \tilde{u}_i^{q_i-1} \\ \text{in } B_0\Big(\delta u_i(x_i)^{(q_i-2)/2}\Big). \end{aligned} \tag{3.4}$$

Moreover, we have that

$$\tilde{u}_i \leq \tilde{u}_i(0) = 1 \quad \text{in } B_0\Big(\delta u_i(x_i)^{(q_i-2)/2}\Big). \tag{3.5}$$

Standard elliptic theory (see, e.g., [12]) then gives that, up to a subsequence,

$$\tilde{u}_i \longrightarrow \tilde{U} \text{ in } C^2_{loc}(\mathbb{R}^n) \quad \text{as } i \longrightarrow +\infty, \tag{3.6}$$

where

$$\Delta_\xi \tilde{U} = \tilde{U}^{q_0 - 1} \quad \text{in } \mathbb{R}^n. \tag{3.7}$$

Here, $q_0 = \lim_{i \to +\infty} q_i$, which does exist up to extracting a new subsequence. Thanks to [11], this is possible if and only if $q_0 = 2^\star$. This ends the proof of Claim 3.1. ■

Claims 3.2 and 3.3 are a way to exhaust roughly some of the concentration points of u_i together with a weak pointwise estimate. These claims should be compared with [10, Theorem 4.1] where the exhaustion of concentration points in that way is precise and complete when the energy of the u_i's is bounded.

Claim 3.2. Fix $R > 0$. There exists $D_0 > 2R$ and $i_0 \in \mathbb{N}$ such that for any $i \geq i_0$, for any compact set $\mathcal{S}_i \subset M$, if

$$\max_M \left(d_g(x, \mathcal{S}_i) u_i(x)^{(q_i - 2)/2}\right) \geq D_0, \tag{3.8}$$

then u_i possesses a local maximum $y_i \in M \backslash \mathcal{S}_i$ which satisfies

$$\begin{aligned} & d_g(y_i, \mathcal{S}_i) u_i(y_i)^{(q_i - 2)/2} \geq \frac{3D_0}{4}, \\ & d_g(y_i, x) u_i(x)^{(q_i - 2)/2} \leq \frac{D_0}{4} \quad \text{in } B_{y_i}\left(2R u_i(y_i)^{-(q_i - 2)/2}\right), \\ & \int_{B_{y_i}(R u_i(y_i)^{-(q_i - 2)/2})} u_i^{q_i} \, dv_g \geq \frac{1}{D_0}. \end{aligned} \tag{3.9}$$

We allow $\mathcal{S}_i$ to be the empty set with the convention that $d_g(y, \emptyset) = 1$ for all $y \in M$. □

Proof of Claim 3.2. Fix $R > 0$. We prove the claim by contradiction. We assume that, for some subsequence, there exists $D_i \to +\infty$ as $i \to +\infty$ and there exists a compact set $\mathcal{S}_i \subset M$, possibly empty, such that

$$\max_M \left(d_g(x, \mathcal{S}_i) u_i(x)^{(q_i - 2)/2}\right) \geq D_i \tag{3.10}$$

and such that there is no local maximum point of u_i satisfying the conclusion of the claim with D_i and $\mathcal{S}_i$. We let $z_i \in M \backslash \mathcal{S}_i$ be such that

$$d_g(z_i, \mathcal{S}_i) u_i(z_i)^{(q_i - 2)/2} = \max_M \left(d_g(x, \mathcal{S}_i) u_i(x)^{(q_i - 2)/2}\right) \tag{3.11}$$

and we set

$$u_i(z_i) = \varepsilon_i^{-2/(q_i - 2)}. \tag{3.12}$$

Since M is compact, we get, thanks to (3.10) and (3.11), that

$$\varepsilon_i \longrightarrow 0 \quad \text{as } i \longrightarrow +\infty. \tag{3.13}$$

We also have, thanks to (3.10) and (3.11), that

$$\frac{d_g(z_i, \mathcal{S}_i)}{\varepsilon_i} \longrightarrow +\infty \quad \text{as } i \longrightarrow +\infty. \tag{3.14}$$

Fix $\delta > 0$ small. We set for $x \in B_0(\delta\varepsilon_i^{-1})$ the Euclidean ball of center 0 and radius $\delta\varepsilon_i^{-1}$,

$$\bar{u}_i(x) = \varepsilon_i^{2/(q_i-2)} u_i\big(\exp_{z_i}(\varepsilon_i x)\big), \qquad \bar{g}_i(x) = \exp_{z_i}^\star g(\varepsilon_i x), \tag{3.15}$$

so that

$$\Delta_{\bar{g}_i}\bar{u}_i + \varepsilon_i^2 a_i\big(\exp_{z_i}(\varepsilon_i x)\big)\bar{u}_i = \bar{u}_i^{q_i-1} \quad \text{in } B_0\big(\delta\varepsilon_i^{-1}\big). \tag{3.16}$$

Thanks to (3.13), we also have that

$$\bar{g}_i \longrightarrow \xi \text{ in } C^2_{loc}\big(\mathbb{R}^n\big) \quad \text{as } i \longrightarrow +\infty. \tag{3.17}$$

We let $R > 0$ and we let $(\bar{z}_i)$ be a sequence of points in $B_0(R)$. Since

$$d_g\big(z_i, \exp_{z_i}(\varepsilon_i \bar{z}_i)\big) \le R\varepsilon_i, \tag{3.18}$$

we get, thanks to (3.14), that

$$d_g\big(\exp_{z_i}(\varepsilon_i\bar{z}_i), \mathcal{S}_i\big) = d_g\big(z_i, \mathcal{S}_i\big)\big(1 + o(1)\big). \tag{3.19}$$

This leads, thanks to (3.11), to

$$\bar{u}_i\big(\bar{z}_i\big) \le \bar{u}_i(0)\big(1 + o(1)\big) = 1 + o(1). \tag{3.20}$$

This proves that $(\bar{u}_i)$ is locally uniformly bounded in $\mathbb{R}^n$. Standard elliptic theory (see, e.g., [12]) then gives that, after passing to a subsequence,

$$\bar{u}_i \longrightarrow \bar{U} \text{ in } C^2_{loc}\big(\mathbb{R}^n\big) \quad \text{as } i \longrightarrow +\infty, \tag{3.21}$$

with $\Delta_\xi \bar{U} = \bar{U}^{2^\star-1}$ (since $q_i \to 2^\star$ by Claim 3.1) and $\bar{U}(0) = \max_{\mathbb{R}^n} \bar{U} = 1$. Thanks to [5], we have that

$$\bar{U} = \left(1 + \frac{|x|^2}{n(n-2)}\right)^{-(n-2)/2}. \tag{3.22}$$

This clearly proves that for i large, u_i possesses a local maximum point y_i satisfying that $d_g(z_i, y_i) = o(\varepsilon_i)$. One then easily checks that $y_i \in M \backslash \mathcal{S}_i$,

$$\begin{aligned} & d_g(y_i, \mathcal{S}_i) u_i(y_i)^{(q_i-2)/2} = D_i(1+o(1)) \geq \frac{3D_i}{4}, \\ & d_g(y_i, x) u_i(x)^{(q_i-2)/2} \leq \left(\max_{s \in [0,R]} \frac{s}{1 + \frac{s^2}{n(n-2)}} \right) (1+o(1)) \leq \frac{D_i}{4} \end{aligned} \tag{3.23}$$

in $B_{y_i}(Ru_i(y_i)^{-(q_i-2)/2})$, and

$$\begin{aligned} \int_{B_{y_i}(Ru_i(y_i)^{-(q_i-2)/2})} u_i^{q_i} \, dv_g & \geq (1+o(1)) \left(\int_{B_0(R)} \bar{U}^{2^\star} \, dx \right) \varepsilon_i^{n-2-4/(q_i-2)} \\ & \geq (1+o(1)) \left(\int_{B_0(R)} \bar{U}^{2^\star} \, dx \right) \geq \frac{1}{D_i} \end{aligned} \tag{3.24}$$

for i large. We thus constructed a local maximum of u_i satisfying the conclusion of the claim with D_i and $\mathcal{S}_i$. This is a contradiction. Claim 3.2 is proved. ■

Claim 3.3. There exist $D_1 > 0$ and $D_2 > 0$ such that for all i large enough, there exist $N(i) \in \mathbb{N}^\star$ and $N(i)$ local maxima of u_i, $x_{1,i}, \ldots, x_{N(i),i}$ such that

$$\begin{aligned} & d_g(x_{\alpha,i}, x_{\beta,i}) u_i(x_{\alpha,i})^{(q_i-2)/2} \geq D_1 \quad \forall \alpha, \beta = 1, \ldots, N(i), \ \alpha \neq \beta, \\ & \left(\min_{\alpha=1,\ldots,N(i)} d_g(x_{\alpha,i}, x) \right) u_i(x)^{(q_i-2)/2} \leq D_2 \end{aligned} \tag{3.25}$$

for all i large and all $x \in M$. □

Proof of Claim 3.3. We fix $R > 0$. We let $D_0 > 2R$ and $i_0 \in \mathbb{N}$ be given by Claim 3.2. We fix $i \geq i_0$ large enough such that

$$\left(\max_M u_i \right)^{(q_i-2)/2} \geq D_0. \tag{3.26}$$

Note that this is always possible thanks to (3.1). For $(x_1, \ldots, x_k)$, $k \in \mathbb{N}$, a family of local maxima of u_i, we consider the following assertions:

$(\mathcal{P}_k)$

$$B_{x_\alpha}\left(Ru_i(x_\alpha)^{-(q_i-2)/2}\right)\cap B_{x_\beta}\left(Ru_i(x_\beta)^{-(q_i-2)/2}\right)=\varnothing \quad \forall\alpha,\beta=1,\dots,k,\ \alpha\neq\beta; \tag{3.27}$$

$$\int_{B_{x_\alpha}(Ru_i(x_\alpha)^{-(q_i-2)/2})} u_i^{q_i}\,dv_g \geq \frac{1}{D_0} \quad \forall\alpha=1,\dots,k; \tag{3.28}$$

$$d_g(x_\alpha,x)u_i(x)^{(q_i-2)/2} \leq \frac{D_0}{4} \quad \text{in } B_{x_\alpha}\left(2Ru_i(x_\alpha)^{-(q_i-2)/2}\right) \quad \forall\alpha=1,\dots,k. \tag{3.29}$$

We say that $(\mathcal{P}_k)$ holds for u_i if there exists a family $(x_1,\dots,x_k)$ of local maxima of u_i such that the above assertions (3.27), (3.28), and (3.29) hold for this family.

We note first that $(\mathcal{P}_1)$ holds for (u_i). This is a consequence of Claim 3.2: thanks to (3.26), we can apply Claim 3.2 with $\mathcal{S}_i=\varnothing$. Let $k\geq 1$ be such that $(\mathcal{P}_k)$ holds for some family $(x_1,\dots,x_k)$ of local maxima of u_i. Then either $(\mathcal{P}_{k+1})$ holds for u_i or

$$d_g\left(x,\bigcup_{\alpha=1}^{k} B_{x_\alpha}\left(Ru_i(x_\alpha)^{-(q_i-2)/2}\right)\right)u_i(x)^{(q_i-2)/2}\leq D_0 \quad \text{in } M. \tag{3.30}$$

We now prove (3.30) For that purpose, we assume that

$$d_g\left(y,\bigcup_{\alpha=1}^{k} B_{x_\alpha}\left(Ru_i(x_\alpha)^{-(q_i-2)/2}\right)\right)u_i(y)^{(q_i-2)/2}\geq D_0 \tag{3.31}$$

for some $y\in M$. Thus we can apply Claim 3.2 with

$$\mathcal{S}_i=\bigcup_{\alpha=1}^{k} B_{x_\alpha}\left(Ru_i(x_\alpha)^{-(q_i-2)/2}\right). \tag{3.32}$$

This gives a local maximum $x_{k+1}\in M\backslash\mathcal{S}_i$ of u_i which satisfies

$$\begin{aligned} &d_g(x_{k+1},\mathcal{S}_i)u_i(x_{k+1})^{(q_i-2)/2}\geq \frac{3D_0}{4},\\ &d_g(x_{k+1},x)u_i(x)^{(q_i-2)/2}\leq\frac{D_0}{4} \quad \text{in } B_{x_{k+1}}\left(2Ru_i(x_{k+1})^{-(q_i-2)/2}\right),\\ &\int_{B_{x_{k+1}}(Ru_i(x_{k+1})^{-(q_i-2)/2})} u_i^{q_i}\,dv_g\geq\frac{1}{C_0}. \end{aligned} \tag{3.33}$$

We prove that assertions (3.27), (3.28), and (3.29) of $(\mathcal{P}_{k+1})$ hold for the family $(x_1,\dots,x_{k+1})$. Assertions (3.28) and (3.29) hold for $x_1,\dots,x_k$ thanks to $(\mathcal{P}_k)$, while they also hold for x_{k+1} thanks to (3.33). Thanks to assertion (3.27), it just remains to prove that for any $\alpha \in \{1,\dots,k\}$,

$$B_{x_\alpha}\left(Ru_i(x_\alpha)^{-(q_i-2)/2}\right)\bigcap B_{x_{k+1}}\left(Ru_i(x_{k+1})^{-(q_i-2)/2}\right)=\emptyset. \tag{3.34}$$

Thanks to (3.33), since $D_0 > 2R$, we have

$$d_g(x_{k+1},\mathcal{S}_i) \geq \frac{3}{2}Ru_i(x_{k+1})^{-(q_i-2)/2}. \tag{3.35}$$

Definition (3.32) of $\mathcal{S}_i$ then clearly gives the equation we were looking for. This ends the proof of (3.30).

We apply now (3.30) by induction of k. The process will necessarily stop for some $k = N(i)$ since assertions (3.27) and (3.29) imply that

$$\int_M u_i^{q_i}\,dv_g \geq \frac{k}{D_0}. \tag{3.36}$$

Then we have the existence of $(x_1,\dots,x_{N(i)})$, a family of local maxima of u_i, such that assertions (3.27), (3.28), and (3.29) of $(\mathcal{P}_{N(i)})$ hold for this family and that

$$d_g\left(x, \bigcup_{\alpha=1}^{k} B_{x_\alpha}\left(Ru_i(x_\alpha)^{-(q_i-2)/2}\right)\right)u_i(x)^{(q_i-2)/2} \leq D_0 \quad \text{in } M. \tag{3.37}$$

Thanks to assertion (3.27) of $(\mathcal{P}_{N(i)})$, we have that

$$d_g(x_\alpha,x_\beta)u_i(x_\alpha)^{(q_i-2)/2} \geq R \quad \forall\alpha,\beta=1,\dots,N(i),\ \alpha\neq\beta. \tag{3.38}$$

Let $x \in M$. If

$$x \in \bigcup_{\alpha=1}^{N(i)} B_{x_\alpha}\left(2Ru_i(x_\alpha)^{-(q_i-2)/2}\right), \tag{3.39}$$

then

$$\left(\min_{\alpha=1,\dots,N(i)} d_g(x_\alpha,x)\right)u_i(x)^{(q_i-2)/2} \leq \frac{D_0}{4} \tag{3.40}$$

thanks to assertion (3.29) of $(\mathcal{P}_{N(i)})$. If

$$x \notin \bigcup_{\alpha=1}^{N(i)} B_{x_\alpha}\left(2Ru_i(x_\alpha)^{-(q_i-2)/2}\right), \tag{3.41}$$

we let $\beta \in \{1, \dots, N(i)\}$ be such that

$$d_g\left(x, \bigcup_{\alpha=1}^{k} B_{x_\alpha}\left(Ru_i(x_\alpha)^{-(q_i-2)/2}\right)\right) = d_g\left(x, B_{x_\beta}\left(Ru_i(x_\beta)^{-(q_i-2)/2}\right)\right) \tag{3.42}$$

and we write

$$\begin{aligned}\left(\min_{\alpha=1,\dots,N(i)} d_g(x_\alpha, x)\right)u_i(x)^{(q_i-2)/2} &\le d_g(x_\beta, x)u_i(x)^{(q_i-2)/2}\\ &\le 2d_g\left(x, B_{x_\beta}\left(Ru_i(x_\beta)^{-(q_i-2)/2}\right)\right)u_i(x)^{(q_i-2)/2}\\ &\le 2D_0\end{aligned} \tag{3.43}$$

thanks to (3.37). Thus we have proved that Claim 3.3 holds with $D_1 = R$ and $D_2 = 2D_0$. ■

Now let $d_i > 0$ be defined by

$$d_i = \min_{\alpha,\beta=1,\dots,N(i),\ \alpha\neq\beta} d_g(x_{\alpha,i}, x_{\beta,i}). \tag{3.44}$$

Claim 3.5 will assert that $d_i \ge d > 0$, that is, that the concentration points are isolated. The next claim is a technical step toward this result.

Claim 3.4. We let $1 \le \alpha_i \le N(i)$. If

$$d_i u_i(x_{\alpha_i,i})^{(q_i-2)/2} = O(1), \tag{3.45}$$

then

$$d_i\left(\sup_{B_{x_{\alpha_i,i}}(d_i/2)} u_i\right)^{(q_i-2)/2} = O(1). \tag{3.46}$$

□

Proof of Claim 3.4. Up to reordering the $x_{i,\alpha}$'s, we may assume that $\alpha_i = 1$ for all i. We assume that

$$d_i u_i(x_{1,i})^{(q_i-2)/2} = O(1). \tag{3.47}$$

We let $y_i \in \overline{B_{x_{1,i}}(d_i/2)}$ be such that

$$\sup_{B_{x_{1,i}}(d_i/2)} u_i = u_i(y_i) \tag{3.48}$$

and assume by contradiction that

$$d_i u_i(y_i)^{(q_i-2)/2} \longrightarrow +\infty \quad \text{as } i \longrightarrow +\infty. \tag{3.49}$$

By Claim 3.3 and thanks to definition (3.44) of d_i, we have that

$$d_g(x_{1,i}, y_i) u_i(y_i)^{(q_i-2)/2} \leq D_2 \tag{3.50}$$

so that

$$d_g(x_{1,i}, y_i) = o(d_i). \tag{3.51}$$

We set

$$\widehat{\mu}_i = u_i(y_i)^{-(q_i-2)/2} \tag{3.52}$$

and we set for $x \in B_0(\delta\widehat{\mu}_i^{-1})$ the Euclidean ball of center 0 and radius $\delta\widehat{\mu}_i^{-1}$, with $\delta > 0$ small fixed,

$$\begin{aligned} \widehat{u}_i(x) &= \widehat{\mu}_i^{2/(q_i-2)} u_i\big(\exp_{y_i}(\widehat{\mu}_i x)\big), \\ \widehat{g}_i(x) &= \exp_{y_i}^{\star} g(\widehat{\mu}_i x), \\ \widehat{a}_i(x) &= a_i\big(\exp_{y_i}(\widehat{\mu}_i x)\big). \end{aligned} \tag{3.53}$$

Since $\widehat{\mu}_i \to 0$ as $i \to +\infty$ (thanks to (3.49)), we obtain that $\widehat{g}_i \to \xi$ in $C^2_{loc}(\mathbb{R}^n)$ as $i \to +\infty$. Thanks to (3.48), (3.49), and (3.51), we also have that $(\widehat{u}_i)$ is uniformly bounded in all compact subsets of $\mathbb{R}^n$. Since $\widehat{u}_i$ verifies

$$\Delta_{\widehat{g}_i}\widehat{u}_i + \widehat{\mu}_i^2 \widehat{a}_i \widehat{u}_i = \widehat{u}_i^{q_i-1} \quad \text{in } B_0(\delta\widehat{\mu}_i^{-1}), \tag{3.54}$$

we get by standard elliptic theory that $\widehat{u}_i \to \widehat{U}$ in $C^2_{loc}(\mathbb{R}^n)$ as $i \to +\infty$, where $\widehat{U}$ is a solution of $\Delta_\xi \widehat{U} = \widehat{U}^{2^\star-1}$ in $\mathbb{R}^n$, $\widehat{U}(0) = 1$. Then $\widehat{U} > 0$ in $\mathbb{R}^n$. By (3.50), $((1/\widehat{\mu}_i)\exp_{y_i}^{-1}(x_{1,i}))$ is a bounded sequence of points in $\mathbb{R}^n$ so that

$$\liminf_{i\to+\infty} \frac{u_i(x_{1,i})}{u_i(y_i)} > 0. \tag{3.55}$$

This is in contradiction with (3.47) and (3.49). This proves Claim 3.4. ■

Claim 3.5. There exists $d > 0$ such that $d_i \geq d$ for all i. □

Proof of Claim 3.5. Up to reordering the $x_{\alpha,i}$'s, we may assume that

$$d_i = d_g\left(x_{1,i}, x_{2,i}\right). \tag{3.56}$$

We assume by contradiction that

$$d_i \longrightarrow 0 \quad \text{as } i \longrightarrow +\infty. \tag{3.57}$$

We set for $x \in B_0(\delta d_i^{-1})$ the Euclidean ball of center 0 and radius δd_i^{-1}, with $\delta > 0$ small fixed,

$$\begin{aligned} \check{u}_i(x) &= d_i^{2/(q_i-2)} u_i\left(\exp_{x_{1,i}}(d_i x)\right), \\ \check{g}_i(x) &= \exp_{x_{1,i}}^{\star} g\left(d_i x\right), \\ \check{a}_i(x) &= a_i\left(\exp_{x_{1,i}}(d_i x)\right). \end{aligned} \tag{3.58}$$

By (3.57), we have that $\check{g}_i \to \xi$ in $C^2_{loc}(\mathbb{R}^n)$ as $i \to +\infty$. Independently, we have that $\check{u}_i$ verifies

$$\Delta_{\check{g}_i} \check{u}_i + d_i^2 \check{a}_i \check{u}_i = \check{u}_i^{q_i-1} \quad \text{in } B_0\left(\delta d_i^{-1}\right). \tag{3.59}$$

We let

$$\check{x}_{2,i} = \frac{1}{d_i} \exp_{x_{1,i}}^{-1}\left(x_{2,i}\right) \tag{3.60}$$

so that $|\check{x}_{2,i}| = 1$. Up to a subsequence, $\check{x}_{2,i} \to \check{x}_2$ as $i \to +\infty$. For $R > 0$, we set

$$\check{S}_{R,i} = \left\{ \check{x}_{\alpha,i} = \frac{1}{d_i} \exp_{x_{1,i}}^{-1}\left(x_{\alpha,i}\right),\ \alpha = 1, \dots, N(i) : x_{\alpha,i} \in B_{x_{1,i}}\left(R d_i\right) \right\}. \tag{3.61}$$

Thanks to the definition of d_i and to (3.56), we have that, up to a subsequence,

$$\check{S}_{R,i} \longrightarrow \check{S}_R \quad \text{as } i \longrightarrow +\infty, \tag{3.62}$$

with $\check{S}_R$ a finite set which contains 0 and $\check{x}_2$. Also let

$$\check{S} = \bigcup_{R>0} \check{S}_R. \tag{3.63}$$

We assume that

$$\text{there exists } \beta_i = 1, \dots, N(i) \text{ such that } \begin{cases} d_g\left(x_{1,i}, x_{\beta_i,i}\right) = O\left(d_i\right), \\ \check{u}_i\left(x_{\beta_i,i}\right) = O(1). \end{cases} \tag{3.64}$$

We claim that

$$(3.64) \Longrightarrow (\check{u}_i) \text{ is uniformly bounded in all compact subsets of } \mathbb{R}^n. \tag{3.65}$$

In order to prove (3.65), we first note that, for a sequence $\alpha_i = 1, \dots, N(i)$ such that $d_g(x_{1,i}, x_{\alpha_i,i}) = O(d_i)$, two situations can occur: either $\check{u}_i(\check{x}_{\alpha_i,i})$ is bounded and then, thanks to Claim 3.4, $(\check{u}_i)$ is uniformly bounded in $B_{\check{x}_{\alpha_i,i}}(1/2)$ or $\check{u}_i(\check{x}_{\alpha_i,i}) \to +\infty$ as $i \to +\infty$ and then we can apply the results of Section 2 with $x_i = x_{\alpha_i,i}$ and $\rho_i = d_i/6$ thanks to Claim 3.3. Assume now that for some $\alpha_i = 1, \dots, N(i)$, $d_g(x_{1,i}, x_{\alpha_i,i}) = O(d_i)$ and $\check{u}_i(\check{x}_{\alpha_i,i}) \to +\infty$ as $i \to +\infty$. Applying Claim 2.7 with $x_i = x_{\alpha_i,i}$ and $\rho_i = d_i/6$, we obtain that $\check{u}_i \to 0$ in $C^2_{loc}(B_{\check{x}}(1/9)\backslash\{\check{x}\})$, where $\check{x} = \lim_{i\to+\infty} \check{x}_{\alpha_i,i}$. We let $R > 2|\check{x}|$. We know, thanks to Claim 3.3 and to definition (3.44) of d_i, that $(\check{u}_i)$ is uniformly bounded in all compact subsets of $B_0(R)\backslash\check{S}_R$. But, thanks to (3.64) and to Claim 3.4, $(\check{u}_i)$ is uniformly bounded on $B_{\check{y}}(1/2)$, where $\check{y} = \lim_{i\to+\infty} \check{x}_{\beta_i,i}$. We thus obtain, thanks to Harnack's inequality, that $\check{u}_i(\check{x}_{\beta_i,i}) \to 0$ as $i \to +\infty$. This is in contradiction with the first assertion of Claim 3.3. Thus we have proved that, for all $\alpha_i = 1, \dots, N(i)$ such that $d_g(x_{1,i}, x_{\alpha_i,i}) = O(d_i)$, $\check{u}_i(\check{x}_{\alpha_i,i}) = O(1)$. Thanks to Claim 3.4, this proves that $(\check{u}_i)$ is uniformly bounded in a neighborhood of $\check{S}_R$ for all $R > 0$. Thanks to Claim 3.3, $(\check{u}_i)$ is also uniformly bounded in all compact subsets of $B_0(R)\backslash\check{S}_R$ for all $R > 0$. This clearly proves (3.65). Then we can pass to the limit in equation (3.59) thanks to standard elliptic theory: this gives that $\check{u}_i \to \check{U}$ in $C^2_{loc}(\mathbb{R}^n)$ as $i \to +\infty$ with $\Delta_\xi \check{U} = \check{U}^{2^\star-1}$. Thanks to the first part of Claim 3.3, we know that $\check{U}(0) \geq C_1^{(n-2)/2}$. Thanks to Claim 3.3, we also know that $\check{U}$ possesses at least two local maxima, namely 0 and $\check{x}_2$. By the work of Caffarelli, Gidas, and Spruck [5], this is impossible. Thus (3.64) leads to a contradiction.

Thus, for any $\alpha_i = 1, \dots, N(i)$ such that $d_g(x_{1,i}, x_{\alpha_i,i}) = O(d_i)$, $\check{u}_i(\check{x}_{\alpha_i,i}) \to +\infty$ as $i \to +\infty$ and we can apply the results of Section 2 with $x_i = x_{\alpha_i,i}$ and $\rho_i = d_i/6$. Applying, in particular, Claim 2.7, we obtain that

$$\check{u}_i(0)\check{u}_i \longrightarrow \check{H} \text{ in } C^2_{loc}(\mathbb{R}^n\backslash\check{S}) \quad \text{as } i \longrightarrow +\infty, \tag{3.66}$$

where $\check{S}$ is as in (3.63) and

$$\check{H} = \frac{\lambda_1}{|x|^{n-2}} + \frac{\lambda_2}{|x - \check{x}_2|^{n-2}} + \check{h} \tag{3.67}$$

with $\check{h}$ a nonnegative harmonic function in $\mathbb{R}^n\backslash\{\check{S}\backslash\{0, \check{x}_2\}\}$, $\lambda_1 > 0$, and $\lambda_2 > 0$. Then we can write that

$$\check{H} = \frac{\lambda_1}{|x|^{n-2}} + A + o(1) \tag{3.68}$$

around 0 with $A > 0$. This is easily checked to be in contradiction with the last part of Claim 2.7. Thus this second situation also leads to a contradiction. This clearly proves that (3.57) is absurd. Claim 3.5 is proved. ■

Now, that we know that $d_i \geq d > 0$, we are ready to end the proof of the theorem. The arguments are really similar to those used at the end of [7]. We recall them briefly here. Up to a subsequence, we may assume that $N(i) = N$ for all i. We let $(x_{\alpha,i})_{\alpha=1,\ldots,N}$ be the family of local maxima of (u_i) given by Claim 3.3. Let $\alpha \in \{1,\ldots,N\}$. If $u_i(x_{\alpha,i}) = O(1)$, then, by Claim 3.4, (u_i) is uniformly bounded in $B_{x_{\alpha,i}}(\delta/2)$. In this case, the assertions of Claim 3.3 continue to hold even if we remove $x_{\alpha,i}$ from the family $\{x_{\beta,i}\}_{\beta=1,\ldots,N}$, up to changing the constants D_1 and D_2. Thus we may assume without loss of generality that

$$u_i(x_{\alpha,i}) \longrightarrow +\infty \quad \text{as } i \longrightarrow +\infty \ \ \forall \alpha \in \{1,\ldots,N\}. \tag{3.69}$$

Applying the results of Section 2 successively to $x_i = x_{\alpha,i}$, $\alpha = 1,\ldots,N$, with $\rho_i = d/6$, we get then, thanks to standard elliptic theory, that there exists $C > 1$ such that

$$\frac{1}{C} u_i(x_{1,\alpha}) \leq u_i(x_{\alpha,i}) \leq C u_i(x_{1,\alpha}) \quad \forall \alpha = 1,\ldots,N. \tag{3.70}$$

Setting

$$x_\alpha = \lim_{i\to+\infty} x_{\alpha,i} \quad \text{for } \alpha = 1,\ldots,N, \tag{3.71}$$

we get, by standard elliptic theory and thanks to the results of Section 2, that, after passing to a subsequence,

$$u_i(x_{1,i})u_i \longrightarrow H \text{ in } C^2_{loc}(M\backslash\{x_1,\ldots,x_N\}) \quad \text{as } i \longrightarrow +\infty, \tag{3.72}$$

where

$$H(x) = \frac{(n-2)\omega_{n-1}}{\big(n(n-2)\big)^{(n+2)/2}} \sum_{\alpha=1}^{N} \lambda_\alpha G(x_\alpha, x) \tag{3.73}$$

with

$$\lambda_\alpha = \lim_{i\to+\infty} \left(\frac{u_i(x_{1,i})}{u_i(x_{\alpha,i})}\right). \tag{3.74}$$

Here, G is the Green function of the limit operator $\Delta_g + a_\infty$. Note also that $\lambda_\alpha > 0$ for all $\alpha = 1,\ldots,N$ thanks to (3.70). Now we let $\varphi \in C^\infty(M)$, $\varphi > 0$, be such that

$$\varphi(x_1) = 0, \qquad \nabla\varphi(x_1) = 0, \tag{3.75}$$

and such that the metric $h = \varphi^{-4/(n-2)} g$ verifies

$$\mathrm{Ric}_h(x_1) = 0. \tag{3.76}$$

It is always possible to find such a φ (see, e.g., [16]). We set $w_i = u_i \varphi$ so that w_i verifies

$$\Delta_h w_i + \alpha_i w_i = \varphi^{2^\star - q_i} w_i^{q_i - 1} \quad \text{in } M, \tag{3.77}$$

with

$$\alpha_i = c_n S_h + \left(a_i - c_n S_g\right) \varphi^{2^\star - 2}. \tag{3.78}$$

Thanks to Claim 2.1 applied to u_i (with $x_i = x_{1,i}$ and $\rho_i = d/8$), it is clear that there exists $y_{1,i} \in M$, a local maximum of w_i which satisfies

$$d_g\left(x_{1,i}, y_{1,i}\right) u_i\left(x_{1,i}\right)^{2/(q_i - 2)} = o(1). \tag{3.79}$$

It is then easily checked that we can apply the results of Section 2 to w_i with $x_i = y_{1,i}$ and $\rho_i = d/8$. Note that (3.78) implies that $\alpha_i \le c_n S_h$ since $a_i \le c_n S_g$. Applying Claim 2.6, we obtain, in particular, that

$$\begin{aligned} M_\infty(\delta) \mu_i^{n-2} \le{} & -\left(\frac{(n-2)^2}{4n} K_n^{-n/2} + o(1)\right)\left(2^\star - q_i\right) \\ & + C\delta \mu_i^{n-2} + o\left(\mu_i^{n-2}\right) \\ & + \left(\alpha_i(x_i) - c_n S_h(x_i)\right) \begin{cases} \left(64\omega_3 + o(1)\right)\mu_i^2 \ln \dfrac{1}{\mu_i}, & \text{if } n = 4, \\ \left(16 K_5^{-5/2} + o(1)\right)\mu_i^2, & \text{if } n = 5, \end{cases} \end{aligned} \tag{3.80}$$

where $w_i(y_{1,i}) = \mu_i^{-2/(q_i - 2)}$ and

$$\begin{aligned} M_\infty(\delta) = {} & \frac{1}{2} \int_{\partial B_{x_1}(\delta)} (\nabla f, \nu)_h \left|\nabla(\varphi H)\right|_h^2 d\sigma_h \\ & + \frac{1}{2} \int_{\partial B_{x_1}(\delta)} (\nabla f, \nu)_h \alpha_\infty (\varphi H)^2 d\sigma_h \\ & - \frac{n-2}{2} \int_{\partial B_{x_1}(\delta)} \left(\nabla(\varphi H), \nu\right)_h \varphi H \, d\sigma_h \\ & - \int_{\partial B_{x_1}(\delta)} \left(\nabla(\varphi H), \nabla f\right)_h \left(\nabla(\varphi H), \nu\right)_h d\sigma_h \\ & + \frac{1}{2} \int_{\partial B_{x_1}(\delta)} \left(\Delta_h f + n\right)\left(\nabla(\varphi H), \nu\right)_h \varphi H \, d\sigma_h \\ & - \frac{1}{4} \int_{\partial B_{x_1}(\delta)} \left(\nabla\left(\Delta_h f\right), \nu\right)_h (\varphi H)^2 d\sigma_h. \end{aligned} \tag{3.81}$$

Note that we used Claim 2.3, (3.72), and (3.75) to estimate M_i given by (2.149). In (3.80), C is some constant independent of i and δ. In (3.81), $x_1 = \lim_{i\to+\infty} x_{1,i}$ and $f(x) = (1/2)d_h(x_1, x)^2$. Estimate (3.80) clearly implies that

$$\alpha_\infty(x_1) = c_n S_h(x_1), \tag{3.82}$$

where

$$\alpha_\infty = c_n S_h + \left(a_\infty - c_n S_g\right)\varphi^{2^\star-2}. \tag{3.83}$$

Using $q_i \le 2^\star$ and $\alpha_i \le c_n S_g$, we also get from (3.80) that

$$\limsup_{\delta\to 0} M_\infty(\delta) \le 0. \tag{3.84}$$

We write that, in a neighborhood of x_1,

$$\varphi(x)H(x) = \frac{(n-2)\omega_{n-1}}{\left(n(n-2)\right)^{(n+2)/2}}\tilde{G}(x_1, x) + \beta(x), \tag{3.85}$$

where $\tilde{G}$ is the Green function of $\Delta_h + \alpha_\infty$ and β, C^2, in a neighborhood of x_1, verifies $\Delta_h\beta + \alpha_\infty\beta = 0$ and $\beta(x_1) \ge 0$. Note that $\beta(x_1) = 0$ if and only if $x_{1,i}$ is the only concentration point of u_i. We Let $\bar{G}$ be the Green function of $\Delta_h + c_n S_h$. Then

$$\tilde{G} = \bar{G} + \tilde{\beta}, \tag{3.86}$$

where $\tilde{\beta}$ verifies that

$$\Delta_h\tilde{\beta} + \alpha_\infty\tilde{\beta} = \left(c_n S_h - \alpha_\infty\right)\bar{G} \tag{3.87}$$

in M in the sense of distributions. Since $\alpha_\infty \le c_n S_h$ and $\alpha_\infty(x_1) = c_n S_h(x_1)$ with (2.1), (3.78), and (3.82), we have by standard properties of Green's functions that

$$0 \le \left(c_n S_h - \alpha_\infty\right)\bar{G} \le C \begin{cases} d_h\left(x_1, x\right)^{-1}, & \text{if } n = 3, \\ 1, & \text{if } n = 4, \\ d_h\left(x_1, x\right)^{-1}, & \text{if } n = 5, \end{cases} \tag{3.88}$$

so that $\tilde{\beta} \in C^{0,\eta}(M) \cap C^2(M\backslash\{x_1\})$ for all $0 < \eta < 1$. It comes also from standard elliptic estimates that

$$\delta \sup_{\partial B_{x_1}(\delta)} \left|\nabla\tilde{\beta}\right|_h \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0. \tag{3.89}$$

At last, the maximum principle gives that either $\tilde{\beta} > 0$ in M or $\tilde{\beta} \equiv 0$ in M, and $\alpha_\infty \equiv c_n S_h$ in M.

Thanks to the choice of h we made, see (3.76), we know that (see [16]), in a neighborhood of x,

$$\bar{G}(x_1, x) = \frac{1}{(n-2)\omega_{n-1} d_g(x_1, x)^{n-2}} + \bar{\beta}(x) \tag{3.90}$$

for some $\bar{\beta} \in C^{0,\eta}(M) \cap C^2(M \backslash \{x_1\})$ for all $0 < \eta < 1$ verifying that

$$\delta \sup_{\partial B_{x_1}(\delta)} |\nabla \bar{\beta}|_h \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0. \tag{3.91}$$

Moreover, we have that $\bar{\beta}(x_1) > 0$ except if (M, h) is conformally diffeomorphic to the standard sphere (S^n, can).

This result comes from the positive mass theorem and has been proved by [25, 26]. Summarizing, we arrive at

$$\varphi H = \frac{1}{\big(n(n-2)\big)^{(n+2)/2}} \frac{1}{d_h(x_1, x)^{n-2}} + R_0(x) \tag{3.92}$$

in a neighborhood of x_1 with $R_0 = (n-2)\omega_{n-1}(\tilde{\beta} + \bar{\beta}) + \beta$. It is then rather easily checked, thanks to the estimates on β, $\tilde{\beta}$, and $\bar{\beta}$ above, that

$$\lim_{\delta \to 0} M_\infty(\delta) = \frac{1}{\big(n(n-2)\big)^{(n+2)/2}} \frac{(n-2)^2}{2} \omega_{n-1} R_0(x_1). \tag{3.93}$$

Thanks to the above discussion, we have $R_0(x_1) > 0$ except if there is only one concentration point, $a_\infty \equiv c_n S_g$, and (M, g) is conformally diffeomorphic to (S^n, can). This ends the proof of the theorem thanks to (3.84).

Remark 3.6. Note that the above proof gives the compactness of sequences (u_i) of solutions of equation (1.7) in all dimensions if $a_\infty < c_n S_g$ in (1.6). In other words, when the limit of the linear term is strictly below the linear term of the Yamabe equation, compactness holds for (1.7). This can be seen by noticing that the leading term in the formula of Claim 2.6 will always be the term involving the scalar curvature in this case. And this is true whatever the dimension is. With this remark, it is easily checked that the subsequent arguments of the proof continue to hold in all dimensions and lead to a contradiction.

Appendix

A Pohozaev identity

We prove the Pohozaev identity we repeatedly used in this paper. We let (M, g) be a complete Riemannian manifold and let Ω be a compact subset of M with smooth boundary. We let $x_0 \in M$ and $R > 0$ be such that $\Omega \subset B_{x_0}(R)$ and we assume that u is a smooth positive function such that

$$\Delta_g u + a u = \psi u^{q-1} \tag{A.1}$$

in $B_{x_0}(R)$ for some $a \in C^\infty(B_{x_0}(R))$ and some $2 < q \leq 2^\star$. At last, we let $f \in C^\infty(B_{x_0}(R))$. Integrating by parts, we have that

$$\begin{aligned}\int_\Omega &(\nabla u, \nabla f)_g \Delta_g u \, dv_g \\ &= \int_\Omega \left(\nabla\left((\nabla u, \nabla f)_g\right), \nabla u\right)_g dv_g - \int_{\partial\Omega} (\nabla u, \nabla f)_g (\nabla u, \nu)_g \, d\sigma_g,\end{aligned} \tag{A.2}$$

where ν denotes the unit outer normal of $\partial\Omega$ and $d\sigma_g$ is the induced Riemannian measure on $\partial\Omega$. Noting that

$$\left(\nabla\left((\nabla u, \nabla f)_g\right), \nabla u\right)_g = \nabla^2 f(\nabla u, \nabla u) + \frac{1}{2}\left(\nabla f, \nabla\left(|\nabla u|_g^2\right)\right)_g, \tag{A.3}$$

we obtain by integration by parts that

$$\begin{aligned}\int_\Omega (\nabla u, \nabla f)_g \Delta_g u \, dv_g = \frac{1}{2}\int_\Omega \Delta_g f |\nabla u|_g^2 dv_g + \int_\Omega \nabla^2 f(\nabla u, \nabla u) dv_g \\ + \frac{1}{2}\int_{\partial\Omega} (\nabla f, \nu)_g |\nabla u|_g^2 d\sigma_g - \int_{\partial\Omega} (\nabla u, \nabla f)_g (\nabla u, \nu)_g \, d\sigma_g\end{aligned} \tag{A.4}$$

so that

$$\begin{aligned}\int_\Omega &(\nabla u, \nabla f)_g \Delta_g u \, dv_g + \frac{n-2}{2}\int_\Omega |\nabla u|_g^2 dv_g \\ &= \frac{1}{2}\int_{\partial\Omega} (\nabla f, \nu)_g |\nabla u|_g^2 d\sigma_g - \int_{\partial\Omega} (\nabla u, \nabla f)_g (\nabla u, \nu)_g \, d\sigma_g \\ &\quad + \frac{1}{2}\int_\Omega \left(\Delta_g f + n\right)|\nabla u|_g^2 dv_g + \int_\Omega \left(\nabla^2 f - g\right)(\nabla u, \nabla u) dv_g.\end{aligned} \tag{A.5}$$

Now, we use the equation satisfied by u to get that

$$\begin{aligned}
\int_\Omega |\nabla u|_g^2 dv_g &= \int_{\partial\Omega} u(\nabla u, \nu)_g d\sigma_g + \int_\Omega \psi u^q dv_g - \int_\Omega a u^2 dv_g, \\
\int_\Omega (\nabla u, \nabla f)_g \Delta_g u \, dv_g &= \int_\Omega \Delta_g f \left(\frac{1}{q}\psi u^q - \frac{1}{2} a u^2\right) dv_g + \frac{1}{2}\int_\Omega (\nabla f, \nabla a)_g u^2 dv_g \\
&\quad - \frac{1}{q}\int_\Omega (\nabla f, \nabla \psi)_g u^q dv_g + \int_{\partial\Omega} (\nabla f, \nu)_g \left(\frac{1}{q}\psi u^q - \frac{1}{2} a u^2\right) d\sigma_g,
\end{aligned} \tag{A.6}$$

which gives that

$$\begin{aligned}
&\int_\Omega (\nabla u, \nabla f)_g \Delta_g u \, dv_g + \frac{n-2}{2}\int_\Omega |\nabla u|_g^2 dv_g \\
&\quad = \left(\frac{n-2}{2} - \frac{n}{q}\right)\int_\Omega \psi u^q dv_g - \frac{1}{q}\int_\Omega (\nabla f, \nabla\psi)_g u^q dv_g \\
&\quad + \int_{\partial\Omega} (\nabla f, \nu)_g \left(\frac{1}{q}\psi u^q - \frac{1}{2} a u^2\right) d\sigma_g + \frac{n-2}{2}\int_{\partial\Omega} (\nabla u, \nu)_g u \, d\sigma_g \\
&\quad + \int_\Omega (\Delta_g f + n)\left(\frac{1}{q}\psi u^q - \frac{1}{2} a u^2\right) dv_g + \int_\Omega \left(a + \frac{1}{2}(\nabla a, \nabla f)_g\right) u^2 dv_g.
\end{aligned} \tag{A.7}$$

Thus we have obtained that

$$\begin{aligned}
&\int_B \left(a + \frac{1}{2}(\nabla a, \nabla f)_g\right) u^2 dv_g + \left(\frac{n-2}{2} - \frac{n}{q}\right)\int_\Omega \psi u^q dv_g \\
&\quad = \int_\Omega (\Delta_g f + n)\left(\frac{1}{2}|\nabla u|_g^2 + \frac{1}{2} a u^2 - \frac{1}{q}\psi u^q\right) dv_g + \frac{1}{q}\int_\Omega (\nabla f, \nabla \psi)_g u^q dv_g \\
&\quad + \int_\Omega (\nabla^2 f - g)(\nabla u, \nabla u) dv_g - \int_{\partial\Omega} (\nabla f, \nu)_g \left(\frac{1}{q}\psi u^q - \frac{1}{2} a u^2\right) d\sigma_g \\
&\quad + \frac{1}{2}\int_{\partial\Omega} (\nabla f, \nu)_g |\nabla u|_g^2 d\sigma_g - \int_{\partial\Omega} (\nabla u, \nabla f)_g (\nabla u, \nu)_g d\sigma_g \\
&\quad - \frac{n-2}{2}\int_{\partial\Omega} u(\nabla u, \nu)_g d\sigma_g.
\end{aligned} \tag{A.8}$$

Integrating by parts and using the equation satisfied by u, we have that

$$\begin{aligned}
&\int_\Omega (\Delta_g f + n)|\nabla u|_g^2 dv_g \\
&\quad = \int_\Omega \big(\nabla\big((\Delta_g f + n)u\big), \nabla u\big)_g dv_g - \frac{1}{2}\int_\Omega \big(\nabla(\Delta_g f), \nabla u^2\big)_g dv_g \\
&\quad = \int_{\partial\Omega} (\Delta_g f + n)(\nabla u, \nu)_g u \, d\sigma_g - \frac{1}{2}\int_{\partial\Omega} \big(\nabla(\Delta_g f), \nu\big)_g u^2 d\sigma_g \\
&\quad + \int_\Omega (\Delta_g f + n)(\psi u^q - a u^2) dv_g - \frac{1}{2}\int_\Omega (\Delta_g^2 f) u^2 dv_g.
\end{aligned} \tag{A.9}$$

Thus we get that

$$\begin{aligned}
&\int_\Omega (\Delta_g f + n)\left(\frac{1}{2}|\nabla u|_g^2 + \frac{1}{2}au^2 - \frac{1}{q}\psi u^q\right) dv_g \\
&\quad = \left(\frac{1}{2} - \frac{1}{q}\right)\int_\Omega (\Delta_g f + n)\psi u^q \, dv_g - \frac{1}{4}\int_\Omega (\Delta_g^2 f) u^2 dv_g \\
&\quad + \frac{1}{2}\int_{\partial\Omega} (\Delta_g f + n)(\nabla u, \nu)_g u \, d\sigma_g - \frac{1}{4}\int_{\partial\Omega} \left(\nabla(\Delta_g f), \nu\right)_g u^2 d\sigma_g.
\end{aligned} \tag{A.10}$$

This finally leads to the following:

$$\begin{aligned}
&\int_\Omega \left(a + \frac{1}{2}(\nabla f, \nabla a)_g + \frac{1}{4}(\Delta_g^2) f\right) u^2 dv_g + \left(\frac{n-2}{2} - \frac{n}{q}\right)\int_\Omega \psi u^q \, dv_g \\
&\quad = \left(\frac{1}{2} - \frac{1}{q}\right)\int_\Omega (\Delta_g f + n)\psi u^q \, dv_g + \int_\Omega (\nabla^2 f - g)(\nabla u, \nabla u) dv_g \\
&\quad + \frac{1}{q}\int_\Omega (\nabla f, \nabla \psi)_g u^q \, dv_g + A,
\end{aligned} \tag{A.11}$$

where A is the boundary term:

$$\begin{aligned}
A = &\frac{1}{2}\int_{\partial\Omega} (\nabla f, \nu)_g |\nabla u|_g^2 d\sigma_g - \int_{\partial\Omega} (\nabla u, \nabla f)_g (\nabla u, \nu)_g d\sigma_g \\
&- \frac{n-2}{2}\int_{\partial\Omega} (\nabla u, \nu)_g u \, d\sigma_g - \int_{\partial\Omega} (\nabla f, \nu)_g \left(\frac{1}{q}\psi u^q - \frac{1}{2}au^2\right) d\sigma_g \\
&+ \frac{1}{2}\int_{\partial\Omega} (\Delta_g f + n)(\nabla u, \nu)_g u \, d\sigma_g - \frac{1}{4}\int_{\partial\Omega} \left(\nabla(\Delta_g f), \nu\right)_g u^2 d\sigma_g.
\end{aligned} \tag{A.12}$$

This is the relation we referred to as the Pohozaev identity, with test function f, applied in Ω to a function u which verifies the above equation.

Acknowledgments

The author thanks Emmanuel Hebey for useful discussions and comments on this work. The author also thanks the organizers of The Abdus Salam International Centre for Theoretical Physics (ICTP) conference held in Trieste in May 2003 where this work was presented for the first time.

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