PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 95, Number 2, October 1985

COMPACTNESS IN L^2 AND THE FOURIER TRANSFORM

ROBERT L. PEGO

ABSTRACT. The Riesz-Tamarkin compactness theorem in $L^p(\mathbf{R}^n)$ employs notions of L^p-equicontinuity and uniform L^p-decay at ∞. When $1 \leqslant p \leqslant 2$, we show that these notions correspond under the Fourier transform, and establish new necessary and sufficient criteria for compactness in $L^2(\mathbf{R}^n)$.

An oft-quoted classical result characterizing compact sets in $L^p(\mathbf{R}^n)$ is due to M. Riesz and J. D. Tamarkin (see [**1**, **2**, **4**]):

THEOREM. *A bounded subset K of $L^p(\mathbf{R}^n)$, $1 \leqslant p < \infty$, is conditionally compact if and only if*

(I) $\int_{\mathbf{R}^n} |f(x+y) - f(x)|^p \, dx \to 0$ *as* $y \to 0$ *uniformly for f in K, and*

(II) $\int_{|x|>R} |f(x)|^p \, dx \to 0$ *as* $R \to \infty$ *uniformly for f in K.*

Property (I) is a uniform smoothness property. By analogy with the terminology of Arzela-Ascoli, we say the functions in K are L^p-equicontinuous if (I) holds. Property (II) is a uniform decay property. The connection between smoothness and decay through the Fourier transform has been well explored [**6**]. Yet the following nice equivalence seems to be new:

THEOREM 1. *Let K be a bounded subset of $L^2(\mathbf{R}^n)$ and let $\hat{K}$ be the Fourier transform of K, $\hat{K} = \{\hat{f} \mid f \in K\}$. The functions of K are L^2-equicontinuous if and only if the functions of $\hat{K}$ decay uniformly in L^2, and vice versa. That is, K satisfies* (I) *in L^2 if and only if $\hat{K}$ satisfies* (II) *in L^2, and vice versa.*

Combining this result with the Riesz-Tamarkin theorem, we obtain two alternative characterizations of compact sets in $L^2(\mathbf{R}^n)$:

THEOREM 2. *A bounded subset K of $L^2(\mathbf{R}^n)$ is conditionally compact if and only if* $\int |f(x+y) - f(x)|^2 \, dx \to 0$ *as* $y \to 0$, *and* $\int |\hat{f}(\xi + \omega) - \hat{f}(\xi)|^2 \, d\xi \to 0$ *as* $\omega \to 0$, *both uniformly for f in K.*

THEOREM 3. *A bounded subset K of $L^2(\mathbf{R}^n)$ is conditionally compact if and only if* $\int_{|x|>R} |f(x)|^2 \, dx \to 0$ *and* $\int_{|\xi|>R} |\hat{f}(\xi)|^2 \, d\xi \to 0$ *as* $R \to \infty$, *both uniformly for f in K.*

Received by the editors November 7, 1984.
1980 *Mathematics Subject Classification*. Primary 46E30, 42B99.
Key words and phrases. Compactness, Fourier transform, L^p, L^2, L^p-equicontinuity.

Theorem 1 is an easy consequence of the theorem below, which offers some results in L^p, $1 \leqslant p \leqslant 2$.

THEOREM 4. *Let K be a bounded subset of L^p, $1 \leqslant p \leqslant 2$. If K satisfies* (I) (*resp.* (II)) *in L^p, then $\hat{K}$ satisfies* (II) (*resp.* (I)) *in L^q, where $1/p + 1/q = 1$.* (*If $q = \infty$, conditions* (I) *and* (II) *are to be stated in the obvious way using the* sup *norm.*)

Let us set our notation and recall some basic results. For $f \in L^1(\mathbf{R}^n)$,

$$\hat{f}(\xi) = \int_{R^n} f(x) e^{-i\xi \cdot x}\, dx.$$

Recall [3]:

(1) The Fourier transform above extends to a bounded linear map $f \to \hat{f}$ from L^p to L^q, for $1 \leqslant p \leqslant 2$ and $1/p + 1/q = 1$, so $\|\hat{f}\|_q \leqslant C_p \|f\|_p$ for f in L^p.

(2) For f in L^p, ω in $\mathbf{R}^n$, we have $[e^{-i\omega x} f(x)]\hat{}\,(\xi) = \hat{f}(\xi + \omega)$ in L^q.

(3) For f in L^p, ψ in the Schwartz class $\mathscr{S}$, $(f * \psi)\hat{}\,(\xi) = \hat{f}(\xi)\hat{\psi}(\xi)$ in L^q, where $f * \psi(x) = \int_{\mathbf{R}^n} f(x - y)\psi(y)\, dy$.

PROOF OF THEOREM 4. First, we assume K satisfies (II) in L^p. Let M be a bound for K in L^p. For f in K,

$$\hat{f}(\xi + \omega) - \hat{f}(\xi) = \left[(e^{-i\omega \cdot x} - 1) f(x)\right]\hat{}\,(\xi),$$

whence

$$\begin{aligned}\|\hat{f}(\xi + \omega) - \hat{f}(\xi)\|_q &\leqslant C_p \left\|(e^{-i\omega \cdot x} - 1) f(x)\right\|_p \\ &\leqslant C_p \left(\int_{|x| \leqslant R} \left(|\omega|\, |x|\, |f(x)|\right)^p dx + 2\int_{|x| > R} |f(x)|^p\, dx \right)^{1/p}.\end{aligned}$$

Let $\varepsilon > 0$. Because of (II) we may choose R so large that the second term here is less than $\frac{1}{2}(\varepsilon/C_p)^p$ independent of f in K. Then since $\int_{|x| \leqslant R} (|x|\, |f(x)|)^p\, dx \leqslant (RM)^p$ for f in K, we have $\|\hat{f}(\xi + \omega) - \hat{f}(\xi)\|_q < \varepsilon$ if ω is sufficiently small, $|\omega|^p < \frac{1}{2}(\varepsilon/C_p RM)^p$, independent of f in K. So $\hat{K}$ satisfies (I) in L^q.

Now assume K satisfies (I) in L^p. We seek to show that functions in $\hat{K}$ decay uniformly in L^q. Let $\psi(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$, $\psi_R(x) = \psi(Rx)R^n$, so that ψ_R and $\hat{\psi}_R(\xi) = \hat{\psi}(\xi/R)$ are in $\mathscr{S}$, with $\hat{\psi}(\xi) = e^{-|\xi|^2/2}$, $\hat{\psi}_R(0) = \int \psi_R(y)\, dy = 1$. Now for $|\xi| \geqslant 2R$, $\frac{1}{2} \leqslant 1 - \hat{\psi}_R(\xi)$, so for $f \in K$,

$$\begin{aligned}\frac{1}{2}\left(\int_{|\xi| > 2R} |\hat{f}(\xi)|^q\, d\xi \right)^{1/q} &\leqslant \|\hat{f}(\xi)(1 - \hat{\psi}_R(\xi))\|_q \\ &\leqslant C_p \|f(x) - f * \psi_R(x)\|_p \\ &= C_p \left[\int \left| \int (f(x) - f(x - y))\psi_R(y)\, dy \right|^p dx \right]^{1/p}.\end{aligned}$$

By Jensen's inequality and Fubini's theorem, this is

$$\leqslant C_p \left[\int \left[\int \left| f(x) - f\left(x - \frac{y}{R}\right) \right|^p dx \right] \psi(y)\, dy \right]^{1/p}.$$

Now define a uniform L^p modulus of continuity for K,

$$H(y) = \sup_{f \in K} \int |f(x) - f(x - y)|^p \, dx.$$

By (I), $H(y) \to 0$ as $y \to 0$, and $H(y) \leqslant (2M)^p$ for all y. From above, we have

$$\left[\int_{|\xi| > 2R} |\hat{f}(\xi)|^q \, d\xi \right]^{1/q} \leqslant 2C_p \left[\int H\left(\frac{y}{R}\right) \psi(y) \, dy \right]^{1/p} \to 0$$

as $R \to \infty$ uniformly for f in K. Hence, $\hat{K}$ satisfies (II).

We conclude with a small application, which illustrates a principle known in information theory (see [5]) that an operator in L^2 that is "band limited and time limited" is compact.

Fix any $\phi_1(x)$, $\phi_2(x)$ bounded functions on $\mathbf{R}^n$ which satisfy $\lim_{|x| \to \infty} \phi_i(x) = 0$, $i = 1, 2$, and let ϕ_i denote the multiplication operator on L^2 given by $u(x) \to \phi_i(x) u(x)$, $i = 1, 2$. Let F denote the Fourier transform operator $u \to Fu = \hat{u}$. Define an operator T on L^2 by $T = \phi_1 F \phi_2$. Assume $\phi_1(x)$ is continuous.

PROPOSITION. *T is a compact operator on L^2.*

PROOF. Let K be a bounded set in L^2. Clearly, the set $\phi_2 K$ has the uniform decay property (II) in L^2. From Theorem 1, the set $F\phi_2 K$ is L^2-equicontinuous (has property (I)). The set $TK = \phi_1 F \phi_2$ is also L^2-equicontinuous, and also has the uniform decay property (II). By Riesz-Tamarkin, it follows that TK is precompact. Q.E.D.

ACKNOWLEDGEMENT. The author thanks Jonathan Goodman for pointing out this application.

REFERENCES

1. R. Adams, *Sobolev spaces*, Academic Press, New York, 1975.

2. N. Dunford and J. T. Schwartz, *Linear operators*, Part I, Wiley-Interscience, New York, 1966.

3. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton N. J., 1971.

4. K. Yosida, *Functional analysis*, 2nd ed., Springer-Verlag, New York, 1968.

5. H. J. Landau and H. O. Pollack, *Prolate spherodial wave functions, Fourier analysis, and uncertainty*. III, Bell System Tech. J. **41** (1962), 1295–1336.

6. M. Murata, *A theorem of Liouville type for partial differential equations with constant coefficients*, J. Fac. Sci. Univ. Tokyo IA **21** (1974), 395–404.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48109