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COMPACTNESS OF COMPOSITION OPERATORS ON BMOA

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ABSTRACT. A function theoretic characterization is given of when a composition operator is compact on BMOA, the space of analytic functions on the unit disk having radial limits that are of bounded mean oscillation on the unit circle. When the symbol of the composition operator is univalent, compactness on BMOA is shown to be equivalent to compactness on the Bloch space, and a characterization in terms of the geometry of the image of the disk under the symbol of the operator results.

§1. INTRODUCTION

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map of the unit disk $\mathbb{D} = \{z : |z| < 1\}$. The *composition operator* C_φ induced by such a φ is the linear map on the space of all analytic functions on the unit disk defined by $C_\varphi(f) = f \circ \varphi$. A fundamental problem concerning composition operators is to relate function theoretic properties of φ to operator theoretic properties of the restrictions of C_φ to various Banach spaces of analytic functions. This problem is addressed here for the Banach space BMOA of analytic functions on $\mathbb{D}$ that are of bounded mean oscillation on the unit circle.

There are many ways to define BMOA; see Chapter 6 of [G]. For the purposes of this paper, it will be defined as a Möbius invariant version of the Hardy space H^2; see [Ba]. Recall that an analytic function f on $\mathbb{D}$ belongs to H^p, $0 < p < \infty$, provided

$$\|f\|_{H^p}^p = \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

For $a \in \mathbb{D}$, let $\sigma_a(z) = (a-z)/(1-\overline{a}z)$, so that σ_a is an automorphism of $\mathbb{D}$ that exchanges the points 0 and a. Then $f \in H^2$ belongs to BMOA provided

$$\|f\|_* = \sup_{a\in\mathbb{D}} \|f\circ\sigma_a - f(a)\|_{H^2} < \infty.$$

It is clear that $\|\cdot\|_*$ does not distinguish between functions differing by a constant, but with the norm $\|f\|_{BMOA} = |f(0)| + \|f\|_*$, BMOA is a Banach space.

The main goal of this paper is to provide a function theoretic characterization of when C_φ is compact on BMOA. Recall that an operator is said to be compact provided it takes bounded sets to sets with compact closure. P. S. Bourdon, J. A. Cima and A. L. Matheson have recently shown that compactness of C_φ on

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BMOA is equivalent to a little-oh Carleson measure condition holding uniformly for all functions in the unit ball of BMOA; see Theorem 3.1 in [BCM]. M. Tjani [Tj, Theorem 3.11] had previously shown that a similar condition is equivalent to compactness of C_φ on the closed subspace VMOA (defined below) of BMOA. The characterization of compactness given below involves only the symbol φ of the operator.

Some background is required before we can state our characterization of compactness of C_φ on BMOA. It is clear that if σ is any automorphism of $\mathbb{D}$ and $f \in$ BMOA, then $\|f \circ \sigma\|_* = \|f\|_*$, and so σ induces a bounded composition operator on BMOA. It is well known that in fact $\|f \circ \varphi\|_* \leq \|f\|_*$ for every analytic self-map φ of $\mathbb{D}$, and so C_φ is bounded on BMOA. This appears in section 4 of [St], where it was shown to be a consequence of Littlewood's Subordination Principle; see also [AFP, Theorem 12] for another proof. To motivate our characterization of compactness of C_φ on BMOA, we sketch a slightly different proof that uses a formula for the H^2 norm of $C_\psi(f)$ from [Sh1].

The *Nevanlinna counting function* of an analytic self-map ψ of $\mathbb{D}$ is defined by

$$N(\psi, w) = \sum_{z \in \psi^{-1}\{w\}} \log(1/|z|), \quad w \in \mathbb{D} \setminus \{\psi(0)\}.$$

In [Sh1], J. H. Shapiro used $N(\psi, \cdot)$ to give a formula for the essential norm of C_ψ on H^2. The importance of $N(\psi, \cdot)$ in the study of C_ψ on H^2 comes from the fundamental formula

$$\|f \circ \psi - f(\psi(0))\|_{H^2}^2 = 2 \int_{\mathbb{D}} |f'(w)|^2 N(\psi, w) d\mathcal{A}(w), \tag{1.1}$$

where $d\mathcal{A} = dx\,dy/\pi$ is two-dimensional Lebesgue measure on $\mathbb{D}$, normalized so that $\mathcal{A}(\mathbb{D}) = 1$; see Corollary 4.4 in [Sh1]. That C_φ is always bounded on BMOA can be seen from this and what is known as Littlewood's Inequality (see, for example, [Sh1, p. 380]), which asserts that

$$N(\psi, w) \leq \log(1/|\sigma_{\psi(0)}(w)|) = N(\sigma_{\psi(0)}, w), \quad w \in \mathbb{D} \setminus \{\psi(0)\},$$

for every analytic self-map ψ of $\mathbb{D}$. Then, for $f \in$ BMOA, we have

$$\|f \circ \varphi\|_*^2 = \sup_{a \in \mathbb{D}} \|f \circ \varphi \circ \sigma_a - f(\varphi(a))\|_{H^2}^2 = 2 \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(w)|^2 N(\varphi \circ \sigma_a, w) d\mathcal{A}(w). \tag{1.2}$$

Since $N(\varphi \circ \sigma_a, w) \leq N(\sigma_{\varphi(a)}, w)$ by Littlewood's Inequality, using (1.1) again gives that

$$\|f \circ \varphi\|_*^2 \leq \sup_{a \in \mathbb{D}} \|f \circ \sigma_{\varphi(a)} - f(\varphi(a))\|_{H^2}^2 \leq \|f\|_*^2. \tag{1.3}$$

A general principle in operator theory is that if a big-oh condition such as Littlewood's Inequality determines boundedness of an operator, then a corresponding little-oh condition should determine compactness. As will be seen in Lemma 2.1, Theorem 1.1 (i) below can be viewed as a conformally invariant little-oh version of Littlewood's Inequality. It will be used to control $\|f \circ \varphi \circ \sigma_a - f(\varphi(a))\|_{H^2}$ when $|\varphi(a)|$ is near 1.

A second condition is required to handle points a with $|\varphi(a)|$ bounded away from 1. For φ an analytic self-map of $\mathbb{D}$ and $t \in (0, 1)$, define

$$E(\varphi, t) = \{e^{i\theta} : |\varphi(e^{i\theta})| > t\}.$$

Here $\varphi(e^{i\theta})$ represents the radial limit of φ at $e^{i\theta}$, which exists for almost all θ. We use $m(A)$ to denote the Lebesgue one-dimensional measure of a measurable set $A \subset \partial\mathbb{D}$.

Theorem 1.1. *Let φ be an analytic self-map of $\mathbb{D}$. Then C_φ is compact on* BMOA *if and only if*

(i) $\displaystyle\lim_{|\varphi(a)|\to 1} \sup_{0<|w|<1} |w|^2 N(\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a, w) = 0$

and, for all $R < 1$,

(ii) $\displaystyle\lim_{t\to 1} \sup_{\{a:|\varphi(a)|\le R\}} m(\sigma_a(E(\varphi, t))) = 0.$

Bourdon, Cima and Matheson [BCM, Theorem 4.3] have shown that compactness of C_φ on BMOA implies compactness on H^2. Hence Theorem 1.1 (i) and (ii) imply that $N(\varphi, w) = o(\log(1/|w|))$ as $|w| \to 1$, since this is Shapiro's criterion for C_φ to be compact on H^2; see [Sh1, Theorem 2.3]. I do not know a direct proof of this.

Condition (ii) can be motivated by taking the viewpoint of C_φ as a mapping of BMOA into H^2. Since BMOA is a Möbius invariant version of H^2, it is reasonable to expect that a Möbius invariant version of compactness criteria for C_φ : BMOA $\to H^2$ will be relevant to the compactness problem for C_φ : BMOA $\to$ BMOA. This was the viewpoint taken by the author and R. Zhao in studying composition operators from the Bloch space to various Möbius invariant spaces in [SZ].

C_φ : BMOA $\to H^p$ is always bounded, $0 < p < \infty$, since C_φ is bounded on BMOA and BMOA $\subset H^p$. It is not, however, always compact. This can be seen by taking $\varphi(z) = z$, so that C_φ is the identity operator, and considering $f_n(z) = z^n$. Then $\|f_n\|_* \le \|f_n\|_{H^2} = 1$ but $C_\varphi f_n$ has no convergent subsequence in H^p, and so C_φ : BMOA $\to H^p$ is not compact.

For φ an analytic self-map of $\mathbb{D}$, define

$$E(\varphi) = \{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}.$$

H. Jarchow has recently shown that the compactness of $C_\varphi : H^p \to H^q$, where $0 < q < p \le \infty$, is equivalent to $m(E(\varphi)) = 0$; see Theorem 1 in [J]. It is easy to see that the same condition determines compactness of C_φ : BMOA $\to H^p$.

Proposition 1.2. *Let φ be an analytic self-map of $\mathbb{D}$ and let $0 < p < \infty$. Then C_φ* : BMOA $\to H^p$ *is compact if and only if $m(E(\varphi)) = 0$.*

For completeness, we sketch the proof. First suppose that $m(E(\varphi)) = 0$. Then $C_\varphi : H^{2p} \to H^p$ is compact by Jarchow's theorem, and compactness of C_φ : BMOA $\to H^p$ follows from the continuity of the inclusion BMOA $\subset H^{2p}$. The proof of the converse is essentially the same as that in [J]. If C_φ : BMOA $\to H^p$ is compact, then

$$m(E(\varphi)) = \lim_{n\to\infty} \|\varphi^n\|_{H^p}^p = 0,$$

since $C_\varphi z^n = \varphi^n$ and the sequence $\{z^n\}$ is bounded and tends weakly to 0 in BMOA. This completes the sketch of the proof of the proposition.

Note that if $m(E(\varphi)) = 0$, then $\lim_{t\to 1} m(E(\varphi, t)) = 0$, and so Theorem 1.1 (ii) can be viewed as a Möbius invariant version of the compactness criteria for C_φ : BMOA $\to H^2$.

We say $f \in \mathrm{BMOA}$ has vanishing mean oscillation, and write $f \in \mathrm{VMOA}$ if

$$\lim_{|a|\to 1} \|f \circ \sigma_a - f(a)\|_{H^2} = 0.$$

VMOA is the closure in BMOA of the analytic polynomials, and is a Banach space with the norm it inherits from BMOA. Also, BMOA = VMOA**; see Chapter 6 of [G].

A characterization of when C_φ is compact on VMOA is an easy corollary to Theorem 1.1. Indeed, if $\varphi \in \mathrm{VMOA}$, then $C_\varphi P \in \mathrm{VMOA}$ for every polynomial P, and so C_φ maps VMOA into itself. Hence C_φ will be compact on VMOA if it is compact as an operator on BMOA. Conversely, if C_φ is compact on VMOA, then φ, being the image of the identity function under C_φ, is in VMOA. Further, it is easily checked that the second adjoint of C_φ, viewed as an operator on VMOA, is C_φ on BMOA. Hence C_φ is also compact on BMOA. The following corollary results from these observations and Theorem 1.1.

Corollary 1.3. *Let φ be an analytic self-map of $\mathbb{D}$. Then C_φ is compact on* VMOA *if and only if $\varphi \in$* VMOA *and Theorem 1.1 (i) and (ii) hold.*

Background is given in the next section, and then Theorem 1.1 will be proved in §3. Finally, it is known that if C_φ is compact on BMOA, then it is also compact on the Bloch space $\mathcal{B}$. In §4 we will specialize to the symbol φ being univalent, and show that in this case the converse holds. Thus, when φ is univalent, a geometric characterization of the compactness of C_φ on $\mathcal{B}$ due to K. Madigan and A. Matheson applies to BMOA as well.

Finally, I thank Paul Bourdon for his careful reading of and comments on a preliminary version of this paper.

§2. Background

In the proof of Theorem 1.1, we will need to use a refinement of the estimates for the counting functions implicit in Theorem 1.1 (i). This refinement is based on the fact that while $N(\psi, \cdot)$ need not be subharmonic, it is an increasing limit of subharmonic functions. For $0 < r < 1$, define the partial Nevanlinna counting function for ψ by

$$N_r(\psi, w) = \sum_{z \in \psi^{-1}\{w\}} \log^+(r/|z|), \quad w \in \mathbb{D} \setminus \{\psi(0)\},$$

where $\log^+ x = \max\{\log x, 0\}$, so only z with $|z| < r$ contribute to the sum. Then $N_r(\psi, \cdot)$ is subharmonic in $\mathbb{D} \setminus \{\psi(0)\}$, and $N(\psi, w) = \lim_{r\to 1} N_r(\psi, w)$. See §4 of [Sh1] or Chapter 10 of [Sh2] for these facts about the partial Nevanlinna counting functions.

Lemma 2.1. *Let ψ be an analytic self-map of $\mathbb{D}$ with $\psi(0) = 0$. If*

$$\sup_{0<|w|<1} |w|^2 N(\psi, w) < \varepsilon,$$

then

$$N(\psi, w) \le \begin{cases} \log(1/|w|), & 0 < |w| < \varepsilon^{1/4}; \\ \varepsilon^{1/2}, & \varepsilon^{1/4} \le |w| < \frac{1}{2}; \\ \frac{4\varepsilon}{\log 2} \log(1/|w|), & \frac{1}{2} \le |w| < 1. \end{cases}$$

Proof. The estimate for $\varepsilon^{1/4} \leq |w| < \frac{1}{2}$ is immediate from the assumption, and for $0 < |w| < \varepsilon^{1/4}$, it is just Littlewood's Inequality. In the remaining case, for each $r \in (0,1)$ we have that $N_r(\psi, w)$ is subharmonic on $\mathbb{D} \setminus \{0\}$. By assumption this function is bounded above by 4ε when $|w| = 1/2$, and by Littlewood's Inequality tends uniformly to 0 as $|w| \to 1$. Since it is subharmonic, it is bounded above by the harmonic function on the annulus $\{w : 1/2 < |w| < 1\}$ having these boundary values, and so

$$N(\psi, w) = \lim_{r\to 1} N_r(\psi, w) \leq \frac{4\varepsilon}{\log 2} \log \frac{1}{|w|}, \quad \frac{1}{2} < |w| < 1.$$

□

The next lemma shows how the counting functions transform under composition. It is a simple consequence of the definitions and the fact that $\sigma_a^{-1} = \sigma_a$.

Lemma 2.2 ([Sh2, p.192])**.** *Let ψ be an analytic self-map of D and let $a \in D$. Then*

$$N(\psi, \sigma_a(w)) = N(\sigma_a \circ \psi, w), \qquad w \in \mathbb{D}.$$

A final fact required in the proof of Theorem 1.1 is that Nevanlinna counting functions satisfy a submean value property.

Lemma 2.3 ([ESS, §2], [Sh1, §4])**.** *Let ψ be an analytic self-map of $\mathbb{D}$. If $\psi(0) \neq 0$ and $0 < r < |\psi(0)|$, then*

$$N(\psi, 0) \leq \frac{1}{r^2} \int_{r\mathbb{D}} N(\psi, w) d\mathcal{A}(w).$$

§3. Proof of Theorem 1.1

Proof of necessity. Suppose that (i) fails. Then there exists $\varepsilon > 0$, and $a_n, w_n \in \mathbb{D}$ such that $|\varphi(a_n)| \to 1$ and

$$|w_n|^2 N(\sigma_{\varphi(a_n)} \circ \varphi \circ \sigma_{a_n}, w_n) \geq \varepsilon.$$

Since $N(\sigma_{\varphi(a_n)} \circ \varphi \circ \sigma_{a_n}, w_n) \leq \log(1/|w_n|)$, it follows that there exists R such that $|w_n| \leq R < 1$, for all n. Thus $|b_n| \to 1$, where $b_n = \sigma_{\varphi(a_n)}(w_n)$. Define $f_n(z) = (1 - |b_n|^2)/(1 - \overline{b_n} z)$. Then $\{f_n\}$ converges uniformly on compact subsets of $\mathbb{D}$ to 0 and it is easy to check that $\|f_n\|_{BMOA} \leq C$. Throughout the paper C will be used to denote an absolute constant whose value may change from line to line. If C_φ were compact on BMOA, it would follow that $C_\varphi f_n \to 0$ in BMOA. From (1.2),

$$\|f_n \circ \varphi\|_*^2 \geq \|f_n \circ \varphi \circ \sigma_{a_n} - f_n(\varphi(a_n))\|_{H^2}^2 = 2 \int_{\mathbb{D}} |f_n'|^2 N(\varphi \circ \sigma_{a_n}, w) d\mathcal{A}(w).$$

We now introduce the notation

$$D(a, r) = \{z \in \mathbb{D} : |\sigma_a(z)| < r\}$$

for the pseudohyperbolic disk with center a and radius r, where $r \in (0,1)$ and $a \in \mathbb{D}$. For n sufficiently large, $2|f_n'(z)|^2 = 2|b_n|^2|\sigma_{b_n}'(z)|^2 \geq |\sigma_{b_n}'(z)|^2$, and so

$$\begin{aligned}\|f_n \circ \varphi\|_*^2 &\geq \int_{D(b_n,|w_n|/2)} |\sigma_{b_n}'|^2 N(\sigma_{b_n} \circ \varphi \circ \sigma_{a_n}, \sigma_{b_n}(w)) d\mathcal{A}(w) \\ &= \int_{D(0,|w_n|/2)} N(\sigma_{b_n} \circ \varphi \circ \sigma_{a_n}, z) d\mathcal{A}(z).\end{aligned}$$

The inequality used the identity $N(\varphi \circ \sigma_{a_n}, w) = N(\sigma_{b_n} \circ \varphi \circ \sigma_{a_n}, \sigma_{b_n}(w))$ from Lemma 2.2, and the next step was just the change of variable $z = \sigma_{b_n}(w)$.

Note that $|\sigma_{b_n} \circ \varphi \circ \sigma_{a_n}(0)| = |\sigma_{\varphi(a_n)}(b_n)| = |w_n|$, and so Lemma 2.3 gives

$$\|f_n \circ \varphi\|_*^2 \geq \frac{1}{4}|w_n|^2 N(\sigma_{b_n} \circ \varphi \circ \sigma_{a_n}, 0) = \frac{1}{4}|w_n|^2 N(\sigma_{\varphi(a_n)} \circ \varphi \circ \sigma_{a_n}, w_n) \geq \frac{\varepsilon}{4}.$$

For the equality, we used Lemma 2.2 again and the definition of b_n. Hence $C_\varphi f_n \not\to 0$ in BMOA, and it follows that (i) is necessary for C_φ to be compact on BMOA.

To prove the necessity of (ii), suppose there exist $R < 1$, $\varepsilon > 0$, $t_n \to 1$, and $a_n \in \mathbb{D}$ such that $|\varphi(a_n)| \leq R$ and $m(\sigma_{a_n}(E(\varphi, t_n))) \geq \varepsilon$. By passing to a subsequence if necessary, we may assume that $t_n^n \to 1$. Let $f_n(z) = z^n$, so that $\|f_n\|_* \leq \|z^n\|_{H^2} = 1$ and the sequence $\{f_n\}$ converges uniformly on compact subsets of $\mathbb{D}$ to 0. Once again, it suffices to show that $C_\varphi f_n \not\to 0$ in BMOA. Now,

$$\begin{aligned}\|f_n \circ \varphi\|_*^2 &\geq \|f_n \circ \varphi \circ \sigma_{a_n} - f_n(\varphi(a_n))\|_{H^2}^2 \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\varphi^n \circ \sigma_{a_n}(e^{i\theta})|^2 d\theta - |\varphi^n(a_n)|^2 \\ &\geq \frac{1}{2\pi}\int_{\sigma_{a_n}(E(\varphi,t_n))} |\varphi^n \circ \sigma_{a_n}(e^{i\theta})|^2 d\theta - R^{2n} \\ &\geq t_n^{2n}\varepsilon - R^{2n}.\end{aligned}$$

The last step used that σ_{a_n} is its own inverse. Hence $\liminf_{n\to\infty} \|f_n \circ \varphi\|_*^2 \geq \varepsilon$, and it follows that (ii) is necessary for C_φ to be compact on BMOA. □

Proof of sufficiency. Suppose (i) and (ii) hold, $\|f_n\|_{BMOA} \leq 1$ and $\{f_n\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. Since the unit ball in BMOA is a normal family, compactness of C_φ on BMOA will follow by a standard argument from showing that $f_n \circ \varphi \to 0$ in BMOA as $n \to \infty$. Let $\varepsilon \in (0,1)$ and use (i) to choose R such that

$$\sup_{0<|w|<1} |w|^2 N(\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a, w) < \varepsilon \tag{3.1}$$

whenever $|\varphi(a)| > R$.

First consider $a \in \mathbb{D}$ such that $|\varphi(a)| \leq R$. Use (ii) to choose $t_0 \in (0,1)$ such that $m(\sigma_a(E(\varphi, t_0))) < \varepsilon$ for all such a. Then there exists $N(\varepsilon)$ such that $n \geq N(\varepsilon)$ implies

$$\int_{\partial\mathbb{D}\setminus\sigma_a(E(\varphi,t_0))} |f_n \circ \varphi \circ \sigma_a(e^{i\theta}) - f_n(\varphi(a))|^2 d\theta < \varepsilon,$$

since $|\varphi \circ \sigma_a| \leq t_0$ a.e. on the set of integration, and $\{f_n\}$ converges to 0 uniformly on $t_0\mathbb{D}$. Also, from the John-Nirenberg Theorem (see [G, p. 233]) and (1.3) we have

$$\|f_n \circ \varphi \circ \sigma_a - f_n(\varphi(a))\|_{H^4}^2 \leq C\|f_n \circ \varphi \circ \sigma_a\|_*^2 \leq C\|f_n\|_*^2 \leq C.$$

Hence we get from the Schwarz inequality that

$$\int_{\sigma_a(E(\varphi,t_0))} |f_n\circ\varphi\circ\sigma_a(e^{i\theta}) - f_n(\varphi(a))|^2 d\theta$$
$$\leq \|f_n\circ\varphi\circ\sigma_a(e^{i\theta}) - f_n(\varphi(a))\|_{H^4}^2 m(\sigma_a(E(\varphi,t_0)))^{1/2} < C\varepsilon^{1/2}.$$

Combining these estimates, we see that if $|\varphi(a)|\leq R$ and $n\geq N(\varepsilon)$, then

$$\|f_n\circ\varphi\circ\sigma_a - f_n(\varphi(a))\|_{H^2}^2 < \varepsilon + C\varepsilon^{1/2}. \tag{3.2}$$

Next, consider $a\in\mathbb{D}$ such that $|\varphi(a)|>R$. From (3.1) and Lemma 2.1,

$$\begin{aligned} N(\varphi\circ\sigma_a, w) &= N(\sigma_{\varphi(a)}\circ\varphi\circ\sigma_a, \sigma_{\varphi(a)}(w)) \\ &\leq \frac{4\varepsilon}{\log 2}\log\frac{1}{|\sigma_{\varphi(a)}(w)|} = \frac{4\varepsilon}{\log 2} N(\sigma_{\varphi(a)}, w), \end{aligned}$$

whenever $1/2 < |\sigma_{\varphi(a)}(w)| < 1$ or equivalently $w\in\mathbb{D}\backslash D(\varphi(a),1/2)$. (Recall that $D(z,r)$ is the pseudohyperbolic disk with center z and radius r.) Hence

$$\begin{aligned} \int_{\mathbb{D}\backslash D(\varphi(a),1/2)} |f_n'(w)|^2 N(\varphi\circ\sigma_a,w)d\mathcal{A}(w) &\leq C\varepsilon\int |f_n'(w)|^2 N(\sigma_{\varphi(a)},w)d\mathcal{A}(w) \\ &\leq C\varepsilon\|f_n\|_*^2 \leq C\varepsilon. \end{aligned} \tag{3.3}$$

Now we use that $|f_n'(w)|\leq C(1-|w|^2)^{-1}$, since $\|f_n\|_{\mathcal{B}}\leq C\|f_n\|_{BMOA}\leq C$; see for example [P, p. 172]. Here $\|f_n\|_{\mathcal{B}} = |f_n(0)| + \sup_{w\in\mathbb{D}}(1-|w|^2)|f_n'(w)|$ is the Bloch norm of f_n. This gives

$$\begin{aligned} &\int_{D(\varphi(a),1/2)} |f_n'(w)|^2 N(\varphi\circ\sigma_a,w)d\mathcal{A}(w) \\ &\leq C\int_{D(\varphi(a),1/2)} N(\varphi\circ\sigma_a,w)\frac{d\mathcal{A}(w)}{(1-|w|^2)^2} \\ &= C\int_{D(0,1/2)} N(\sigma_{\varphi(a)}\circ\varphi\circ\sigma_a,z)\frac{d\mathcal{A}(z)}{(1-|z|^2)^2}. \end{aligned} \tag{3.4}$$

The equality above used the change of variable $w=\sigma_a(z)$, the conformal invariance of the measure $(1-|w|^2)^{-2}d\mathcal{A}(w)$, and the identity $N(\sigma_{\varphi(a)}\circ\varphi\circ\sigma_a,z) = N(\varphi\circ\sigma_a,\sigma_{\varphi(a)}(z))$ from Lemma 2.2. To estimate the last integral above, we first use (3.1) and Lemma 2.1 to get that

$$\int_{\varepsilon^{1/4}\mathbb{D}} N(\sigma_{\varphi(a)}\circ\varphi\circ\sigma_a,z)d\mathcal{A}(z) \leq C\int_0^{\varepsilon^{1/4}} r\log\frac{1}{r}dr \leq C\varepsilon^{1/4},$$

and

$$\int_{D(0,1/2)\backslash\varepsilon^{1/4}\mathbb{D}} N(\sigma_{\varphi(a)}\circ\varphi\circ\sigma_a,z)d\mathcal{A}(z) \leq \varepsilon^{1/2}\mathcal{A}(\mathbb{D}) = \varepsilon^{1/2}.$$

Since $1-|z|^2$ is comparable to 1 for $z\in D(0,1/2)$, from (3.4) we conclude that

$$\int_{D(\varphi(a),1/2)} |f_n'(w)|^2 N(\varphi\circ\sigma_a,w)d\mathcal{A}(w) \leq C\varepsilon^{1/4} + C\varepsilon^{1/2} \leq C\varepsilon^{1/4}.$$

Together with (1.1) and (3.3), this shows that

$$\|f_n\circ\varphi\circ\sigma_a - f_n(\varphi(a))\|_{H^2}^2 = 2\int |f_n'(w)|^2 N(\varphi\circ\sigma_a,w)d\mathcal{A}(w) \leq C\varepsilon^{1/4},$$

whenever $|\varphi(a)| > R$. With (3.2), this gives

$$\|f_n \circ \varphi\|_*^2 = \sup_{a \in \mathbb{D}} \|f_n \circ \varphi \circ \sigma_a - f_n(\varphi(a))\|_{H^2}^2 \leq C\varepsilon^{1/4},$$

whenever $n \geq N(\varepsilon)$. Since we also have that $f_n(\varphi(0)) \to 0$, this proves $f_n \circ \varphi \to 0$ in BMOA, and the proof is complete. □

§4. Geometric characterization

In this section we specialize to the case that the symbol φ of the composition operator is univalent. Compactness of C_φ on BMOA in this case turns out to be equivalent to C_φ being compact on the Bloch space. Recall that the Bloch space $\mathcal{B}$ consists of the analytic functions on $\mathbb{D}$ satisfying

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in D} (1 - |z|^2)|f'(z)| < \infty,$$

and that the little Bloch space $\mathcal{B}_0$ is the closed subspace of functions $f \in \mathcal{B}$ such that

$$\lim_{|z| \to 1} (1 - |z|^2)|f'(z)| = 0.$$

It is well known that BMOA $\subset \mathcal{B}$ and that $\|f\|_{\mathcal{B}} \leq C\|f\|_{BMOA}$; see for example [P, p.172].

K. Madigan and A. Matheson have recently proved that C_φ is compact on $\mathcal{B}$ if and only if

$$\lim_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2} = 0, \tag{4.1}$$

and C_φ is compact on $\mathcal{B}_0$ if and only if

$$\lim_{|z| \to 1} \frac{(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2} = 0; \tag{4.2}$$

see [MM]. The proof in [MM] that (4.1) is necessary for compactness of C_φ on $\mathcal{B}$ involved consideration of univalent test functions in $\mathcal{B}$. A univalent function in $\mathcal{B}$ is also in BMOA, and has comparable norm there; see for example section 6 of [ALXZ]. Thus it is a corollary to the proof in [MM] that if C_φ is compact on BMOA, then it also is compact on $\mathcal{B}$. A different proof of this has been given by M. Tjani [Tj, Proposition 3.2].

We say that an analytic self-map φ of $\mathbb{D}$ belongs to the hyperbolic little Bloch class, and write $\varphi \in \mathcal{B}_0^h$ if (4.2) is satisfied by φ. In [Sm], the author constructed inner functions that belong to $\mathcal{B}_0^h$. Since the sequence $\{z^n\}$ converges weakly to 0 in VMOA, an inner function cannot induce a compact composition operator on VMOA or on BMOA. Thus compactness of C_φ on these spaces is strictly stronger than compactness on $\mathcal{B}_0$ and on $\mathcal{B}$, respectively. However, when φ is univalent, it turns out to be equivalent.

It is well known that every bounded univalent function is in VMOA, and so by Corollary 1.3 compactness of C_φ on BMOA and on VMOA are equivalent when φ is univalent. When φ is univalent, conditions (4.1) and (4.2) have a simple geometric interpretation. We use $\delta_\Omega(z)$ to denote the distance from a point z in a region Ω

to the boundary of Ω, and we set $\delta_\Omega(z) = 0$ for $z \notin \Omega$. It is an easy consequence of the Koebe distortion theorem that if φ is univalent, then

$$\frac{1}{4}(1-|z|^2)|\varphi'(z)| \le \delta_{\varphi(\mathbb{D})}(\varphi(z)) \le (1-|z|^2)|\varphi'(z)|; \tag{4.3}$$

see Corollary 1.4 in [P]. Thus the numerator of the expression in (4.1) and (4.2) is comparable to $\delta_{\varphi(\mathbb{D})}(\varphi(z))$, while the denominator is comparable to $\delta_{\mathbb{D}}(\varphi(z))$. It is immediate from this that (4.1) and (4.2) are equivalent when φ is univalent, and can be restated as

$$\lim_{|w|\to 1} \frac{\delta_{\varphi(\mathbb{D})}(w)}{\delta_{\mathbb{D}}(w)} = 0.$$

This says, in particular, that $\varphi(\mathbb{D})$ can only reach the unit circle through a cusp.

Theorem 4.1. *Let φ be a univalent self-map of $\mathbb{D}$. Then the following are equivalent:*

(1) *C_φ is compact on* BMOA*;*
(2) *C_φ is compact on* VMOA*;*
(3) *C_φ is compact on $\mathcal{B}$;*
(4) *C_φ is compact on $\mathcal{B}_0$;*
(5) $\displaystyle\lim_{|w|\to 1} \frac{\delta_{\varphi(\mathbb{D})}(w)}{\delta_{\mathbb{D}}(w)} = 0.$

M. Tjani [Tj, Theorem 3.15] has shown that statements (1) through (4) are equivalent when φ is boundedly valent and $\varphi(\mathbb{D})$ is contained in a polygon inscribed in the unit circle.

Proof. From the discussion above, we know that with φ univalent the first two statements are equivalent and that they imply the last three statements, which we also know to be equivalent when φ is univalent. Thus it suffices to prove that (1) is a consequence of (3) and (5). We assume that (3) and (5) hold and show that Theorem 1.1 (i) and (ii) then hold. Let $0 < \varepsilon < 1/10$. For $a \in \mathbb{D}$ let $\psi_a = \sigma_{\varphi(a)} \circ \varphi \circ \sigma_a$. It is easily checked that $|\psi_a'(0)| = \frac{(1-|a|^2)|\varphi'(a)|}{1-|\varphi(a)|^2}$, and so by the equivalence of (3) and (4.1) there exists $R < 1$ such that $|\varphi(a)| > R$ implies $|\psi_a'(0)| < \varepsilon$. Now assume that $w = \psi_a(z)$ and $2\varepsilon < |w| < 1$. By the Koebe distortion theorem [P, Theorem 1.3],

$$|w| = |\psi_a(z)| \le |\psi_a'(0)|(1-|z|)^{-2} < \varepsilon(1-|z|)^{-2}.$$

Note also that $|z|$ is bounded away from 0, since $(1-|z|)^{-2} \ge |w|/\varepsilon > 2$, so $\log(1/|z|) \le C(1-|z|)$ and

$$|w|^2 N(\psi_a, w) = |w|^2 \log(1/|z|) \le |w|^2 C(1-|z|) \le C\varepsilon^{1/2}, \quad 2\varepsilon < |w| < 1.$$

Next we consider w satisfying $0 < |w| \le 2\varepsilon$. From Littlewood's Inequality, we have

$$|w|^2 N(\psi_a, w) \le |w|^2 \log(1/|w|) \le C\varepsilon^2 \log(1/\varepsilon), \quad |w| \le 2\varepsilon.$$

Hence $|w|^2 N(\psi_a, w) < C\varepsilon^{1/2}$, $0 < |w| < 1$, whenever $|\varphi(a)| > R$, and we have shown that Theorem 1.1 (i) holds.

The proof that Theorem 1.1 (ii) holds uses the notion of harmonic measure. We use $\omega(z, F, G)$ to denote the harmonic measure of a set F contained in the closure of a region G, evaluated at a point $z \in G$. Roughly speaking, it is the harmonic function on $G \setminus F$ that is equal to 1 on F and equal to 0 on $\partial G \setminus F$. When φ is univalent, conformal invariance of harmonic measure tells us that

$$\begin{aligned} m(\sigma_a(E(\varphi,t)))/2\pi &= \omega(0, \sigma_a(E(\varphi,t)), \mathbb{D}) \\ &= \omega(a, E(\varphi,t), \mathbb{D}) = \omega(\varphi(a), \partial\varphi(\mathbb{D}) \setminus t\mathbb{D}, \varphi(\mathbb{D})). \end{aligned}$$

Thus to prove Theorem 1.1 (ii) holds, we must show that for all $R \in (0,1)$,

$$\lim_{t\to 1} \sup_{\{z_0 \in \varphi(\mathbb{D}) : |z_0| \le R\}} \omega(z_0, \partial\varphi(\mathbb{D}) \setminus t\mathbb{D}, \varphi(\mathbb{D})) = 0.$$

Fix $R \in (0,1)$ and let $\eta > 0$. Next, use (5) to choose $t_0 \in (R,1)$ such that $\delta_{\varphi(\mathbb{D})}(w)/\delta_{\mathbb{D}}(w) < \eta$ whenever $w \in \varphi(\mathbb{D})$ and $|w| \ge t_0$. If $(1+t_0)/2 < t < 1$, and $|z_0| \le R$, then the maximum principle for harmonic functions tells us that

$$\omega(z_0, \partial\varphi(\mathbb{D}) \setminus t\mathbb{D}, \varphi(\mathbb{D})) \le \sup_{\{z_1 \in \varphi(\mathbb{D}) : |z_1| = t_0\}} \omega(z_1, \partial\varphi(\mathbb{D}) \setminus t\mathbb{D}, \varphi(\mathbb{D})).$$

There are many ways to see that this is small. We will use the following lemma, which is an easy consequence of the Carleman-Tsuji estimate for harmonic measure; see Theorem III.67 in [Ts]. We use the notation $B(z,r) = \{w : |w-z| < r\}$ for the Euclidean ball with center z and radius r.

Lemma 4.2. *Let Ω be a simply connected domain properly contained in the plane and let $z \in \Omega$. Then there exists an absolute constant C such that, for any $M < \infty$,*

$$\omega(z, \partial\Omega \setminus B(z, M\delta_\Omega(z)), \Omega) < C \cdot M^{-1/2}.$$

Let η, t_0 and t be as above. If $z_1 \in \varphi(\mathbb{D})$ and $|z_1| = t_0$, then

$$t\,\mathbb{D} \supset B(z_1, 2^{-1}\delta_{\mathbb{D}}(z_1)) \supset B(z_1, 2^{-1}\eta^{-1}\delta_{\varphi(\mathbb{D})}(z_1)),$$

and so from Lemma 4.2 we see that

$$\omega(z_1, \partial\varphi(\mathbb{D}) \setminus t\,\mathbb{D}, \varphi(\mathbb{D})) \le \omega(z_1, \partial\varphi(\mathbb{D}) \setminus B(z_1, 2^{-1}\eta^{-1}\delta_{\varphi(\mathbb{D})}(z_1)), \varphi(\mathbb{D})) \le C\eta^{1/2}.$$

Hence if $|z_0| \le R$ and $(1+t_0)/2 < t < 1$, then

$$\omega(z_0, \partial\varphi(\mathbb{D}) \setminus t\,\mathbb{D}, \varphi(\mathbb{D})) < C\eta^{1/2},$$

and since $\eta > 0$ was arbitrary the proof is complete. □

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