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COMPARABILITY OF PARTIAL DIFFERENTIAL OPERATORS AND MANIFOLD LINEARITY

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Abstract. The comparison of differential operators is a problem that was formulated in sixties by Lars Hormander, it is a part of the theory of partial differential operators with constant coefficients, this problem up to now does not have a complete solution.The urgency of this problem is stipulated by the applications of the solvability theory of differential equations in special spaces of generalized functions.

In this paper we prove some results in the manifold linearity of a polynomial, these results play an important role in the comparison of two linear partial differential operators problem.

1. Introduction

Let R^n be the n-dimensional Euclidean space and let $\mathbb{C}^n = \mathbb{C} \times ...... \times \mathbb{C}$ be the Cartesian product of n-complex planes $\mathbb{C}$, for $x = (x_1, ..., x_n)$ from R^n , set

$|x| = (x_1^2 + ... + x_n^2)^{\frac{1}{2}}$, the scalar product in R^n is denoted by $(.,.)_{R^n}$.

Let $Pol_{\mathbb{C}}(n,m)$ be the set of all polynomials of n variables with complex coefficients of order at most m. Each such polynomial $P(\xi)$ can be written in the form

$$P(\xi) = \sum_{\alpha} C_\alpha \xi^\alpha, \tag{1.1}$$

and by $Pol_{\mathbb{R}}\ (n,m)$ we denote the set of all polynomials of n variables with real coefficients of order m.

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$Pol_{\mathbb{C}}$ (n,m) and $Pol_{\mathbb{R}}$ (n,m) are vector spaces of $\frac{(m+n)!}{m!n!}$ dimension over the fields $\mathbb{C}$ and $\mathbb{R}$ respectively. We have the vector space $Pol_{\mathbb{C}}$ (n,m) as a complexification of the space $Pol_{\mathbb{R}}$ (n,m). It is helpful to take into account the following construction. On the direct sum $Pol_{\mathbb{R}}$ $(n,m) \oplus Pol_{\mathbb{R}}$ (n,m) we introduce the complex lattice $\mathfrak{L}$ using the formula

$$\mathfrak{L}\ (P(\xi), Q(\xi)) = (-Q(\xi), P(\xi)).$$

Identifying $Pol_{\mathbb{R}}$ (n,m) with the subset of vectors of the form $(P(\xi), \theta)$ in $Pol_{\mathbb{R}}$ $(n,m) \oplus Pol_{\mathbb{R}}$ (n,m) and using the fact that

$$i(P(\xi), \theta) = \mathfrak{L}(P(\xi), \theta) = (\theta, P(\xi)),$$

we can write any vector from $Pol_{\mathbb{C}}$ (n,m) in the form

$$(P(\xi), Q(\xi)) = P(\xi) + iQ(\xi).$$

The last sum is a line over $\mathbb{R}$, but not over $\mathbb{C}$.

The standard notation of such constructions is

$$Pol_{\mathbb{C}}(n,m) = \mathbb{C} \oplus_R Pol_{\mathbb{R}}(n,m).$$

Lemma 1.1 Any basis $Pol_{\mathbb{R}}(n,m)$ over $\mathbb{R}$ will be basis for $Pol_{\mathbb{C}}(n,m)$ over $\mathbb{C}$.It is sometimes said that R (subspace of $Pol_{\mathbb{R}}(n,m)$) is a real form of the space $Pol_{\mathbb{C}}(n,m)$.

Linear partial differential operators with complex coefficients take the form

$$P(\partial) = \sum_{\alpha} C_{\alpha} \partial^{\alpha} \tag{1.2}$$

where C_{α} are complex numbers.These operators can be obtained from the polynomial $P(\xi) \in Pol_{\mathbb{C}}(n,m)$ by a formal substitution $\xi_i \to \partial_i$, $1 \le i \le n$. On the functions of C^{∞} class its action is defined by the following rule

$$C^{\infty} \ni \varphi \to P(\overline{\partial})\varphi = \sum_{\alpha} C_{\alpha} \partial^{\alpha} \varphi \in C^{\infty}.$$

The correspondence

$Pol_{\mathbb{C}}(n,m) \ni P(\xi) \to P(\partial) \in Diff_{\mathbb{C}}(n,m)$

is linear and isomorphic by virtue of the formula

$$P(\xi) = e^{-(x,\xi)_{R^n}} P(\partial) e^{(x,\xi)_{R^n}} \tag{1.3}$$

where $(x,\xi)_{R^n}$ is the scalar product in R^n. $Diff_{\mathbb{C}}(n,m)$ is by definition the set of all linear partial differential operators with constant complex coefficients.

Definition 1.1[2,5] Hormander's function of the polynomial $P(\xi)$ is called the function $\widetilde{P}(\xi)$ defined by the formula

(1.4)
$$\widetilde{P}(\xi)=\sqrt{\sum_{\alpha}\left|\partial^{\alpha}P(\xi)\right|^{2}}.$$

Definition 1.2 Let $P(\partial),Q(\partial)\in Diff_{\mathbb{C}}(n,m)$. If $\frac{\widetilde{Q}(\xi)}{\widetilde{P}(\xi)}<C,\xi\in\mathbb{R}$, we shall say that Q is weaker than P and write $Q<P$, or that P is stronger than Q and write $P>Q$. If $P<Q<P$, the operators are called equally strong.

Definition 1.2 assigns a partial order on the set $Pol_{\mathbb{C}}(n,m)$. Not all polynomials are comparable with each other.

The motivation of introducing definition 1.2 is in the following :

Definition 1.3 A positive function k defined in R^n will be called a temperate weight function if there exist positive constants C and N such that

$$k(\xi+\zeta)\leq K(1+C\left|\xi\right|)^{N}k(\zeta);\qquad \xi,\zeta\in R^{n}.$$

The set of all such functions K will be denoted by $\mathcal{K}$.

The main examples of such functions are Hormander's functions. With the help of weight functions of $\mathcal{K}$ class, an important class of functional spaces in the theory of differential equations can be constructed, these classes of functional spaces are denoted by $B_{P,k}(R^n)$ and their local versions are denoted by $B^{loc}_{p,k}(R^n)$.

We remind here the definition of the $B_{p,k}(R^n)$ space . The definition of other spaces that we will use can be found in [1,2,3].

Definition 1.4 If $k\in\mathcal{K}$ and $1\leq P\leq\infty$, we denote by $B_{p,k}(R^n)$ the set of all distributions $u\in S'(R^n)$ such that the Fourier transform $\hat{u}$ is a function and

$$\left\|u\right\|_{p,k}=((2\pi)^{-n}\int\limits_{R^n}\left|k(\xi)\hat{u}(\xi)\right|^{p}d\xi)^{1/p}<+\infty.$$

When $p=\infty$, we shall interpret $\left\|u\right\|_{p,k}$ as

$$ess\sup\left|k(\xi)\hat{u}(\xi)\right|.$$

Consider the equation

(1.5)
$$P(\partial)u=f,\ f\in E'(R^n).$$

The role of Hormander's weight function $\widetilde{P}(\xi)$ of the operator $P(\partial)$ is explained by the following result [1].

Theorem 1.1 Let $u\in E'$, $k\in\mathcal{K}$ and $1\leq p\leq\infty$. The inclusion $u\in B_{p,k}$ holds if and only if $P(\partial)u\in B_{p,k}$.Let $B^{c}_{p,k}=B^{Loc}_{p,k}\cap E'(R^n)$.

Theorem 1.2 Let $f\in B^{c}_{p,k}$. If $Q<P$, then, for the solution $u=E*f$ of equation (1.5), where E is a fundamental solution, the following inclusion

$Q(\partial)u \in B^{Loc}_{p,k}$ is correct. The converse is true, if for some $k \in \mathcal{K}$, and $1 \le p \le \infty$, equation (1.5) has the solution u such that $Q(\partial)u \in B^{Loc}_{p,k}$ for each $f \in B^{c}_{p,k}$, then $Q < P$.

This theorem and the next completely motivate the role of definition 1.2 in the theory of differential equations with constant coefficients, see [2].

Theorem 1.3 If $k \in \mathcal{K}$ and $1 \le p \le \infty$, then the condition $Q < P$ is equivalent to the following statement: the inclusions $u \in E'$ and $P(\partial)u \in B_{p,k}$ imply $Q(\partial)u \in B_{p,k}$. In Summary we can say that the distribution $Q(\partial)u$ is vaild if and only if it has the same smoothness of $P(\partial)u$, when $Q < P$.

2. Definitions and Auxiliary Constructions

Let $\mathbb{C}\{x_1,\dots,x_n\}$ be the set of all formal power series of the unknowns $x_1,\dots,x_n$ with complex coefficients. Any element $f \in \mathbb{C}\{x_1,\dots,x_n\}$ has the form

$$f = \sum_{\alpha} a_{\alpha} x^{\alpha}.$$

For each element $f \in \mathbb{C}\{x_1,\dots,x_n\}$ we set

$$\|f\|_t = \sum |a_{\alpha}|\, t_1^{\alpha_1} \dots t_n^{\alpha_n},$$

where $t = (t_1,\dots,t_n) \in \mathbb{R}^n_+$ is a fixed collection from n positive numbers.

Let $B_t < x_1,\dots,x_n >= \{f \in \mathbb{C}\{x_1,\dots,x_n\} : \|f\|_t < +\infty\}$. It's known that B_t is Banach $\mathbb{C}-$algebra, that is, B_t is a complex normed space over $\mathbb{C}$ with the norm $\|.\|_t$ and for any $f, g \in B_t$, we have

$$\|f \cdot g\|_t \le \|f\|_t \cdot \|g\|_t$$

The algebra B_t does not have zero divisors. $\mathbb{C}[x_1,\dots,x_n]$ is the algebra of polynomials, that is, a subspace $\mathbb{C}\{x_1,\dots,x_n\}$ composed from these formal series f which have only a finite number of coefficients $\{a_{\alpha}\}_{\alpha\in\mathbb{Z}}$, other than zero.

The algebra of polynomials is dense in any algebra $B_t < x_1,\dots,x_n >$ see[4].

Definition 2.1 The formal series $f \in \mathbb{C}\{x_1,\dots,x_n\}$ is consedered convergent if $f \in B_t$ for some $t \in \mathbb{R}^n_+$. The set of all convergent power series is denoted by

$$\mathbb{C} < x_1,\dots,x_n >= \mathbb{C}_n < x >, n > 0.$$

The system which forms the algebra Diff $\mathbb{C}(n,m)$ will be denoted by $\overline{\partial}_1,\dots,\overline{\partial}_n$. It's clear that the symbols $\overline{\partial}_1,\dots,\overline{\partial}_n$ are pairwise commutative and

$\overline{\partial}_i^{m+1} = 0, \quad i = 1,\dots,n.$

The space $Pol_{\mathbb{C}}(n,m)$ has the dimension $\upsilon = \frac{(m+n)!}{m!\, n!}$.

We denote by τ_ξ the linear operator of translation on the vector $\xi \in \mathbb{C}_n$ acting by the formula

$$Pol_{\mathbb{C}}(n,m) \ni P(\xi) \to P(\xi+\zeta) \in Pol_{\mathbb{C}}(n,m).$$

The following formula is correct :

$$\left\|\tau_\xi P\right\|_0 = \sqrt{\sum_\alpha \frac{1}{\alpha!}\left|(\overline{\partial}^\alpha P)(\xi)\right|^2}\ . \tag{2.1}$$

The space $Pol_{\mathbb{C}}(n,m)$ provides the standard structure of a *Hilbert space.* For any two elements $P(\xi)$ and $Q(\xi)$ from $Pol_{\mathbb{C}}(n,m)$ we put

$$(P,Q)_0 = \sum_\alpha \frac{1}{\alpha!}(\overline{\partial}^\alpha P)(0)\overline{(\overline{\partial}^\alpha Q)(0)}\ ,$$

which is the scalar product that will generate on $Pol_{\mathbb{C}}(n,m)$ the norm

$$\|P\|_0 = \sqrt{\sum_\alpha \frac{1}{\alpha!}\left|(\overline{\partial}^\alpha P)(0)\right|^2}\ .$$

Lemma 2.1 The polynomial Q is weaker than the polynomial P if and only if

$$\|\tau Q\|_0 \leq C\,\|\tau P\|_0$$

for all $\xi \in \mathbb{R}^n$ with $C > 0$, where C is a constant depending only on P and Q.

Proof. The proof is immediately implied from (2.1) and definition 1.2.

Definition 2.2:[5] $\mathbb{C}-$ mainfold linearity of the polynomial $P \in Pol_{\mathbb{C}}(n,m)$ is called the set

$$\Lambda_{\mathbb{C}^n}(P) = \{\zeta \in \mathbb{C}^n : P(\xi+\zeta) = P(\xi) \text{ for all } \xi \in \mathbb{R}^n\}$$

$\mathbb{R}-$ mainfold linearity of the polynomial P is called the set

$$\Lambda_{\mathbb{R}^n}(P) = \{\xi \in \mathbb{R}^n : P(x+\xi) = P(x) \text{ for all } x \in \mathbb{R}^n\}\,.$$

For justification of the terminology introduced in definition 2.1 we will prove that $\Lambda_{\mathbb{C}^n}(P)$ and $\Lambda_{\mathbb{R}^n}(P)$ are vector subspaces of $\mathbb{C}^n$ and $\mathbb{R}^n$ respectively.

In the algebra $\mathrm{Diff}_{\mathbb{C}}(n,m)$ we consider the operators of class

$$V(\zeta) = \sum_{|\alpha|\leq m} \frac{\zeta_1^{\alpha_1}\ldots\zeta_n^{\alpha_n}}{\alpha_1!\ldots\alpha_n!}\overline{\partial}_1^{\alpha_1}\ldots\overline{\partial}_n^{\alpha_n}, \zeta \in \mathbb{C}^n.$$

By virtue of Taylor's formula we have

$$\begin{aligned} P(\xi+\zeta) &= \sum_{k=0}^{m} \frac{1}{k!}(\zeta_1\overline{\partial}_1+\ldots+\zeta_n\overline{\partial}_n)^k P(\xi) = \\ &= V(\zeta)P(\xi) = (e^{\zeta_1\overline{\partial}_1+\ldots+\zeta_n\overline{\partial}_n}P)(\xi). \end{aligned}$$

Lemma 2.2 The sets $\Lambda_{\mathbb{C}^n}(P)$ and $\Lambda_{\mathbb{R}^n}(P)$ are vector subspaces.

Proof. We will prove that $\Lambda_{\mathbb{C}^n}(P)$ is a vector subspace in $\mathbb{C}^n$. In the algebra $\mathrm{Diff}_{\mathbb{C}}(n,m)$ we consider the equality

$$e^{\zeta_1\overline{\partial}_1+\ldots+\zeta_n\overline{\partial}_n} = \sum_{k=0}^{m} \frac{1}{k!}(\zeta_1\overline{\partial}_1+\ldots+\zeta_n\overline{\partial}_n)^k.$$

Taking the logarithm for both sides we have

$$\zeta_1\overline{\partial}_1+\ldots+\zeta_n\overline{\partial}_n = Log(\overline{1}+e^{\zeta_1\overline{\partial}_1+\ldots+\zeta_n\overline{\partial}_n}-\overline{1}) =$$

$$= (e^{\zeta_1\overline{\partial}_1+\ldots+\zeta_n\overline{\partial}_n}-1)(\sum_{k=0}^{m}\frac{(-1)^{k-1}}{k}(e^{\zeta_1\overline{\partial}_1+\ldots+\zeta_n\overline{\partial}_n}-\overline{1})^{k-1}).$$

The operator

$$\sigma_\zeta(\overline{\partial}) = \sum_{k=0}^{m}\frac{(-1)^{k-1}}{k}(e^{\zeta_1\overline{\partial}_1+\ldots+\zeta_n\overline{\partial}_n}-\overline{1})^{k-1}$$

is invertible in the algebra since $\sigma_\zeta(0)\neq 0$. Consequently,

$$\begin{aligned} \Lambda_{\mathbb{C}^n}(P) &= \left\{\zeta\in\mathbb{C}^n : (e^{\zeta_1\overline{\partial}_1+\ldots+\zeta_n\overline{\partial}_n}-\overline{1})P(\xi)\equiv 0,\ \text{for all } \xi\in\mathbb{R}^n\right\} = \\ &= \{\zeta\in\mathbb{C}^n : \zeta_1(\partial_1 P)(\xi)+\ldots+\zeta_n(\partial_n P)(\xi)=0,\ \text{for all } \xi\in\mathbb{R}^n\} \end{aligned}$$

by virtue of invertibility of the operator $\sigma_\zeta(\overline{\partial})$ for each $\zeta\in\mathbb{C}^n$. Hence directly implies the conclusion of the lemma ■.

Let $\Lambda'_{\mathbb{R}^n}(P)$ be some subspace in $\mathbb{R}^n$ complementary to $\Lambda_{\mathbb{R}^n}(P)$, so that the direct sum holds

$$\mathbb{R}^n = \Lambda_{\mathbb{R}^n}(P)\oplus\Lambda'_{\mathbb{R}^n}(P).$$

We choose a basis $a^1,\ldots,a^n$ in $\mathbb{R}^n$ so that the vectors $a^1,\ldots,a^p$ form the basis of $\Lambda'_{\mathbb{R}^n}(P)$ and the vectors $a^{p+1},\ldots,a^n$ form the basis of $\Lambda_{\mathbb{R}^n}(P)$. By $l_1,\ldots,l_n$ we denote the natural basis of the space $\mathbb{R}^n$, by $x_1,\ldots,x_n$ we denote the coordinates of vector x in an expansion in the basis $l_1,\ldots,l_n$, and the coordinates of the same vector x, in the basis $a^1,\ldots,a^n$ we denote by $(x)_1,\ldots,(x)_n$.

Hence

$$x=\sum_{i=1}^{n}x_i l_i, \qquad x=\sum_{i=1}^{n}(x)_i a^i$$

from where we have

$$\begin{aligned}(x,a^j)_{\mathbb{R}^n} &= \sum_{i=1}^{n}x_i(l_i,a^j)_{\mathbb{R}^n}=(x)_j\\ (x,l_j)_{\mathbb{R}^n} &= \sum_{i=1}^{n}(x)_i(a^i,l_j)_{\mathbb{R}^n}=x_j.\end{aligned}$$

Let A be the linear operator corresponding to the square matrix

$$\left\{(a^j,l_i)_{\mathbb{R}^n}\right\} \qquad \begin{matrix}1\le i\le n\\ 1\le j\le n\end{matrix},$$

A^T is the transposed operator. By definition we assume

$$\left[a^i,\overline{\partial}\right]=(l_1,a^i)_{\mathbb{R}^n}\overline{\partial}_1+\ldots+(l_n,a^i)_{\mathbb{R}^n}\overline{\partial}_n$$

With each linear operator $A:\mathbb{R}^n\to\mathbb{R}^n$ we connect the linear operator $A_n: Pol_{\mathbb{C}}(n,m)\to Pol_{\mathbb{C}}(n,m)$ assuming that

$$Pol_{\mathbb{C}}(n,m)\ \ni P(\xi)\to P(A\xi)\in Pol_{\mathbb{C}}(n,m).$$

Lemma 2.3 The polynomial Q is weaker than the polynomial P if and only if the polynomial A_nQ is weaker than the polynomial A_nP, if the matrix A is invertible.

Proof. If Q is weaker than P, then

$$\left\|\tau_\xi Q\right\|_0\le C\left\|\tau_\xi P\right\|_0, \xi\in\mathbb{R}^n$$

hence

$$\left\|\tau_{A\xi} Q\right\|_0\le C\left\|\tau_{A\xi} P\right\|_0, \xi\in\mathbb{R}^n.$$

Since A_n is an automorphism of $Pol_{\mathbb{C}}(n,m)$, then

$$\left\|A_n\tau_{A\xi}Q\right\|_0\le C'\left\|\tau_{A\xi}Q\right\|_0\le C''\left\|\tau_{A\xi}P\right\|_0\le C'''\left\|A_n\tau_{A\xi}Q\right\|_0. \tag{2.2}$$

Now note that

$$\begin{aligned}\tau_\xi A_nQ &= A_n\tau_{A\xi}Q\\ \tau_\xi A_nP &= A_n\tau_{A\xi}P\end{aligned}$$

from (2.3) implies that

$\left\|\tau_\xi A_nQ\right\|_0\le C\left\|\tau_\xi A_nP\right\|_0$, for all $\xi\in\mathbb{R}^n$■.

Now we choose the operator A which defines the square matrix

$$\left\{(a^j,l_i)_{\mathbb{R}^n}\right\} \qquad \begin{matrix}1\le i\le n\\ 1\le j\le n\end{matrix},$$

and we put $A_nP(\xi) = P_\Delta((\xi)_1, \ldots, (\xi)_n)$, where $(\xi)_1, \ldots, (\xi)_n$ are the coordinates of the vector ξ in the basis $a^1, \ldots, a^n$.

Corollary 2.1 The polynomial $Q(\xi)$ is weaker than the polynomial $P(\xi)$ if and only if the polynomial $Q_\Delta((\xi))$ is weaker than the polynomial $P_\Delta((\xi))$.

Lemma 2.4 If the polynomial Q is weaker than the polynomial P, then

$$\Lambda_{\mathbb{R}^n}(P) \subseteq \Lambda_{\mathbb{R}^n}(Q).$$

Proof. By virtue of the lemma proposition

$$\left\|\tau_\xi Q\right\|_0 \leq C \left\|\tau_\xi P\right\|_0, \qquad \xi \in \mathbb{R}^n$$

let $\xi \in \Lambda_{\mathbb{R}^n}(P)$, then $\left\|\tau_\xi Q\right\|_0 \leq C \left\|P\right\|_0$, $\xi \in \mathbb{R}^n$, from which

$$\left\|\tau_\xi Q - Q\right\|_0^2 \leq (2C)^2 \left\|P\right\|_0^2$$

for any $\xi \in \Lambda_{\mathbb{R}^n}(P)$. If $\Lambda_{\mathbb{R}^n}(P) = \{\theta\}$, then there is nothing to prove. Otherwise the mapping

$$\Lambda_{\mathbb{R}^n}(P) \quad \ni \quad \xi \to \left\|\tau_\xi Q - Q\right\|_0^2$$

is a polynomial on the vector space $\Lambda_{\mathbb{R}^n}(P)$ which is bounded above. By the Liouville theorem it must be reduced to a constant. It is clear that this constant equals zero, since at the point $\xi = \theta$ this polynomial equals zero. So $\tau_\xi Q = Q$ if $\xi \in \Lambda_{\mathbb{R}^n}(P)$■.

Thus, if $Q < P$, we can always choose the basis $a^1, \ldots, a^n$ so that the part $a^1, \ldots, a^s$ of the basis $a^1, \ldots, a^p$, $s \leq p$ form the basis of $\Lambda'_{\mathbb{R}^n}(Q)$.

According to this we have $\dim \Lambda'_{\mathbb{R}^n}(Q) = s, \dim\, \Lambda'_{\mathbb{R}^n}(P) = p$.

Now note that

$$P_\Delta((\xi)_1, \ldots, (\xi)_p, (\xi)_{p+1} + (\zeta)_{p+1}, \ldots, (\xi)_n + (\zeta)_n) = P_\Delta((\xi)_1, \ldots, (\xi)_n)$$

for all $(\xi)_1, \ldots, (\xi)_n$ and $((\zeta)_{p+1}, \ldots, (\zeta)_n)$.

Consequently

$$P_\Delta((\xi)_1, \ldots, (\xi)_n) = P_\Delta((\xi)_1, \ldots, (\xi)_p, 0, \ldots, 0)$$

Similarly,

$$Q_\Delta((\xi)_1, \ldots, (\xi)_n) = Q_\Delta((\xi)_1, \ldots, (\xi)_s, 0, \ldots, 0).$$

Definition 2.3 The polynomial $P(\xi)$ is called complete if

$$\Lambda_{\mathbb{R}^n}(P) = \{\theta\}.$$

Definition 2.4 For each polynomial $P(\xi) \in Pol_{\mathbb{C}}(n, m)$ we denote by $span_{\mathbb{C}}(\tau_{_{\mathbb{R}^n}}(P))$ the linear *span* of translations $\{P(\xi + x) : x \in \mathbb{R}^n\}$.

Definition 2.4 The polynomial $P(\xi)$ is called regular if

$$\dim[span_{\mathbb{C}}(\tau_{\mathbb{R}^n}(P_\Delta))] = \frac{(m+p)!}{m!p!}$$

where $p = \dim \Lambda'_{\mathbb{R}^n}(P)$.

3. The Mainfold Linearity of a Polynomial

By the Definition of $[a^i, \overline{\partial}]$ we have

$$\sum_{i=1}^{n} C_i(e^{[a^i,\overline{\partial}]}p)(\xi) = \sum_{i=1}^{n} C_i p(\xi + a^i)$$

for any $C_1, \ldots\ldots, C_n \in \mathbb{C}$.

Lemma 3.1 If

$$\sum_{i=1}^{n} C_i(e^{[a^i,\overline{\partial}]} - \overline{1})p(\xi) = 0, \ \forall \ \xi \in \mathbb{R}^n, \tag{3.1}$$

then

$$\sum_{i=1}^{n} C_i(e^{[a^i,\overline{\partial}]} - \overline{1})^q p(\xi) = 0, \ \forall \ \xi \in \mathbb{R}^n \tag{3.2}$$

and any natural number $q \in \mathbb{N}$.

Proof. Newton's binomial formula

$$(e^{[a^i,\overline{\partial}]} - \overline{1})^q = \sum_{j=0}^{q} \binom{q}{j} (-1)^j e^{(q-j)\cdot[a^i,\overline{\partial}]}$$

allows us to prove this lemma by mathematical induction.

We assume that the conclusion of the lemma holds for all $s \leq q$, that is

$$\sum_{i=1}^{n} C_i(e^{[a^i,\overline{\partial}]} - \overline{1})^s p(\xi) \equiv \sum_{i=1}^{n} C_i[\sum_{j=0}^{s} \binom{s}{j}(-1)^j p(\xi + (s-j)a^j)] \equiv 0 \tag{3.3}$$

for all $\xi \in \mathbb{R}^n$and $s \leq q$. We will prove (3.3) for $s = q+1$.

We will use that $\binom{q+1}{j} = \binom{q}{j} + \binom{q}{j-1}$, Then for any $i, 1 \leq i \leq n$ we have

$$\begin{aligned}
&\sum_{j=0}^{q+1}(-1)^j\binom{q+1}{j}p(\xi+(q-j+1)a^i)\\
=&\sum_{j=0}^{q}(-1)^j\binom{q+1}{j}p(\xi+(q-j+1)a^i)+(-1)^{q+1}p(\xi)\\
=&\sum_{j=1}^{q}(-1)^j\binom{q+1}{j}p(\xi+(q-j+1)a^i)+p(\xi+(q+1)a^i)+(-1)^{q+1}p(\xi)\\
=&\sum_{j=1}^{q}(-1)^j\binom{q}{j}p(\xi+(q-j+1)a^i)+\sum_{j=1}^{q}(-1)^j\binom{q}{j-1}p(\xi+(q-j+1)a^i)\\
&+p(\xi+(q+1)a^i)+(-1)^{q+1}p(\xi). \qquad (3.4)
\end{aligned}$$

In the second sum we make the following exchange $j-1=s$ and we transform it to the following form

$$\begin{aligned}
&\sum_{j=1}^{q}(-1)^j\binom{q}{j-1}p(\xi+(q-j+1)a^i)\\
=&\ (-1)\sum_{j=0}^{q-1}\binom{q}{j}(-1)^j p(\xi+(q-j)a^i)
\end{aligned}$$

which gives

$$\begin{aligned}
&\sum_{j=1}^{q}(-1)^j\binom{q}{j-1}p(\xi+(q-j+1)a^i)+(-1)^{q+1}p(\xi)\\
=&\ (-1)\sum_{j=0}^{q}\binom{q}{j}(-1)^j p(\xi+(q-j)a^i) \qquad (3.5)
\end{aligned}$$

Furthermore

$$\begin{aligned}
&\sum_{j=1}^{q}\binom{q}{j}(-1)^j p(\xi+(q-j+1)a^i)+p(\xi+(q+1)a^i)\\
=&\sum_{j=0}^{q}\binom{q}{j}(-1)^j p(\xi-j+1)a^i)\\
=&\sum_{j=0}^{q}\binom{q}{j}(-1)^j p((\xi+a^i)+(q-j)a^i). \qquad (3.6)
\end{aligned}$$

Thus (3.4) is transformed into the sums of the right-side parts of (3.5) and (3.6) .

By the assumption of induction we have

$$(-1)\sum_{i=1}^{n} C_i[\sum_{j=0}^{q}\binom{q}{j}(-1)^j p(\xi+(q-j)a^i)]\equiv 0$$

for all $\xi\in\mathbb{R}^n$.

Multiplying both parts of (3.6) by C_i and taking the sum by i from 1 to n we get

$$\sum_{i=1}^{n} C_i[\sum_{j=0}^{q}\binom{q}{j}(-1)^j p(\xi+a^i)+(q-j)a^i)]\equiv 0$$

for all $\xi\in\mathbb{R}^n$ (also by the assumption of induction)■.

We put

$$L_i(\overline{\partial})=\sum_{k=1}^{m}\frac{(-1)^k}{k+1}\left(e^{[a^i,L_1]\overline{\partial}_1+.........+[a^i,L_n]\overline{\partial}_n}-\overline{1}\right)^{k+1},$$

then

$$[a^i,\overline{\partial}]=(e^{[a^i,\overline{\partial}]}-\overline{1})+L_i(\overline{\partial}) \tag{3.7}$$

for any i, $i=1,\ldots\ldots,n$.

Lemma 3.2 For any complex $\zeta_1,\ldots\ldots,\zeta_n$ the following formula holds

$$\sum_{i=1}^{n}\zeta_i(\overline{\partial}_i p)(\xi)=\sum_{i=1}^{n}(\zeta)_i(p(\xi+a^i)-p(\xi))+\sum_{i=1}^{n}(\zeta)_i L_i(\overline{\partial})p(\xi)$$

Proof. The proof is facilitated by elementary calculations taking into account (3.7)■.

Theorem 3.1 The maps

$$\gamma_{\mathbb{C}}:\sum_{i=1}^{n}\zeta_i(\partial_i p)(\xi)\to\sum_{i=1}^{n}(\zeta)_i(p(\xi+a^i)-p(\xi))$$

$$\gamma_{\mathbb{R}}:\sum_{i=1}^{n}x_i(\partial_i p)(\xi)\to\sum_{i=1}^{n}(x)_i(p(\xi+a^i)-p(\xi))$$

are correct, moreover

$$Ker\ \gamma_{\mathbb{C}}=\Lambda_{\mathbb{C}^n}(p),\ Ker\ \gamma_{\mathbb{R}}=\Lambda_{\mathbb{R}^n}(p).$$

Proof. Consider the map $\gamma_{\mathbb{C}}$.

Let $\sum\limits_{i=1}^{n} \zeta_i(\partial_i p)(\xi) \equiv 0$, for all $\xi \in \mathbb{R}^n$.By virtue of Lemma 2.2 we have $\zeta = (\zeta_1, \ldots\ldots, \zeta_n) \ni \Lambda_{\mathbb{C}^n}(p)$. Consequently,

$$(\zeta)_1 = (\zeta)_2 = \ldots\ldots = (\zeta)_p = 0.$$

Note that

$$\sum_{i=1}^{n} (\zeta)_i (p(\xi + a^i) - p(\xi)) \equiv \sum_{i=1}^{p} (\zeta)_i (p(\xi + a^i) - p(\xi))$$

for all $\xi \in \mathbb{R}^n$, since

$$p(\xi + a^i) = p(\xi), \forall \xi \in \mathbb{R}^n$$

for $i = p+1, \ldots\ldots, n$. ($a^i \in \Lambda_{\mathbb{R}^n}(p)$, $i = p+1, \ldots\ldots, n$) So,

$$\sum_{i=1}^{n} (\zeta)_i (p(\xi + a^i) - p(\xi)) \equiv 0, \forall\ \xi \in \mathbb{R}^n.$$

This means that $\gamma_{\mathbb{C}}$is correct ;

$\gamma_{\mathbb{C}}(\theta) = \theta$. $Ker\ \gamma_{\mathbb{C}} = \Lambda_{\mathbb{C}^n}(p)$ is implied by lemmas (3.1) and (3.2)■.

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