

COMPARING DNR AND WWKL

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Abstract. In Reverse Mathematics, the axiom system DNR, asserting the existence of diagonally non-recursive functions, is strictly weaker than WWKL₀ (weak weak König's Lemma).

§1. Introduction. Reverse mathematics is a branch of proof theory which involves proving the equivalence of mathematical theorems with certain collections of axioms over a weaker base theory. In the form adopted by Harvey Friedman (see, e.g., [3]) and Stephen G. Simpson, expounded in the monograph [9] and numerous papers, it involves formulating “countable mathematics” in second-order arithmetic and proving mathematical theorems φ equivalent to suitable axioms (or axiom systems) ψ over a weaker base axiom system T , usually RCA₀. (Here, the subscript 0 denotes restricted induction, i. e., RCA₀ does not include the full second order induction scheme.) Since the model that we shall construct in order to prove our main theorem does satisfy this scheme, subtleties of restricted induction will have no bearing on the arguments in this paper.

Let $T_1 < T_2$ express that the theory T_2 proves all the axioms of the theory T_1 , but not conversely. Simpson points to the chain

$$\text{RCA}_0 < \text{WKL}_0 < \text{ACA}_0 < \text{ATR}_0 < \Pi_1^1\text{CA}_0$$

as consisting of the axiom systems that appear most frequently as $T \cup \psi$.

In [10], Simpson and X. Yu introduced an axiom system WWKL₀ and showed it to be strictly intermediate between RCA₀ and WKL₀ as well as equivalent to some statements on Lebesgue and Borel measure. WWKL₀ was further studied by Giusto and Simpson [4]; and by Brown, Giusto and Simpson [2]. Giusto and Simpson found that a certain version of the Tietze Extension Theorem was provable in WKL₀ and implied the DNR axiom. They pointed out that DNR is intermediate between RCA₀ and WWKL₀, but left open the question whether

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DNR coincides with WWKL_0 , i. e., has the same theorems as WWKL_0 . Simpson conjectured that $\text{DNR} < \text{WWKL}_0$. In the current paper, we confirm Simpson's conjecture.

DEFINITION 1.1. If $\sigma, \tau \in \omega^{<\omega}$ then σ is called a *substring* of τ , $\sigma \subseteq \tau$, if for all x in the domain of σ , $\sigma(x) = \tau(x)$. The length of a string σ is denoted by $|\sigma|$. A string $\langle a_1, \dots, a_n \rangle \in \omega^n$ is denoted (a_1, \dots, a_n) when we find this more natural. The *concatenation* of $\langle a_1, \dots, a_n \rangle$ by $\langle a_n \rangle$ on the right is denoted $((a_1, \dots, a_n), a_{n+1})$ or $\langle a_1, \dots, a_n \rangle * \langle a_{n+1} \rangle = \langle a_1, \dots, a_n \rangle * a_{n+1}$. If $G \in \omega^\omega$ then σ is a substring of G if for all x in the domain of σ , $\sigma(x) = G(x)$.

Given $G : \omega \rightarrow \omega$ and $n \in \omega$, we define the n th column of G to be the function $G_n : \omega \rightarrow \omega$ such that for all $k \in \omega$, $G_n(k) = G(2^n(2k+1))$. On the other hand, if for each $n \in \omega$ we are given a function $G_n : \omega \rightarrow \omega$, then we let $\bigoplus_{n \in \omega} G_n$ denote the function G such that $G(2^n(2k+1)) = G_n(k)$ for all $n, k \in \omega$.

Let Φ_n , $n \in \omega$, be a standard list of the Turing functionals. So if A is recursive in B then for some n , $A = \Phi_n^B$. For convenience, if Φ is a Turing functional and for all B and x , the computation of $\Phi^B(x)$ is independent of x , we sometimes write Φ^B instead of $\Phi^B(x)$. Let $\Phi_{n,t}$ be the modification of Φ_n which goes into an infinite loop after t computation steps if the computation has not ended after t steps. We abbreviate Φ_n^\emptyset by Φ_n . If the computation $\Phi_e(x)$ terminates we write $\Phi_e(x) \downarrow$, otherwise $\Phi_e(x) \uparrow$.

The axiom system DNR corresponds to a class of functions in ω^ω denoted by DNR: Given functions $H, G : \omega \rightarrow \omega$, we say H is DNR^G (*diagonally nonrecursive in G*) if for all $x \in \omega$, $H(x) \neq \Phi_x^G(x)$ or $\Phi_x^G(x) \uparrow$. Given $h : \omega \rightarrow \omega$, we say H is $h\text{-DNR}^G$ if in addition for all n , $H(n) < h(n)$. (This necessitates that $h(n) > 0$ for all n .) We say H is DNR if H is DNR^\emptyset . If H is DNR^G and σ is a substring of H then σ is called a DNR^G string. In this article $G : \omega \rightarrow \omega$ will be called *relatively DNR* if there are no x, y such that $G_{2y}(x) = \Phi_x^{G_0 \oplus \dots \oplus G_{2y-1}}(x)$.

DEFINITION 1.2. Let A be a real, i. e., a subset of the nonnegative integers ω . A *Martin-Löf test U relative to A* is a sequence of open sets $U_n \subseteq 2^\omega$, $n \in \omega$ uniformly r.e. in A such that $\mu(U_n) \leq 2^{-n}$, where μ denotes the standard measure on 2^ω . Then $\bigcap_n U_n$ is called a *Martin-Löf null set relative to A* . If $A = \emptyset$ then we speak simply of a *Martin-Löf test* and a *Martin-Löf null set*. A set $R \subseteq \omega$ is *Martin-Löf random* if for each Martin-Löf test U , there is an n such that $R \notin U_n$.

For an introduction to Martin-Löf randomness and related concepts the reader may consult [1].

The only fact we need about the axiom systems is the following:

Lemma 1.3. Let \mathcal{I} be a Turing ideal, i. e., a set of subsets of ω whose Turing degrees form an ideal within the upper semilattice of all Turing degrees. Let $N(\mathcal{I})$ be the ω -model of RCA_0 with \mathcal{I} as the interpretation of the power set symbol.

(1) $N(\mathcal{I}) \models \text{DNR}$ if and only if for each $G \in \mathcal{I}$, there is $H \in \mathcal{I}$ such that H is DNR^G .

(2) $N(\mathcal{I}) \models \text{WWKL}_0$ if and only if for each $G \in \mathcal{I}$, there is $H \in \mathcal{I}$ such that H is Martin-Löf random relative to G .

PROOF. For definitions of the DNR and WWKL_0 axioms, see [4]. The equivalence (1) is immediate from the definition of the DNR axiom.

The “if” part of (2) follows from the relativization of a result of Martin-Löf [8]: there is a Martin-Löf test $(U_n)_{n \in \omega}$ (as in Definition 1.2) such that the complement of any U_n is a Π_1^0 class of positive measure containing only Martin-Löf random sets. Namely, let $(U_n)_{n \in \omega}$ be a universal Martin-Löf test.

The “only if” part of (2) follows from the relativization of a result of Kučera [6]: for every Martin-Löf random set R , every Π_1^0 class of positive measure contains some finite modification of R . \dashv

We will prove the following theorem by elaborating on the proof of Proposition 3 of [5]. The proof given there is attributed to Kurtz; the result follows also from a theorem of Kučera [6].

Theorem 1.4. There is a recursive function h such that for each Martin-Löf random real R , there is an h -DNR function f recursive in R .

PROOF. Given any real $A \subseteq \omega$ and $x \in \omega$, let $f_A^*(x)$ be equal to A restricted to x , considered as a number $< 2^x$. Let $h(x) = 2^x$ and let

$$U_n = \{A : \exists x > n. f_A^*(x) = \Phi_x(x)\}.$$

Then the sets U_n define a Martin-Löf test U . Hence no Martin-Löf random set is in all of the U_n . So $f_R^*(x) = \Phi_x(x)$ for at most finitely many x . Let f be a finite modification of f^* such that f is h -DNR. Since R computes f^* , R computes f . \dashv

The following theorem is proved in Section 3.

Theorem 1.5. For any recursive function $h : \omega \rightarrow \omega$, there exists $G : \omega \rightarrow \omega$ which is relatively DNR, such that for each Turing functional Φ and each $i \in \omega$, $\Phi^{G_0 \oplus \dots \oplus G_i}$ is not an h -DNR function.

Lemma 1.6. Let h be as in Theorem 1.4 and let \mathcal{I} be the Turing ideal generated by the functions G_i (for $i \in \omega$) of Theorem 1.5 for this h . Then for each element H of \mathcal{I} , there is an element K of \mathcal{I} such that K is DNR^H .

PROOF. Since H is in \mathcal{I} , there exist y and e such that $H = \Phi_e^{G_0 \oplus \dots \oplus G_{2y-1}}$. Let $K = G_{2y}$. Since G is relatively DNR, the proof is complete. \dashv

Theorem 1.7. DNR is strictly weaker than WWKL_0 .

PROOF. Let h be as in Theorem 1.4 and let \mathcal{I} be the Turing ideal generated by the functions G_i (for $i \in \omega$) of Theorem 1.5 for this h . By Theorem 1.4, \mathcal{I} contains no Martin-Löf random real. By Lemma 1.6, for each element H of \mathcal{I} , there is an element K of \mathcal{I} such that K is DNR^H . Hence, by Lemma 1.3, the ω -model of RCA_0 whose second-order part consists of all the sets in \mathcal{I} is a model of DNR in which WWKL_0 is false. \dashv

The following two theorems will not be used for the proof of Theorem 1.7, but seem to have independent interest. Their proofs are based on the proof of Theorem 2.1.

Theorem 1.8. There exists $G : \omega \rightarrow \omega$ such that G is DNR, but G does not compute any h -DNR function for any recursive function h .

Theorem 1.9. For each recursive function $h : \omega \rightarrow \omega$ there exists a recursive function $h^* : \omega \rightarrow \omega$ and a function $G : \omega \rightarrow \omega$ such that G is h^* -DNR, but for all Turing functionals Φ , Φ^G is not h -DNR.

Throughout the rest of this article, fix a recursive function $h : \omega \rightarrow \omega$.

§2. Warm-up. This section is devoted to the proof of Theorem 2.1, which serves as a warm-up exercise for Theorem 1.5.

Theorem 2.1. There exists $G : \omega \rightarrow \omega$ such that G is DNR, but for all Turing functionals Φ , Φ^G is not h -DNR.

To satisfy the requirement that G be DNR, it will be convenient to use the following definition.

DEFINITION 2.2 (Section 2 only). Let Φ_0 be a Turing functional such that for all $G : \omega \rightarrow \omega$, $\Phi_0^G \downarrow \leftrightarrow \exists x. G(x) = \Phi_0(x)$, and if $\Phi_0^G \downarrow = i \in \omega$ then $i = 0$. Let Φ_n , $n \geq 1$, be the Turing functionals of Definition 1.1.

The following definition is based on concepts in Kumabe's unpublished preprint [7], in which he establishes the existence of a fixed-point free minimal degree.

DEFINITION 2.3 (Good trees). A finite set of incomparable strings in $\omega^{<\omega}$ is called a *tree*. (Note that this differs from some common notions of tree.) Given $a \in \omega$, a nonempty tree T is called *a-good from* $\sigma \in \omega^{<\omega}$ if

- (1) every string $\tau \in T$ extends σ , and
- (2) for each $\tau \in \omega^{<\omega}$, if there exists $\rho \in T$ with $\sigma \subseteq \tau \subset \rho$, then there are at least a many immediate successors of τ which are substrings of elements of T .

If T is *a-good from* σ and $T \subseteq P \subseteq \omega^{<\omega}$, then T is called *a-good from* σ *for* P .

Lemma 2.4. Let $b \geq a \geq 1$, let $T, P \subseteq \omega^{<\omega}$ and $\sigma \in \omega^{<\omega}$. If T is *b-good from* σ for P then T is *a-good from* σ for P .

Lemma 2.4 is immediate from Definition 2.3. Note, however, that a tree that contains an *a-good tree* is not necessarily itself *a-good*.

Lemma 2.5 (Lemma 2.2(v) of [7]). Let $n \geq 1$. Given a tree T that is $(2n-1)$ -good from a string α and given a set $P \subseteq T$, there is a subset S of T which is n -good for P or for $T - P$.

PROOF. Give the elements of T the label 1 (0) if they are in P (not in P , respectively). Inductively, suppose β extends α and is a proper substring of an element of T . Suppose all the immediate successors of β that are substrings of elements of T have received a label. Give β the label 1 if at least half of its labelled immediate successors are labelled 1; otherwise, give β the label 0. This process ends after finitely many steps when α is given some label $i \in \{0, 1\}$. Let S be the set of i -labelled strings in T . If $i = 1$ then S is contained in P , and if $i = 0$ then S is contained in $T - P$, so it only remains to show that S is n -good.

Let L be the set of all labelled strings. Note that L is the set of strings extending α that are substrings of elements of T . For any $\beta \in L - T$, let k be the number of immediate successors of β that are in L . Since T is $(2n-1)$ -good,

$k \geq 2n - 1$. Let $p \leq k$ be the number of immediate successors of β that have the same label as β . By construction, $p \geq k/2$, and hence $p \geq n$. It follows that S is n -good. \dashv

The following lemma is not particularly sharp, but is sufficient for our purposes.

Lemma 2.6. Let $a, n \geq 1$. Let T be a tree which is $2^{a-1}n$ -good from a string α , and let P_1, \dots, P_a be sets of strings such that $T \subseteq \bigcup_i P_i$. Then for some i , T has a subset which is n -good from α for P_i .

PROOF. The case $a = 1$ is trivial; the subset is T itself. So assume $a \geq 2$ and assume that Lemma 2.6 holds with $a - 1$ in place of a . By Lemma 2.5, if there is no $2^{a-2}n$ -good subset of T from α for P_1 then there is a $2^{a-2}n$ -good subset S of T from α for the complement \bar{P}_1 . As $T \cap \bar{P}_1 \subseteq P_2 \cup \dots \cup P_a$, it follows that S is $2^{a-2}n$ -good from α for $P_2 \cup \dots \cup P_a$. By Lemma 2.6 with $a - 1$ in place of a , S has a subset R which is n -good from α for some P_i , $i \geq 2$. As R is also a subset of T , the proof is complete. \dashv

DEFINITION 2.7. Let $\epsilon : \omega \rightarrow \omega$ be a finite partial function and write $e_t = \epsilon(t)$ for each t in the domain of ϵ .

Let Φ be any Turing functional such that for all $G : \omega \rightarrow \omega$,

$$\Phi^G(\epsilon) \downarrow \leftrightarrow \exists t \in \text{dom}(\epsilon) [\Phi_t^G(e_t) \downarrow < h(e_t)]$$

Given $n \in \omega$ and ϵ , let $g(n, \epsilon) = 2^a n$ where

$$a = \sum_{t \in \text{dom}(\epsilon)} h(e_t).$$

Suppose we have a sequence of computations (namely, $\Phi_t(e_t)$ for those t where e_t is defined) that we would like to maintain the divergence of, while specifying more and more of the oracle for the computations. Then we can use Definition 2.7 as follows: Given $n \in \omega$, there exists a number $g = g(n, \epsilon) \in \omega$ such that if none of the computations $\Phi_t^G(e_t)$ converge and take values dominated by h on any n -good tree of strings, then $\Phi^G(\epsilon)$ does not converge on any g -good tree of strings. Lemma 2.8 spells this out.

Lemma 2.8. Let $n \geq 1$, let ϵ be a finite partial function from ω to ω , and let g be the function defined in Definition 2.7.

For each pair (t, i) satisfying $i < h(e_t)$ (where h is as in Section 1) and $t \in \text{dom}(\epsilon)$, let $Q_{(t,i)} = \{\beta : \Phi_t^\beta(e_t) = i\}$. Let $Q = \{\beta : \Phi^\beta(\epsilon) \downarrow\}$.

If there is a $g(n, \epsilon)$ -good tree for Q from some string α , then for some (t, i) , there is an n -good tree from α for $Q_{(t,i)}$.

PROOF. The number of pairs (t, i) such that $Q_{(t,i)}$ is defined is

$$a = \sum_{t \in \text{dom}(\epsilon)} h(e_t).$$

By the assumption that there is a $g(n, \epsilon)$ -good tree for Q , it follows that $a > 0$. So since $2^a n \geq 2^{a-1} n$, every $2^a n$ -good tree is $2^{a-1} n$ -good. Now apply Lemma 2.6 to the properties $Q_{(t,i)}$. \dashv

The following Definition 2.9 will be used in Section 3. We include it here for cross-reference with Lemma 3.9.

DEFINITION 2.9. A tree T is n/m -good (read: n -over- m good) from α if there are m many immediate successors of α which have extensions in T , and for any β having a proper extension in T , β a proper superstring of α , there are n many immediate successors of β which have extensions in T .

Note that if we imagine trees as growing upwards, this means T is n “over” m good in a pictorial sense.

Lemma 2.10. Suppose we are given α and n and a set $P \subseteq \omega^{<\omega}$ such that there is no n -good tree from α for P .

Then if V is an n -good tree from α then there exists β such that

1. β extends an element of V , and
2. there is no n -good tree from β for P .

PROOF. In fact, there exists such β which is an element of V , since otherwise, letting V_β be a counterexample for β ,

$$V^* = \bigcup_{\beta \supseteq \alpha, \beta \in V} V_\beta$$

would be n -good from α for P . ⊥

DEFINITION 2.11. Given a string $\alpha \in \omega^{<\omega}$, $c \in \omega$, and $n \in \omega$, let $f = f_{\alpha, c, n}$ be defined by the condition: $\Phi_{f(e), t}(x) = i$ if in t steps a finite tree T and a number $i < h(e)$ are found such that T is n -good from α for $\{\beta : \Phi_c^\beta(e) = i\}$ (and i is the i occurring for the first such tree found). If such T and i are not found within t steps, then $\Phi_{f(e), t}(x) \uparrow$.

DEFINITION 2.12. *The Construction.*

At any stage $s + 1$, the finite set D_{s+1} will consist of indices $t \leq s$ for computations Φ_t^G that we want to ensure to be divergent. The set A_{s+1} will consist of what we think of as acceptable strings.

Stage 0.

Let $G[0] = \emptyset$, the empty string, and $\epsilon[0] = \emptyset$. Let $n[0] = 2$. Let $D_0 = \emptyset$ and $A_0 = \omega^{<\omega}$.

Stage $s + 1$, $s \geq 0$.

Let $n[s + 1] = g(n[s], \epsilon[s])$, with g as in Definition 2.7.

Below we will define D_{s+1} . Given D_{s+1} , A_{s+1} will be the set of strings τ properly extending $G[s]$ such that for each $t \in D_{s+1}$, there is no pair $\langle T, i \rangle$ such that $i < h(e_t)$ and T is a finite $n[s + 1]$ -good tree from τ for $Q_{(t, i)} = \{\sigma : \Phi_t^\sigma(e_t) \downarrow = i\}$.

Let e be the fixed point of $f = f_{G[s], s, n[s+1]}$ (as defined in Definition 2.11) produced by the Recursion Theorem, i. e., $\Phi_e = \Phi_{f(e)}$.

Case 1. $\Phi_e(e) \downarrow$.

Fix T as in Definition 2.11. Let $D_{s+1} = D_s$. Let $G[s + 1]$ be an extension of $G[s]$ such that $G[s + 1] \in T$ and $G[s + 1] \in A_{s+1}$.

Case 2. $\Phi_e(e) \uparrow$. Let $D_{s+1} = D_s \cup \{s\}$. Let $\epsilon[s + 1] = \epsilon[s] \cup \{(s, e)\}$. In other words, $e_s = \epsilon(s)$ exists and equals e . Let $G[s + 1]$ be any element of A_{s+1} .

Let $G = \bigcup_{s \in \omega} G[s]$.

End of Construction.

We now prove that the Construction satisfies Theorem 2.1 in a sequence of lemmas.

Lemma 2.13. For each $s, t \in \omega$ with $t \leq s$, $n_t[s] \geq 2$.

PROOF. For $s = 0$, we have $n[0] = 2$. For $s+1$, we have $n[s+1] = g(n[s], \epsilon[s]) = 2^a n[s]$ for a certain $a \geq 0$, by Definition 2.8, hence the lemma follows. \dashv

Lemma 2.14. For each $s \geq 0$ the following holds.

- (1) The Construction at stage s is well-defined and $G[s] \in A_s$. In particular, if $s > 0$ then in Case 2, A_s is nonempty, and in Case 1, A_s contains at least one element of T .
- (2) There is no $n[s+1]$ -good tree for $Q = \{\beta : \Phi^\beta(\epsilon[s]) \downarrow\}$ from $G[s]$.
- (3) Every tree V which is $n[s+1]$ -good from $G[s]$, and is not just the singleton of $G[s]$, contains an element of A_{s+1} .

PROOF. It suffices to show that (1) holds for $s = 0$, and that for each $s \geq 0$, (1) implies (2) which implies (3), and moreover that (3) for s implies (1) for $s+1$.

(1) holds for $s = 0$ because $G[0] = \emptyset \in \omega^{<\omega} = A_0$.

(1) implies (2):

By definition of A_s and the fact that $G[s] \in A_s$ by (1) for s , we have that for each $t \in D_s$, and each $i < h(e_t)$, there is no $n[s]$ -good tree from $G[s]$ for $Q_{(t,i)} = \{\beta : \Phi_t^\beta(e_t) \downarrow = i\}$. Hence by Lemma 2.8, there is no $n[s+1]$ -good tree for $Q = \{\beta : \Phi^\beta(\epsilon[s]) \downarrow\}$ from $G[s]$.

(2) implies (3):

Since V is $n[s+1]$ -good, by Lemma 2.10 there is an element β of V from which there is no $n[s+1]$ -good tree for Q , and hence not for any $Q_{(t,i)}$ since $Q_{(t,i)} \subseteq Q$. Moreover, β properly extends $G[s]$, since V is an antichain and is not the singleton of $G[s]$. Hence by definition of A_{s+1} , this element β belongs to A_{s+1} .

(3) for s implies (1) for $s+1$:

If Case 1 holds, let T be the tree found by Φ_e , i. e., T is $n[s+1]$ -good from $G[s]$ (for $Q_{(s,i)}$ for some i). If T is not just the singleton of $G[s]$, and Case 1 holds, then apply (3) for s to T .

If T is just the singleton of $G[s]$ or if Case 2 holds, then apply (3) for s to any $n[s+1]$ -good non-singleton tree from $G[s]$. For example, this could be the set of immediate extensions $G[s] * k$, $k < n[s+1]$. \dashv

Lemma 2.15. For any $s \geq 0$, if $s \in D_{s+1}$ then $\Phi_s^G(e_s) \uparrow$ or $\Phi_s^G(e_s) \geq h(e_s)$.

PROOF. Otherwise for some $t \in \omega$, $\Phi_s^{G[t]}(e_s) \downarrow < h(e_s)$. Since the singleton tree $T = \{G[t]\}$ is n -good from $G[t]$ for all n , hence in particular $n[t]$ -good, this contradicts the fact that by Lemma 2.14(1), $G[t] \in A_t$. \dashv

Lemma 2.16. There is no 2-good tree for $\{\beta : \Phi_0^\beta \downarrow\}$ from $G[0]$.

PROOF. Suppose a string α is DNR and $k_1 \neq k_2$ are integers. Let $x = |\alpha|$ (so x is the first input on which α is undefined). It may or may not be the case that $\varphi_x(x) \downarrow$. In any case, it cannot be that $k_1 = \varphi_x(x) = k_2$. Hence at least one among $\alpha * k_1$ and $\alpha * k_2$ is DNR. This shows that there is no 2-good tree from α for the set of non-DNR strings. By Definition 2.2, $\Phi_0^\beta \downarrow$ iff β is not a DNR string. As $G[0] = \emptyset$ (the empty string) is a DNR string, the lemma follows. \dashv

Lemma 2.17. $0 \in D_1$.

PROOF. By definition of D_1 , it suffices to show that at stage 1 of the Construction, there is no $n[1]$ -good tree from $G[0]$ for $\{\beta : \Phi_0^\beta \downarrow = i\}$ for any $i < h(e)$. As $\{\beta : \Phi_0^\beta \downarrow = i\} \subseteq \{\beta : \Phi_0^\beta \downarrow\}$ and $n[1] = 2$, this is immediate from Lemma 2.16. \dashv

Lemma 2.18. G is a total function, i.e., $G \in \omega^\omega$.

PROOF. By Lemma 2.14(3), $G[s+1] \in A_{s+1}$ for each $s \geq 0$, and hence by definition of A_{s+1} , $G[s+1]$ is a proper extension of $G[s]$. From this the lemma immediately follows. \dashv

Lemma 2.19. G is DNR.

PROOF. By Lemmas 2.15 and 2.17, we have that either $\Phi_0^G \uparrow$ or $\Phi_0^G \geq h(e_0)$. By Definition 1.1, $h(n) > 0$ for all n , whereas by Definition 2.2, $\Phi_0^G \downarrow = i$ implies $i = 0$. Hence the only possibility is that $\Phi_0^G \uparrow$. By Definition 2.2, this means that G is DNR. \dashv

Lemma 2.20. G computes no h -DNR function.

PROOF. Since each Turing functional has infinitely many indices, it suffices to show that for each s , Φ_s^G is not h -DNR where Φ_s is as in Definition 2.2. That is, the fact that we defined our own Φ_0 is not a problem.

If Case 1 of the construction is followed then $\Phi_s^G(e) = \Phi^{G[s+1]}(e) = \Phi_e(e)$ because $G[s+1] \in T$. So Φ_s^G is not h -DNR. If Case 2 of the construction is followed then $s \in D_{s+1}$ and so $\Phi_s^G(e) \uparrow$ or $\Phi_s^G(e) \geq h(e)$ by Lemma 2.15. Hence Φ_s^G is not h -DNR. \dashv

§3. The main theorem. In this section we prove Theorem 1.5, which we restate here.

Theorem 3.1. For any recursive function $h : \omega \rightarrow \omega$, there exists $G : \omega \rightarrow \omega$ (where $G = \bigoplus_{i \in \omega} G_i$) which is relatively DNR, and such that for each Turing functional Φ and each $i \in \omega$, $\Phi^{G_0 \oplus \dots \oplus G_i}$ is not an h -DNR function.

To satisfy the requirement that G be relatively DNR, it will be convenient to use the following definition.

DEFINITION 3.2 (Section 3 only). Let Φ_z , $z \in \omega$ be a sequence of Turing functionals satisfying the following conditions:

- (1) For all z , Φ_z queries its oracle on no column other than columns $0, \dots, z$. So $\Phi_z^G = \Phi_z^{G_0 \oplus \dots \oplus G_z}$ for all $G : \omega \rightarrow \omega$.
- (2) For all $y \in \omega$, $\Phi_{2y}^G \downarrow \leftrightarrow \exists x. G_{2y}(x) = \Phi_x^{G_0 \oplus \dots \oplus G_{2y-1}}(x)$, and if $\Phi_{2y}^G \downarrow = i \in \omega$ then $i = 0$. All other Turing functionals belong to the set $\{\Phi_{2y+1} : y \in \omega\}$.

In Definition 3.2, we note that when $y = 0$, $G_0 \oplus \cdots \oplus G_{2y-1}$ equals \emptyset . Also Φ_{2y}^G only queries G on columns $0, \dots, 2y$, so (2) is in compliance with (1).

We need the following extension of Definition 2.3.

DEFINITION 3.3 (Good systems of trees). Given strings $\sigma_n \in \omega^{<\omega}$, $n \in \omega$, we define $\sigma = \bigoplus_{n \in \omega} \sigma_n$ by $\sigma(2^n(2k+1)) = \sigma_n(k)$. We write $\sigma = \sigma_0 \oplus \cdots \oplus \sigma_k$ if $\sigma_n = \emptyset$ for all $n > k$. Let $\Omega = \omega^{<\omega}$, and let $\Omega^{<\omega}$ be the set

$$\{\sigma_0 \oplus \cdots \oplus \sigma_k \mid k \in \omega \ \& \ \forall i \leq k. \sigma_i \in \omega^{<\omega}\}.$$

Note that $\Omega \subseteq \Omega^{<\omega}$. Conversely, given $\sigma \in \Omega^{<\omega}$, the equation $\sigma(2^n(2k+1)) = \sigma_n(k)$ defines each σ_n . We refer to the elements of $\Omega^{<\omega}$ as pseudostrings. For example, $\langle 0, 1, 1, 0 \rangle \oplus \langle 1, 1, 0 \rangle$ is pictured as being defined on initial segments of the first two columns of ω of length 4 and 3, respectively.

Given $\alpha_0, \dots, \alpha_x \in \Omega$, $x \geq 0$, we use the shorthand notation $\vec{\alpha}_x$ for $(\alpha_0, \dots, \alpha_x)$. Similarly for other mathematical objects: so for example if n_0, \dots, n_x are integers we abbreviate (n_0, \dots, n_x) by \vec{n}_x . $\vec{\alpha}_x$ is also identified with the pseudostring $\alpha_0 \oplus \cdots \oplus \alpha_x$. So given $\alpha \in \Omega^{<\omega}$, the equation $\alpha = \vec{\alpha}_x$ is equivalent to: $\alpha_y = \emptyset$ for all $y > x$.

If $\vec{n}_x = (n_0, \dots, n_x)$ then we can apply operations componentwise, such as writing $2\vec{n}_x - 1$ for $(2n_0 - 1, \dots, 2n_x - 1)$.

Let $x \geq 0$. A *system of trees* $\vec{T} = (T_0, \dots, T_x) = \vec{T}_x$ is a tree T_0 together with, for each $\sigma_0 \in T_0$, a tree $T_1(\sigma_0)$; and recursively for each $\sigma_k \in T_k(\vec{\sigma}_{k-1})$, $0 \leq k < x$, a tree $T_{k+1}(\vec{\sigma}_k)$. If $\sigma_x \in T_x(\vec{\sigma}_{x-1})$, we say $\vec{\sigma}_x \in \vec{T}$. (If $x = 0$, $\vec{\sigma}_{x-1}$ is the empty sequence and $T_x(\vec{\sigma}_{x-1}) = T_0$.)

We say that a pseudostring β extends a pseudostring α if $\beta(x) = \alpha(x)$ whenever $\alpha(x)$ is defined.

Hence if α and β are elements of Ω^{x+1} for some $x \geq 0$ then we have a notion of β extending α .

We call a set $P \subseteq \Omega^{<\omega}$ *open* if for each $\alpha \in P$ and β extending α , $\beta \in P$. Given $x \geq 0$, a subset P of Ω^{x+1} is called *open* if for each $\alpha \in P$ and β extending α , $\beta \in \Omega^{x+1}$, we have $\beta \in P$.

Suppose P is a subset of Ω^{x+1} . A system is said to be a *system for* P if each element of the system is in P . We write $P(\vec{\xi}_x)$ to indicate that $\vec{\xi}_x \in P$; and we write $P(\vec{\xi}_{x-1}, \cdot)$ for $\{\xi_x : P(\vec{\xi}_{x-1}, \xi_x)\}$.

A system \vec{T}_x is \vec{n}_x -*good from* $\vec{\sigma}_x$ if for each $\vec{\beta}_{k-1} \in \vec{T}_{k-1}$, $0 \leq k < x$, $T_k(\vec{\beta}_{k-1})$ is n_k -good from σ_k . For $k = 0$ this means that T_0 is n_0 -good from σ_0 .

A system \vec{T}_x is $(\vec{n}_{x-1}, n_x/m)$ -*good from* $\vec{\sigma}_x$ if

- (1) \vec{T}_{x-1} is \vec{n}_{x-1} -good from $\vec{\sigma}_{x-1}$, and
- (2) for each $\vec{\beta}_{x-1} \in \vec{T}_{x-1}$, $T_x(\vec{\beta}_{x-1})$ is n_x/m -good from σ_x (as in Definition 2.9).

We say that $\vec{\xi}_x$ *componentwise extends* $\vec{\beta}_x$ if for each $0 \leq y \leq x$, ξ_y extends β_y ; in other words $\vec{\xi}_x$ extends $\vec{\beta}_x$ if we consider them both as pseudostrings. If $\vec{\xi}_x$ componentwise extends $\vec{\beta}_x$, and $\vec{\beta}_x$ is an element of a system \vec{T}_x , then $\vec{\beta}_x$ is called the *restriction* of $\vec{\xi}_x$ to \vec{T}_x . This is well-defined since \vec{T}_x is an antichain under the partial order of componentwise extension.

To prove Theorem 3.1 we will extend the results of Section 2 from trees to systems of trees.

Lemma 3.4. Let \vec{m}_x, \vec{n}_x be sequences of positive integers such that $m_i \geq n_i$ for each $0 \leq i \leq x$. Let \vec{T}_x be a system of trees. Let $P \subseteq \Omega^{x+1}$, and let $\vec{\sigma}_x$ be a sequence of elements of Ω . If \vec{T}_x is \vec{m}_x -good from $\vec{\sigma}_x$ for P then \vec{T}_x is \vec{n}_x -good from $\vec{\sigma}_x$ for P .

Lemma 3.4 is immediate from Definition 3.3. The following is a generalization of Lemma 2.5 to systems of trees.

Lemma 3.5. Given $x \geq 0$, a system \vec{T}_x that is $(2\vec{n}_x - 1)$ -good from some sequence of strings $\vec{\sigma}_x$, and a subset P of \vec{T}_x , there is either an \vec{n}_x -good subset of \vec{T}_x for P from $\vec{\sigma}_x$, or an \vec{n}_x -good subset of \vec{T}_x for the complement of P from $\vec{\sigma}_x$.

PROOF. The case $x = 0$ is Lemma 2.5. Suppose $x \geq 1$. All sequences $\vec{\alpha}_y$, $0 \leq y \leq x$, in the following proof are assumed to be in \vec{T}_y . Let $\vec{\alpha}_{-1}$ denote the empty sequence of strings. Call the elements $\vec{\alpha}_x$ that are (not) in P red (blue). So each $\vec{\alpha}_x$ is either red or blue.

Inductively, let $y \leq x$, $y \geq 0$. Call $\vec{\alpha}_{y-1}$ red (blue) if there is an n_y -good tree of α_y from σ_y such that $\vec{\alpha}_y$ is red (blue). Each $\vec{\alpha}_{y-1}$ is either red or blue by Lemma 2.5, since each $\vec{\alpha}_y$ is either red or blue.

Hence $\vec{\alpha}_{-1}$ is either red or blue. Say $\vec{\alpha}_{-1}$ is red. Then there is an \vec{n}_x -good system from $\vec{\sigma}_x$ for which $\vec{\alpha}_x$ is red, namely, the set of all $\vec{\alpha}_x$ such that for each $y \leq x$, $\vec{\alpha}_y$ is red. \dashv

Lemma 2.6 generalizes to Lemma 3.6 below by the same proof.

Lemma 3.6. Let $a \geq 1$ and let \vec{n} be a finite sequence of positive integers. Let \vec{T} be a system of trees which is $2^{a-1}\vec{n}$ -good from some $\vec{\alpha}$, and let P_1, \dots, P_a be sets of sequences of strings such that $\vec{T} \subseteq \bigcup_i P_i$. Then for some i , \vec{T} has a subset which is \vec{n} -good for P_i from $\vec{\alpha}$. \dashv

The following definition extends Definition 2.7.

DEFINITION 3.7. Given a finite sequence of positive integers \vec{n}_x , $x \geq 0$, and a finite partial function ϵ from ω to ω , let $\vec{g}_x(\vec{n}_x, \epsilon) = 2^a \vec{n}_x$ where

$$a = \sum_{t \in \text{dom}(\epsilon)} h(\epsilon(t))$$

and h is as in Section 1.

Lemma 2.8 now generalizes to the following Lemma 3.8. The proof of Lemma 3.8 from Lemma 3.6 is identical to the proof of Lemma 2.8 from Lemma 2.6.

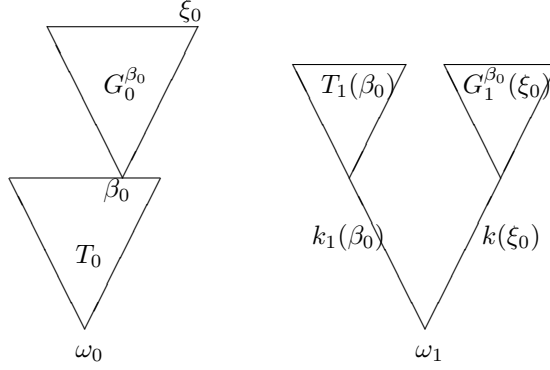
Lemma 3.8. Let \vec{n}_s be a finite sequence of positive integers, let ϵ be a finite partial function from ω to ω , and let \vec{g}_s be the function defined in Definition 3.7.

For each pair (t, i) satisfying $i < h(e_t)$ and $t \in \text{dom}(\epsilon)$, $t \leq s$, let

$$Q_{(t,i)} = \{\vec{\beta}_s : \Phi_t^{\vec{\beta}_t}(\epsilon(t)) = i\}.$$

Let

$$Q = \{\vec{\beta}_s : \Phi_s^{\vec{\beta}_s}(\epsilon) \downarrow\}.$$

FIGURE 1. The case $x = 1$, $m = 1$ of Lemma 3.9.

If there is a $\vec{g}(\vec{n}, \epsilon)$ -good system for Q from some $\vec{\alpha}$, then for some (t, i) , there is an \vec{n} -good system from $\vec{\alpha}$ for $Q_{(t,i)}$. \dashv

In Lemma 3.10 below we will generalize Lemma 2.10. To that end we first prove Lemma 3.9 below. Lemma 3.9 can be viewed as a generalization of the following observation. Recall the notion of b/a -good from Definition 2.9. Suppose $a < b$ and there exists a b/a -good tree T from α for a set P , but there is no $b/a+1$ -good tree from α for P . Suppose k_1, \dots, k_a are a many distinct integers such that for each i , T contains a tree which is n -good from $\alpha * k_i$ for P . Then there is no $k \notin \{k_1, \dots, k_a\}$ such that T contains a tree which is n -good from $\alpha * k$ for P .

Lemma 3.9. Suppose we are given $x \geq 0$, a sequence of strings $\vec{\omega}_x$, a sequence of positive integers \vec{n}_x , and an open set $P \subseteq \Omega^{x+1}$.

Suppose $0 \leq m < n_x$, and \vec{T}_x is a $(\vec{n}_{x-1}, n_x/m)$ -good system from $\vec{\omega}_x$ for P , but there is no $(\vec{n}_{x-1}, n_x/(m+1))$ -good system from $\vec{\omega}_x$ for P .

Given $\vec{\beta}_{x-1} \in \vec{T}_{x-1}$, let $k_i(\vec{\beta}_{x-1})$, $i = 1, \dots, m$ denote m many numbers k for which $T(\vec{\beta}_{x-1})$ is n_x -good from $\omega_x * k$.

Then it is not the case that for every $\vec{\beta}_{x-1} \in \vec{T}_{x-1}$ there exists an \vec{n}_{x-1} -good system $\vec{G}_{x-1}^{\vec{\beta}_{x-1}}$ from $\vec{\beta}_{x-1}$ for which for each $\vec{\xi}_{x-1} \in \vec{G}_{x-1}^{\vec{\beta}_{x-1}}$ there exists $k(\vec{\xi}_{x-1}) \notin \{k_i(\vec{\beta}_{x-1}) : 1 \leq i \leq m\}$ such that there exists $G_x^{\vec{\beta}_{x-1}}(\vec{\xi}_{x-1})$ which is n_x -good for $P(\vec{\xi}_{x-1}, \cdot)$ from $\omega_x * k(\vec{\xi}_{x-1})$.

PROOF. Suppose $\vec{\xi}_{x-1} \in \vec{G}_{x-1}^{\vec{\beta}_{x-1}}$. Since $\vec{G}_{x-1}^{\vec{\beta}_{x-1}}$ is good from $\vec{\beta}_{x-1}$, we know that $\vec{\xi}_{x-1}$ extends $\vec{\beta}_{x-1}$ componentwise. Let $\vec{\beta}_{x-1}$ be the restriction of $\vec{\xi}_{x-1}$ to \vec{T}_{x-1} .

Suppose the lemma fails. Let $\vec{G}_x = \bigcup \{G_x^{\vec{\beta}_{x-1}} : \vec{\beta}_{x-1} \in \vec{T}_{x-1}\}$. Let $\vec{H}_x = \vec{G}_x$ except that

$$H_x(\vec{\xi}_{x-1}) = G_x(\vec{\xi}_{x-1}) \cup T_x(\vec{\beta}_{x-1})$$

for each $\vec{\xi}_{x-1}$ and its restriction $\vec{\beta}_{x-1}$ to \vec{T}_{x-1} .

If i is a number such that $1 \leq i \leq m$, then $T_x(\vec{\beta}_{x-1})$ is an n_x -good tree for $P(\vec{\beta}_{x-1}, \cdot)$ from $\omega_x * k_i$ and hence by openness of P also an n_x -good tree for

$P(\vec{\xi}_{x-1}, \cdot)$ from $\omega_x * k_i$ for each $1 \leq i \leq m$, since $\vec{\xi}_{x-1}$ extends $\vec{\beta}_{x-1}$ componentwise.

But $G_x(\vec{\xi}_{x-1})$ is an n_x -good tree for $P(\vec{\xi}_{x-1}, \cdot)$ from $\omega_x * k(\vec{\xi}_{x-1})$. Hence $H_x(\vec{\xi}_{x-1})$ is an $n_x/(m+1)$ -good tree for $P(\vec{\xi}_{x-1}, \cdot)$ from ω_x . So \vec{H}_x is an $(\vec{n}_{x-1}, n_x/m+1)$ -good system for P from $\vec{\omega}_x$, contradiction. \dashv

Lemma 3.10. Suppose we are given $\vec{\alpha}_x$ and \vec{n}_x and an open set $P \subseteq \Omega^{x+1}$ such that there is no \vec{n}_x -good system from $\vec{\alpha}_x$ for P .

If \vec{V}_x is an \vec{n}_x -good system from $\vec{\alpha}_x$ then there exists $\vec{\beta}_x$ such that

1. β_x extends componentwise an element of \vec{V}_x , and
2. there is no \vec{n}_x -good system from $\vec{\beta}_x$ for P .

PROOF. By Lemma 2.10, it is immediate that Lemma 3.10 holds for $x = 0$. Inductively, suppose $x \geq 1$ is given such that Lemma 3.10 holds for $x - 1$; we will show that Lemma 3.10 holds for x . From the hypothesis of Lemma 3.10, we are given that there is no \vec{n}_x -system from $\vec{\alpha}_x$ for P , and we let \vec{V}_x be as in the statement of Lemma 3.10.

Let P_{x-1} be the property defined by: for all $\vec{\beta}_{x-1}$, $P_{x-1}(\vec{\beta}_{x-1})$ holds iff there is an n_x -good tree from α_x for the property $\{\alpha : P(\vec{\beta}_{x-1}, \alpha)\}$.

We note that Lemma 3.10 for $x - 1$ is applicable to $\vec{\alpha}_{x-1}$, \vec{n}_{x-1} , P_{x-1} and \vec{V}_{x-1} . Indeed if there exists an \vec{n}_{x-1} -good system for P_{x-1} from $\vec{\alpha}_{x-1}$ then there would exist an \vec{n}_x -good system for P from $\vec{\alpha}_x$, by the definition of the notion of a good system, and this would contradict the hypothesis of Lemma 3.10 for x . And \vec{V}_{x-1} is \vec{n}_{x-1} -good from $\vec{\alpha}_{x-1}$.

So by Lemma 3.10 for $x - 1$, there exists $\vec{\sigma}_{x-1}$ extending componentwise an element of \vec{V}_{x-1} , such that there is no \vec{n}_{x-1} -good system from $\vec{\sigma}_{x-1}$ for P_{x-1} . In other words, there is no \vec{n}_x -good system from $(\vec{\sigma}_{x-1}, \alpha_x)$ for P .

Fix such a $\vec{\sigma}_{x-1}$. Let $\vec{\gamma}_{x-1}$ be the element of \vec{V}_{x-1} such that $\vec{\sigma}_{x-1}$ extends $\vec{\gamma}_{x-1}$ componentwise, and let V_x be a shorthand for $V_x(\vec{\gamma}_{x-1})$. Let

$$Q = \{\vec{\tau}_x : \text{there is no } \vec{n}_x\text{-good system for } P \text{ from } \vec{\tau}_x \ \& \ \exists \rho \in V_x(\rho \supseteq \tau_x)\}.$$

To complete the proof of the lemma, we will now construct $\vec{\omega}_x[0], \vec{\omega}_x[1], \dots, \vec{\omega}_x[p]$ for some $p \in \omega$, such that $\vec{\omega}_x[p]$ satisfies the conclusion of Lemma 3.10. To accomplish this we will ensure that for each $0 \leq i \leq p$, $\omega_x[i] \in Q$, and $\omega_x[p] \in V_x$.

Let $\vec{\omega}_x[0] = (\vec{\sigma}_{x-1}, \alpha_x)$. Note $\vec{\omega}_x[0] \in Q$. If $\omega_x[0] \in V_x$ then just let $p = 0$.

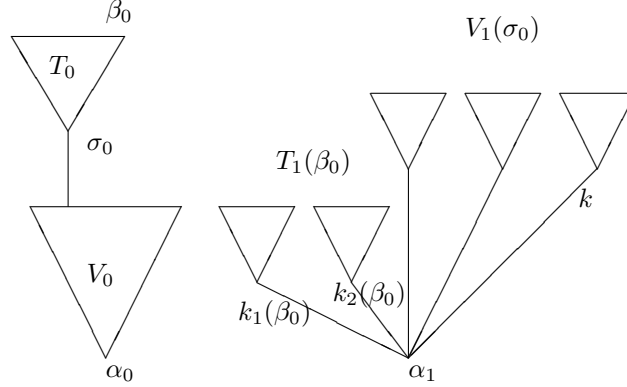
So suppose we are given $\vec{\omega}_x[i] \in Q$ for some $i \geq 0$, such that $\omega_x[i] \notin V_x$.

Let $m \geq 1$ be maximal such that there is a system \vec{T}_x which is $(\vec{n}_{x-1}, n_x/m)$ -good from $\vec{\omega}_x[i]$ for P , if such an m exists. If m exists, then since there is no \vec{n}_x -system from $\vec{\omega}_x[i]$, we have $m < n_x$; let \vec{T}_x be such a system.

If m does not exist, let $\vec{\omega}_x[i+1]$ be $(\vec{\omega}_{x-1}[i], \omega_x[i] * k)$ for some k such that $\omega_x[i] * k \subseteq \rho$ for some $\rho \in V_x$. Such a k exists because $\exists \rho \in V_x. \rho \supseteq \omega_x[i]$ and $\omega_x[i] \notin V_x$. Note that $\omega_x[i+1] \in Q$.

So we may assume m does exist. Given $\vec{\beta}_{x-1} \in \vec{T}_{x-1}$, we use the notation $k_i(\vec{\beta}_{x-1})$, $i = 1, \dots, m$ to list m many numbers k for which $T(\vec{\beta}_{x-1})$ is n_x -good from $\omega_x[i] * k$.

Let us temporarily say that \vec{G}_x is a useful system for $\vec{\beta}_{x-1}$ if \vec{G}_{x-1} is an \vec{n}_{x-1} -good system from $\vec{\beta}_{x-1}$ for which for each $\vec{\xi}_{x-1} \in \vec{G}_{x-1}$ there exists $k(\vec{\xi}_{x-1}) \notin$

FIGURE 2. The case $x = 1$, $m = 2$ of Lemma 3.10.

$\{k_i(\vec{\beta}_{x-1}) : 1 \leq i \leq m\}$ such that there exists $G_x(\vec{\xi}_{x-1})$ which is n_x -good for $P(\vec{\xi}_{x-1}, \cdot)$ from $\omega_x[i] * k(\vec{\xi}_{x-1})$.

By Lemma 3.9, it is not the case that for every $\vec{\beta}_{x-1} \in \vec{T}_{x-1}$ there exists a useful system. Thus, let $\vec{\beta}_{x-1}$ be a counterexample.

Since V_x is n_x -good and $n_x \geq m + 1$, V_x is $m + 1$ -good. We also know that $\exists \rho \in V_x \cdot \rho \supseteq \omega_x[i]$ and $\omega_x[i] \notin V_x$. It follows that there exists $k \notin \{k_i(\vec{\beta}_{x-1}) : 1 \leq i \leq m\}$ such that $\omega_x[i] * k$ is extended by an element of V_x . Fix such a k and let $\vec{\omega}_x[i + 1] = (\vec{\beta}_{x-1}, \omega_x[i] * k)$.

If there existed an \vec{n}_x -good system \vec{D}_x for P from $(\vec{\beta}_{x-1}, \omega_x[i] * k)$, then \vec{D}_x would be a useful system for $\vec{\beta}_{x-1}$ (with $k(\vec{\xi}_{x-1}) := k$ for each $\vec{\xi}_{x-1}$), contradiction. Hence $\vec{\omega}_x[i + 1] \in Q$.

Since V_x is finite, we eventually reach an i such that $\omega_x[i] \in V_x$. Letting $p = i$ completes the proof of the lemma. \dashv

The following definition extends Definition 2.11 to systems of trees.

DEFINITION 3.11. Given $x \geq 0$, a sequence of strings $\vec{\alpha}_x$ where each $\alpha_i \in \Omega$, $c \in \omega$ and a sequence of positive integers \vec{n}_x , let $f = f_{\vec{\alpha}, c, \vec{n}_x}$ be defined by the condition: for all $z, t \in \omega$, $\Phi_{f(e), t}(z) = i$ if in t steps a finite system of trees \vec{T}_x and a number $i < h(e)$ are found such that \vec{T}_x is \vec{n}_x -good from $\vec{\alpha}_x$ for $\{\vec{\beta}_x : \Phi_c^{\vec{\beta}_x}(e) = i\}$ (and i is the i occurring for the first such tree found). If no such \vec{T}_x and i are found within t steps, then $\Phi_{f(e), t}(x)$ is undefined.

DEFINITION 3.12. *The Construction.* At any stage $s + 1$, the finite set D_{s+1} will consist of indices $t \leq s$ for computations Φ_t^G that we want to ensure are divergent. The set A_{s+1} will consist of what we think of as acceptable pseudostrings. At stage s we will define a sequence of positive integers $\vec{n}[s] = \vec{n}_s[s] = (n_0[s], \dots, n_s[s])$.

Stage 0.

Let $G[0] = \emptyset$, the empty pseudostring, and $\epsilon[0] = \emptyset$. Let $\vec{n}[0] = \langle 2 \rangle$. Let $D_0 = \emptyset$ and $A_0 = \Omega$.

Stage $s + 1$, $s \geq 0$.

Below we will define D_{s+1} . Given D_{s+1} , A_{s+1} will be the set of pseudostrings $\tau = \vec{\tau}_s$ such that τ_t properly extends $G_t[s]$ for each $t \leq s$, and for each $t \in D_{s+1}$, there is no pair $\langle \vec{T}_t, i \rangle$ such that $i < h(e_t)$ and \vec{T}_t is a finite $\vec{n}[t]$ -good tree from τ for $Q_{(t,i)} = \{\sigma : \Phi_t^\sigma(e_t) = i\}$.

Let $\vec{n}_{s+1}[s+1] = (\vec{g}_s(\vec{n}_s[s], \epsilon[s]), 2)$, with \vec{g}_s as in Definition 3.7.

Let e be the fixed point of $f = f_{G[s], s, \vec{g}(\vec{n}[s], \epsilon[s])}$ (as in Definition 3.11) produced by the Recursion Theorem, i. e., $\Phi_e = \Phi_{f(e)}$.

Case 1. $\Phi_e(e) \downarrow$.

Fix \vec{T}_{s+1} as in Definition 3.11. Let $D_{s+1} = D_s$. Let $G[s+1]$ be an extension (columnwise, nonempty on columns $\leq s$ only) of $G[s]$ such that $G[s+1] \in \vec{T}_{s+1}$ and $G[s+1] \in A_{s+1}$.

Case 2. $\Phi_e(e) \uparrow$. Let $D_{s+1} = D_s \cup \{s\}$. Let $\epsilon[s+1] := \epsilon[s] \cup \{(s, e)\}$, so $e_s := e$. Let $G[s+1]$ be any element of A_{s+1} .

Let $G = \bigcup_{s \in \omega} G[s]$.

End of Construction.

We now prove that the Construction satisfies Theorem 3.1 in a sequence of lemmas.

Lemma 3.13. For each $s, t \in \omega$ with $t \leq s$, $n_t[s] \geq 2$.

PROOF. For $s = 0$, we have $\vec{n}[0] = (n_0[0]) = (2)$.

For $s + 1$, we have $\vec{n}[s+1] = (\vec{g}_s(\vec{n}[s], \epsilon[s]), 2)$ and $g_s(\vec{n}[s], \epsilon[s]) = 2^a \vec{n}[s]$ for a certain $a \geq 0$, by Definition 3.8, hence the lemma follows. \dashv

Note that $G[s]$, while only nonempty on columns $\leq s-1$, can be considered as defined on all columns, or as many additional columns as desired, in accordance with Definition 1.1. For example, in Lemma 3.14(3) we think of $G[s]$ as

$$G_0[s] \oplus \cdots \oplus G_{s-1}[s] \oplus G_s[s]$$

with $G_s[s] = \emptyset$.

Lemma 3.14. For each $s \geq 0$, the following holds.

- (1) The Construction at stage s is well-defined and $G[s] \in A_s$. In particular, if $s > 0$ then if Case 2 applies then A_s is nonempty, and if Case 1 applies then A_s contains elements of \vec{T} .
- (2) There is no $\vec{g}_s(\vec{n}_s[s], \epsilon[s])$ -good system of trees for

$$Q = \{\vec{\beta}_s : \Phi^{\vec{\beta}_s}(\epsilon[s]) \downarrow\}$$

from $G[s]$.

- (3) Every system \vec{V}_s which is $\vec{g}_s(\vec{n}_s[s], \epsilon[s])$ -good from $G[s]$, and is not just the singleton of $G[s]$, contains an element of A_{s+1} .

PROOF. It suffices to show that (1) holds for $s = 0$, and that for each $s \geq 0$, (1) implies (2) which implies (3), and moreover that (3) for s implies (1) for $s + 1$.

(1) holds for $s = 0$ because $G[0] = \emptyset \in \Omega = A_0$.

(1) implies (2):

Suppose \vec{U}_s is a $\vec{g}_s(\vec{n}_s[s], \epsilon[s])$ -good system for Q from $G[s]$. As each Φ_t only queries columns $\leq t$, and $t \in D_s = \text{dom}(\epsilon[s])$ implies $t < s$, we see that each

Φ_t for $t \in D_s$ only queries columns $\leq s-1$, so $\Phi^X(\epsilon[s])$ only queries columns $\leq s-1$ for any X , and in particular only queries columns $\leq s$. By Lemma 3.8, there is an $\vec{n}[s] = \vec{n}_s[s]$ -good system \vec{V}_s for

$$Q_{(t,i)} = \{\vec{\beta}_s : \Phi_t^{\vec{\beta}_s}(e_t) \downarrow = i\}$$

for some $t \in D_s$ and $i < h(e_t)$ from $G[s]$.

Now $\vec{n}[s] = (\vec{g}_{s-1}(\vec{n}[s-1], \epsilon[s-1]), 2)$, hence the restriction \vec{V}_{s-1} is $\vec{g}_{s-1}(\vec{n}[s-1], \epsilon[s-1])$ -good.

For each $t^* \leq s-1$, $g_{t^*}(\vec{n}[s-1], \epsilon[s-1]) = 2^a n_{t^*}[s-1] \geq n_{t^*}[s-1]$ (for a certain $a \geq 0$). Applying this to $t^* \leq t$ (since $t \leq s-1$), by Lemma 3.4, the further restriction \vec{V}_t is $\vec{n}[t]$ -good.

By (1) for s , $G[s] \in A_s$. Recall that A_s is the set of pseudostrings $\tau = \vec{\tau}_{s-1}$ such that τ_{t^*} properly extends $G_{t^*}[s-1]$ for each $t^* \leq s-1$, and for each $t^* \in D_s$ (hence $t^* \leq s-1$), there is no pair $\langle \vec{T}_{t^*}, i^* \rangle$ such that $i^* < h(e_{t^*})$ and \vec{T}_{t^*} is a finite $\vec{n}[t^*]$ -good tree from τ for $Q_{(t^*, i^*)} = \{\vec{\sigma}_{t^*} : \Phi_{t^*}^{\vec{\sigma}_{t^*}}(e_{t^*}) = i^*\}$.

Applying this with $t^* := t$ and $i^* := i$, we have that $G[s] = G[s]_{s-1}$ and there is no pair $\langle \vec{T}_t, i \rangle$ such that $i < h(e_t)$ and \vec{T}_t is a finite $\vec{n}[t]$ -good tree from $G[s]$ for $Q_{(t,i)} = \{\vec{\sigma}_t : \Phi_t^{\vec{\sigma}_t}(e_t) = i\}$.

But \vec{V}_t is exactly such a tree \vec{T}_t , so we have a contradiction.

(2) *implies* (3):

Since \vec{V}_s is $\vec{g}_s(\vec{n}_s[s], \epsilon[s])$ -good, by Lemma 2.10 there is an element $\vec{\beta}_s$ of \vec{V}_s from which there is no $\vec{g}_s(\vec{n}_s[s], \epsilon[s])$ -good tree for Q , and hence not for any $Q_{(t,i)}$ since $Q_{(t,i)} \subseteq Q$. Moreover $\vec{\beta}_s$ properly extends $G[s]$, since \vec{V}_s is not just the singleton of $G[s]$. So as \vec{V}_s is $\vec{g}_s(\vec{n}_s[s], \epsilon[s])$ -good, as $\vec{n}[s+1] = \vec{n}_{s+1}[s+1] = (\vec{g}_s(\vec{n}_s[s], \epsilon[s]), 2)$ and as by Lemma 3.13, $n_t[s] \geq 2$ for each $t \leq s$, it follows that every column β_t of $\vec{\beta}_s$ extends $G_t[s]$ properly.

Hence by definition of A_{s+1} , this element $\vec{\beta}$ belongs to A_{s+1} .

(3) *for* s *implies* (1) *for* $s+1$:

If Case 1 obtains, let \vec{T}_s be the tree found by Φ_e , i. e., \vec{T}_s is $\vec{g}_s(\vec{n}_s[s], \epsilon[s])$ -good from $G[s]$ (for $Q_{(s,i)}$ for some i). If \vec{T}_s is not just the singleton of $G[s]$, and Case 1 obtains, then apply (3) for s to \vec{T}_s .

If \vec{T}_s is just the singleton of $G[s]$ or if Case 2 obtains, then apply (3) for s to any $\vec{g}_s(\vec{n}_s[s], \epsilon[s])$ -good non-singleton system of trees from $G[s]$. \dashv

Lemma 3.15. For any $s \geq 0$, if $s \in D_{s+1}$ then $\Phi_s^G(e_s) \uparrow$ or $\Phi_s^G(e_s) \geq h(e_s)$.

PROOF. Otherwise for some $t \in \omega$, $\Phi_s^{G[t]}(e_s) \downarrow < h(e_s)$. Since the system whose only element is $G[t]$ is \vec{n}_x -good from $G[t]$ for all \vec{n}_x with $x \geq t$, hence in particular $\vec{n}[t] = \vec{n}_t[t]$ -good, this contradicts the fact that $G[t] \in A_t$. \dashv

For each $x \in \omega$, let $\vec{1}_x = (1_0, \dots, 1_x)$ be the sequence of length $x+1$ consisting of all 1's, i. e., where $1_0 = \dots = 1_x = 1$.

Lemma 3.16. For each $y \geq 0$, there is no $(\vec{1}_{2y-1}, 2)$ -good system from $G[2y]$ for the property $\{\beta : \Phi_{2y}^\beta \downarrow\}$.

PROOF. Suppose there is such a system \vec{T}_{2y} .

First suppose \vec{T}_{2y} has only one element. Then this element is $G[2y]$, by the definition of a good system from $G[2y]$. Hence $\Phi_{2y}^{G[2y]} \downarrow$. But $G_{2y}[2y]$, column $2y$ of G as constructed during stage $2y$, is empty. So by Definition 3.2, $\Phi_{2y}^{G[2y]} \uparrow$, so we have a contradiction.

Now suppose \vec{T}_{2y} has more than one element. Given $G_0 \oplus \cdots \oplus G_{2y-1}$, there is at most one value of $G_{2y}(0)$ such that $\langle G_{2y}(0) \rangle$ is not a $\text{DNR}^{G_0 \oplus \cdots \oplus G_{2y-1}}$ string. Hence for any sequence of positive integers \vec{n}_{2y} , if \vec{T}_{2y} is \vec{n}_{2y} -good from $G[s]$ then $n_{2y} \leq 1$, so $2 \leq 1$, which is a contradiction. \dashv

Lemma 3.17. For each $y \in \omega$, $2y \in D_{2y+1}$.

PROOF. By definition of D_{2y+1} , it suffices to show that at stage $2y+1$ of the Construction, there is no $\vec{g}(\vec{n}[2y], \epsilon[2y])$ -good system from $G[2y]$ for $\{\beta : \Phi_{2y}^\beta \downarrow = i\}$ for any $i < h(e)$. We will show this in fact for $\{\beta : \Phi_{2y}^\beta \downarrow\}$.

Suppose there is such a system \vec{T}_{2y} . By Lemma 3.13, $g_x(\vec{n}[2y], \epsilon[2y]) \geq 2$ for $0 \leq x \leq 2y$. Hence by Lemma 3.4, \vec{T}_{2y} is $(\vec{1}_{2y-1}, 2)$ -good. By Lemma 3.16, we have a contradiction. \dashv

Lemma 3.18. G is a total function, i. e., $G \in \omega^\omega$.

PROOF. By Lemma 3.14(3), $G[s+1] \in A_{s+1}$ for each $s \geq 0$, and hence by definition of A_{s+1} , $G_t[s+1]$ is a proper extension of $G_t[s]$ for each $t \leq s$. From this the lemma immediately follows. \dashv

Lemma 3.19. G is relatively DNR.

The proof of Lemma 3.19 from Definition 3.2, Lemma 3.15 and Lemma 3.17 is formally identical to the proof of Lemma 2.19 from Definition 2.2, Lemma 2.15 and Lemma 2.17.

Lemma 3.20. For each $y \in \omega$, $G_0 \oplus \cdots \oplus G_y$ computes no h -DNR function.

PROOF. It suffices to show that given $y \in \omega$, and a Turing functional Ψ which does not query its oracle beyond column $2y+1$, $\Psi^{G_0 \oplus \cdots \oplus G_{2y+1}}$ is not h -DNR. In the Construction we have been considering the Turing functionals $\Phi_z, z \in \omega$ of Definition 3.2. Since each Turing functional has infinitely many indices, it follows from Definition 3.2 that there are infinitely many odd numbers $s \geq 2y+1$ such that

$$\Psi^{G_0 \oplus \cdots \oplus G_{2y+1}} = \Phi_s^{G_0 \oplus \cdots \oplus G_{2y+1}} = \Phi_s^{G_0 \oplus \cdots \oplus G_s} = \Phi_s^G.$$

Fix such an s and consider stage $s+1$ of the Construction. If Case 1 holds then $\Phi_s^G(e) = \Phi_e(e)$ and so Φ_s^G is not h -DNR. If Case 2 holds then by Lemma 3.15, $\Phi_s^G(e) \uparrow$ or $\Phi_s^G(e) \geq h(e)$. Hence Φ_s^G is not h -DNR. \dashv

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